Mathematica Moravica
Vol. **8–2** (2004), 11–28

Classes of $d(\psi)$-Functions, $d(L^{\psi})$-Spaces, and Orlicz Spaces

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Dedicated to professor Julian Musielak on his 75^{th} birthday.

Abstract. The main purpose of this paper is to give an exposition on fundamental facts and basic notions, and further on some key-results, on new classes of $d(\psi)$-functions and new classes of $d(L^{\psi})$-spaces. This facts and results are directly in connection with Orlicz spaces.

1. Introduction and history

Well known that one of the real attraction of Orlicz spaces is that the subject is sufficiently concrete and yet the spaces have fine structure of importance for applications.

The idea of Orlicz spaces lies in generalizing the space $L^p(a,b)$ of functions, integrable with power $p \geq 1$ in interval (a,b), replacing the power $|u|^p$ by a more general function $\varphi(u)$.

Supposing φ to be an even, convex function equal to 0 only at $u=0$, the set of measurable functions x on (a,b) such that

$$\int_a^b \varphi\big(|x(t)|\big)dt < \infty - \textbf{the Orlicz class} -$$

does not need to be a linear space. The theory of such spaces was introduced and developed by W. Orlicz in early 1930th.

After the war years the study and applications have been vigorous in Poland, Russia and Japan, the latter under the lead of H. Nakano, W. Orlicz and J. Musielak with the name "modulared spaces".

Early results concerning Orlicz spaces may by found in the monograph by M.A. Krasnoselskij and Ya. B. Rutickij in 1958. If further, the assumption of convexity

2000 *Mathematics Subject Classification.* Primary: 26A12, 46F10. Secondary: 42A20, 43A85, 10K30, 60F05.

Key words and phrases. $d(\psi)$-function, φ-function, $d(L^{\psi})$-spaces, L^{φ}-spaces, Orlicz class, Orlicz spaces, $d(\psi)$-class, transversal upper and lower modular spaces, transversal upper and lower modulars, complementary M-function, $d(M)$-class, Δ_2-condition, $d(\Delta_2)$-condition, quasilinear representation.

*Research supported by Science Fund of Serbia under Grant 1457.

was replaced by that of monotony and continuity of φ, thus embracing the case of powers $|u|^p$ for $0 < p < 1$.

In this sense, W. Matuszewska introduced in 1961 the following notion of a **φ-function**, i.e., a nonnegative, even function φ, vanishing only at 0, increasing for $u \geq 0$, i.e., $u \in \mathbb{R}_+^0 := [0, +\infty)$, and such that $\varphi(u) \to \infty$ as $u \to \infty$.

A brief exposition of generalized Orlicz spaces generated by a φ-function may be found in: Orlicz [20]. A more general setting to modular spaces was given in 1959 by W. Orlicz and J. Musielak.

The work of Zaanen, and especially his book on Linear Analysis in 1953 has enabled the western countries to develop the theory and applications in many directions.

In connection with the preceding facts, first, in Tasković [23] we introduced the concept of transversal (upper and lower) normed spaces as a natural extension of normed and Banach spaces.

In this section I give a brief account of the notion of transversal (upper and lower) modular spaces in the following sense.

Let X be a linear space over $\mathbb{K}(:= \mathbb{R}$ or $\mathbb{C})$. The following mapping $\rho : X \to [0, \infty] := \mathbb{R}_+^0 \cup \{+\infty\}$ is called an **upper transversal semimodular** (or *upper semimodular*) iff: $\rho(ax) = \rho(x)$ if $|a| = 1$ and $x \in X$, and if there is a function $g : [0, \infty]^2 \to [0, \infty]$ such that

$$\text{(Mu)} \qquad \rho(ax + by) \leq \max\Big\{\rho(x), \rho(y), g\big(\rho(x), \rho(y)\big)\Big\}$$

for all $x, y \in X$, where $a, b \geq 0$ and $a + b \leq 1$.

Further, $x \mapsto \rho(x)$ is called an **upper transversal modular** (or *upper modular*) iff in addition $\rho(x) = 0$ if and only if $x = 0$.

An **upper transversal modular space** $(X, \rho(x))$ over $\mathbb{K}$ consists of a linear space X over $\mathbb{K}$ together with an upper transversal modular $x \mapsto \rho(x)$.

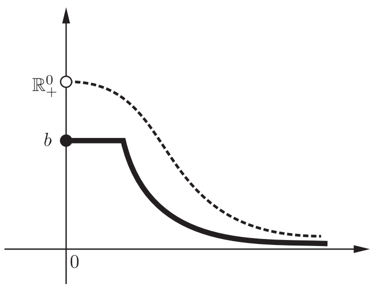

Figure 1

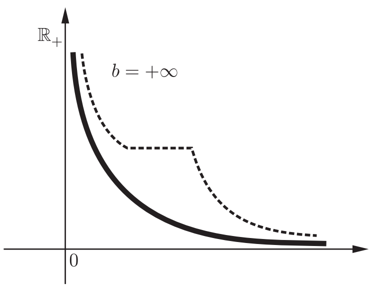

Figure 2

The function $g : [0, \infty]^2 \to [0, \infty]$ in (Mu) is called **upper bisection function**. We notice that the upper transversal modular, de facto, is a general convex function. Otherwise, the general convex functions are introduced in Tasković: *Math. Japonica*, **37** (1992), 367-372.

It is easily seen that the upper transversal norm in an upper transversal normed space is a general convex function and thus an upper transversal (general convex) modular.

Let $g(X_\rho)$ be the set of all $x \in X$ such that there exists a positive number k with property $\rho(kx) < +\infty$. If the upper bisection function g is an increasing function by variables satisfying $g(t,t) \le t$ for every $t \in [0,\infty]$, then the set $g(X_\rho)$ is linear.

On the other hand, the mapping $\rho : X \to [0,\infty]$ is called a **lower transversal semimodular** (or *lower semimodular*) iff: $\rho(ax) = \rho(x)$ if $|a| = 1$ and $x \in X$, and if there is a function $d : [0,\infty]^2 \to [0,\infty]$ such that

$$\rho(ax + by) \ge \min\Big\{\rho(x), \rho(y), d\big(\rho(x), \rho(y)\big)\Big\} \tag{Ml}$$

for all $x, y \in X$, where $a, b \ge 0$ and $a + b \le 1$.

In this sense, $x \mapsto \rho(x)$ is called a **lower transversal modular** (or *lower modular*) iff in addition: $\rho(x) = 0$ if and only if $x = 0$ (or iff in addition: $\rho(x) = +\infty$ if and only if $x = 0$).

A **lower transversal modular space** $(X, \rho(x))$ over $\mathbb{K}$ consists of a linear space X over $\mathbb{K}$ together with a lower transversal modular $x \mapsto \rho(x)$.

The function $d : [0,\infty]^2 \to [0,\infty]$ in (Ml) is called **lower bisection function**. Evidently, the lower transversal modular is a general concave function, from Tasković [23].

It is easily seen that the lower transversal norm in a lower transversal normed space is a general concave function and thus a lower transversal (general concave) modular.

Let $d(X_\rho)$ be the set of all $x \in X$ such that there exists a positive number k with property $\rho(kx) < +\infty$. If the lower bisection function d is an increasing function by variables satisfying $d(t,t) \ge t$ for every $t \in [0,\infty]$, then the set $d(X_\rho)$ is linear. For further facts on transversal upper and lower modular spaces see: Tasković [23].

The purpose of this paper is to give some fundamental facts and an exposition of basic notions, and some key-results on new classes of $d(\psi)$-functions and $d(L^\psi)$-spaces. This facts are directly in connection with modular spaces and Orlicz spaces.

In the theory of Orlicz spaces fundamental notions are *convex functions*, but, in this exposition of $d(L^\psi)$-spaces an important role play *difference of two convex functions.*

2. $d(\psi)$-FUNCTIONS AND AN INEQUALITY

In this paper, by a **$d(\psi)$-function** I shall understand a continuous nonincreasing function $\psi : \mathbb{R}_+^0 \to \mathbb{R}_+^0 := [0,+\infty)$ for which $\psi(u) \to 0$ as $u \to +\infty$ and $\psi(0) = b$ (for some $0 < b \le +\infty$).

In this sense, for the case $b = +\infty$, I shall understand that $\psi(u) \to +\infty$ as $u \to 0$, where $f|\mathbb{R}_+ := (0,+\infty)$. On the Figs. 1, 2, 3 and 4 we have the essential forms of $d(\psi)$-functions.

Class $d(\psi)$-functions appear often in various problems of nonlinear analysis and have a certain analogy (although essential unlike) with the class of φ-functions from Matuszewska [6].

In our context, first time, the forms of $d(\psi)$-functions are appear in the connection with the transversal lower (modular) spaces in 1998 from: Tasković [22] and [24].

The following conditions appear often in various problems in which the $d(\psi)$-functions are of importance:

$$\lim_{u\to 0} u\psi(u) = 0 \text{ and } \lim_{u\to\infty} u\psi(u) = 0,$$

in contrast with appears of the φ-functions in various problems in the forms:

$$\lim_{u\to 0} \frac{\varphi(u)}{u} = 0 \quad \text{and} \quad \lim_{u\to\infty} \frac{\varphi(u)}{u} = +\infty.$$

Let Ω be a nonempty set and let Σ be a σ-algebra of subsets of Ω. Also, let μ be a nonnegative, nontrivial, complete, σ-finite measure on Σ. We take as X the space of all extended real valued, Σ-measurable functions on Ω with equality μ-almost everywhere. Let ψ be a $d(\psi)$-function, then

$$\rho(x) = \int_\Omega \psi(x(t))d\mu$$

is a lower transversal modular in X. Moreover, if ψ is concave, then ρ is concave.

The set of all $x \in X$ for which $\rho(x) < +\infty$ is called the $d(\psi)$**-class**, till the lower transversal modular space $d(X_\rho)$ denoted by $d(L^\psi, (\Omega, \Sigma, \mu))$ or $d(L^\psi)$, i.e., $d(L^\psi)$**-space**. It is evident that $d(L^\psi)$ is the smallest linear space containing the $d(\psi)$-class.

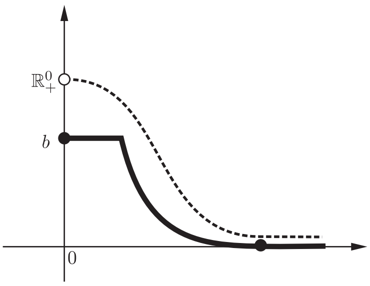

Figure 3

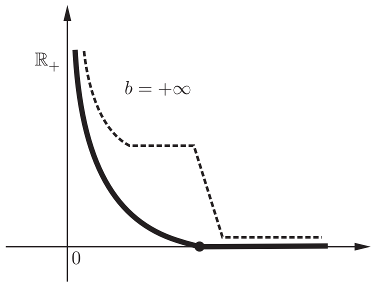

Figure 4

An inequality. *If $f : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ is a continuous and strictly decreasing function with property $f(0) = b \in \mathbb{R}_+$, then for $a > 0$ the following inequality holds in the form*

$$\big(a - f^{-1}(0)\big)b \le \int_0^a f(x)dx - \int_0^b f^{-1}(x)dx, \tag{1}$$

where the equality holds in (1) if and only if $f(a) = 0$. (In (1) f^{-1} is the inverse function of f).

It is easily seen that the function $f : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ from inequality (1) is a function in the class of $d(\psi)$-functions i.e., $f \in d(\psi)$.

On the other hand, from inequality (1), we can defined two complementary functions in the following sense. We set

$$\psi(u) = \int_0^u f(t)dt, \quad \text{and} \quad \psi^*(u) = \int_0^u f^{-1}(t)dt,$$

for $u \geq 0$. Then, the functions $\psi(u)$ and $\psi^*(u)$ are **complementary** in the sense of inequality (1). It is easily proved that they satisfy the following inequality of the form

$$uv - vf^{-1}(0) \leq \psi(u) - \psi^*(v)$$

for all $u, v \geq 0$; and that

$$\psi^*(u) = \inf_{v \geq 0} \Big(\psi(v) - uv + vf^{-1}(0)\Big),$$

and

$$\psi(u) = \sup_{v \geq 0} \Big(uv - vf^{-1}(0) + \psi^*(v)\Big),$$

for every $u \geq 0$, where the infimum being reached at $v = f^{-1}(u)$ in the first case, and the supremum at $v = f(u)$ in the second case. Now, it is easily proved that

$$||x||_\psi^* = \sup \left\{ \int_\Omega x(t)y(t)d\mu : y \in X, \int_\Omega \psi^*(y(t))d\mu \leq 1 \right\},$$

is a lower transversal norm in $d(L^\psi)$, and, on the other hand, that is

$$||x||_\psi = \inf \left\{ u > 0 : \int_\Omega \psi\left(\frac{x(t)}{u}\right) d\mu \leq 1 \right\};$$

also, is a lower transversal norm in $d(L^\psi)$. Obviously is here similarity with the Luxemburg and Orlicz norms.

3. Further on $d(L^\psi)$-spaces

An extension of the $d(L^\psi)$-space is the following. We take a function $\psi : \mathbb{R}_+^0 \times \Omega \to \mathbb{R}_+^0$ such that: $\psi(u,t) = b$ (for some $0 < b \leq +\infty$) if and only if $u = 0$, $\psi(u,t)$ is a continuous and nonincreasing function of $u \geq 0$ for a.e. $t \in \Omega$, $\psi(u,t)$ is Σ-measurable in Ω for every $u \geq 0$ and $\psi(u,t) \to \infty$ as $u \to \infty$ for a.e. $t \in \Omega$. Then

$$\rho(x) = \int_\Omega \psi(x(t),t)d\mu$$

is a lower transversal modular in X. On the other hand, one may consider also $d(L^\psi)$-spaces of vector valued functions with values in a lower transversal normed space $(E, ||\cdot||)$ which is lower complete. The lower transversal modular ρ may be written then in the form

$$\rho(x) = \int_\Omega \psi(||x(t)||,t)d\mu,$$

with suitable measurability for the functions x. Also, this may be still extended to the form

$$\rho(x) = \int_{\Omega} \psi(x(t), t) d\mu,$$

where ψ becomes a map of $E \times \Omega$ in $\mathbb{R}^0_+$. For further facts of this see: Tasković [24].

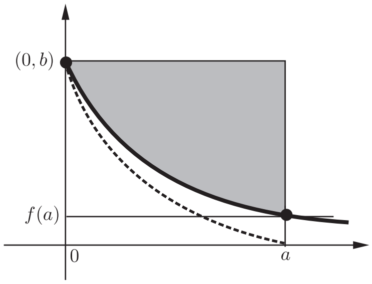

Figure 5

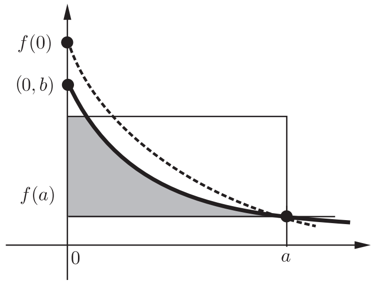

Figure 6

Proof of inequality (1). From the Figures 5 and 6 we have a geometric proof of (1). Meanwhile, for an analytic proof of inequalityu (1), we set

$$g(a) = ab - bf^{-1}(0) - \int_0^a f(x)dx,$$

and consider $b > 0$ as a parametar. Since $g'(a) = b - f(a) + f(0)$, and f is strictly decreasing, we have

$$g'(a) > 0 \quad \text{for} \quad 0 < a < f^{-1}(b - f(0)),$$
$$g'(a) = 0 \quad \text{for} \quad a = f^{-1}(b - f(0)), g'(a) < 0 \quad \text{for} \quad a > f^{-1}(b - f(0))$$

and thus, $g(a)$ is a maximum of g for $a = f^{-1}(b - f(0))$. Therefore, integrating by parts, we obtain

$$g\big(f^{-1}(b - f(0))\big) = f(0)f^{-1}(b - f(0)) - bf^{-1}(0) + \int_{f(0)}^{b-f(0)} f^{-1}(y)dy,$$

i.e., for $b = f(0)$ we have $g(a) \leq g(f^{-1}(0))$, and thus calculating we get (1). The proof is complete.

Further annotations. We notice that Figs. 5 and 6 affirm that inequality (1) is justified. On the other hand, if we consider the areas by Figs. 5 and 6, then directly for a strictly decreasing function $f \in d(\psi)$ we obtain the following inequality of form

$$af(a) \leq \int_0^a f(x)dx - \int_{f(a)}^b f^{-1}(x)dx, \tag{2}$$

where $f(0) \geq b$ and $a > 0$. Equality holds in (2) if and only if $f(0) = b$ and $f(a) = 0$.

We notice that inequality (2) is ties with some inequalities in: Boas-Marcus [2], Mitrinović - Pečarić - Fink [9], and Tasković [24].

4. CLASS OF $d(\psi)$-FUNCTIONS

We shall say that a continuous and nonincreasing[1] function $\psi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ in the class $d(\psi)$ or that is a $d(\psi)$-**function** if

$$\psi(\infty) = \lim_{u\to\infty} \psi(u) = 0$$

and $\psi(0) = b$ $(0 < b \le +\infty)$; where in the case $b = +\infty$ we understand a function of the form that $f : \mathbb{R}^0_+ \to \mathbb{R}^0_+ \cup \{+\infty\}$.

Namely, in the case $b = +\infty$, we can ψ, comprehend and as a restriction on the set $\mathbb{R}_+$, i.e., $\psi|\mathbb{R}_+ \to \mathbb{R}^0_+$. The elements of the class $d(\psi)$ we denoted by ψ, ξ, φ, …and $\xi \in d(\psi)$ is a measurable function. Also, $\xi(u) = \xi \circ u$ is a measurable function for the arbitrary measurable function $u : \mathbb{R}_+ \to \mathbb{R}^0_+$

For the function $\psi \in d(\psi)$ we defined on the **right inverse function** $\psi^* : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ in the folowing sense

$$\psi^*(t) = \sup \psi^{-1}(t, b) = \inf \psi^{-1}([0, t]),$$

where $0 < b \le +\infty$. It is easily proved that $\psi^* \in d(\psi)$ and $\psi^{**} = \psi$, i.e., ψ^* is an idempotent mapping. Precisely, this means that ψ and ψ^* are reciprocally on the right inverse functions.

Proposition 4.1. *Let $\psi \in d(\psi)$. Then the following characteristic facts hold:*

(a) $\psi^*(\psi(s)) \le s$ *for every* $s \ge 0$.

(b) $t > \psi(s)$ *implies that is* $\psi^*(t) < s$.

(c) $\varphi(s) = a\psi(bs)$ *implies* $\varphi^*(t) = \dfrac{1}{b}\psi^*\left(\dfrac{t}{a}\right)$ *for* $a, b > 0$.

(d) $\psi^*(\psi(s) - \varepsilon) \ge s$ *for every* $s > 0$ *and* $0 < \varepsilon < \psi(s)$.

A brief proof of this statement we can to make in the proper manner from book of Musielak [10]. Also see: Matuszewska [7] and Tasković [24].

Annotation. If $\psi \in d(\psi)$ with the values t, then $\psi^*(t) = \inf \psi^{-1}(\{t\})$. Otherwise, if ψ^* (for $\psi \in d(\psi)$) is a continuous function in the point $\psi(s)$, then, from Proposition 1, the following equality holds: $\psi^*(\psi(s)) = s$.

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[1]**Decreasing functions.** For the decreasing function $\psi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ holds the following properties: for every $s \ge 0$ there exist the following expression (we set on convention $\psi(-0) = \psi(0)$):

$$\psi_-(s) = \psi(s-0) = \lim_{h\to 0+} \psi(s-h) = \inf([0,s]) = \inf \psi((t,s)) \quad \text{for} \quad 0 \le t < s \quad \text{and}$$

$$\psi_+(s) = \psi(s+0) = \lim_{h\to 0+} \psi(s+h) = \sup \psi((s,\infty)) = \sup \psi((s,t)) \quad \text{for} \quad s < t;$$

where holds the following inequalities in the forms: $\psi_+ \le \psi \le \psi_-$ and $\psi_-(t) \le \psi_+(s)$ for $s < t$.

On the other hand, a decreasing function $\psi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ is *from the right continuous* if holds the following equality

$$\psi(\sup S) = \inf \psi(S)$$

for every nonempty bounded set $S \subset \mathbb{R}^0_+$. In this sense, if holds and the following equality in the form (*continuity from the left*)

$$\psi(\inf S) = \sup \psi(S),$$

then we say, that the mapping, ψ is continuous. For further facts on the decreasing mappings see: Tasković [24].

The class of $d(M)$-functions. In the further, a function $G : \mathbb{R} \to \mathbb{R}_+^0$ is called an M**-function** if there is a function $\psi \in d(\psi)$ such that

$$G(s) = \int_0^{|s|} \psi(t)dt. \tag{3}$$

By $d(M)$ we shall denote the set of all M-functions. If (3) holds, then we say that the function G determined by the function ψ.

The value of the M-function, itself is the magnitude of the area of the corresponding curvilinear trapezoid. If follows from representation (3) that every M-function is even, thus for $s \in \mathbb{R}_+^0$ and $G \in d(M)$ directly we have the following inequalities

$$as\psi\left(\frac{s}{a}\right) < G(s) < s\psi(0), \tag{4}$$

i.e., in an equivalent form,

$$a^2 s\psi(s) < G(as) < as\psi(0) \tag{5}$$

for $s > 0$ and for $0 < a \leq 1$. Hence, from the preceding inequalities (4) and (5), directly calculating, we obtain the following inequalities in the form

$$\psi_-(\max\{s,t\}) \leq \frac{G(s) - G(t)}{s - t} \leq \psi_+(\min\{s,t\}); \tag{6}$$

and thus, from the preceding facts, we have

$$G(s) = \max_{t \geq 0} \Big\{ G(t) + (s-t)\psi_-(s) \Big\} = \min_{t \geq 0} \Big\{ G(t) + (s-t)\psi_+(s) \Big\}.$$

Proposition 4.2. *The function $f : \mathbb{R} \to \mathbb{R}$ belongs to the class $d(M)$ if and only if the following facts hold: $f(s) = 0$ if and only if $s = 0$, $f \in BC(\mathbb{R}_+^0)^2$, f is continuous even, and*

$$0 \leq \frac{f(\infty)}{\infty} \leq b = \frac{f(0)}{0} \quad (0 < b \leq +\infty).$$

A brief proof of this statement may be found in: Tasković [24]. As an immediate application of this result and the preceding facts we have the following inequalities:

$$G(as) \geq aG(s) \quad \text{for} \quad s \geq 0 \quad \text{and} \quad 0 \leq a \leq 1. \tag{7}$$

$$G(as) \leq aG(s) \quad \text{for} \quad s \geq 0 \quad \text{and} \quad a \geq 1. \tag{8}$$

$$\frac{G(t)}{t} < \frac{G(s)}{s} \quad \text{for} \quad 0 \leq s < t. \tag{9}$$

$$G(s+t) < G(s) + G(t) \quad \text{for} \quad s,t > 0. \tag{10}$$

We notice that properties of the function G directly to bring about properties hers the inverse function defined by $G^{-1} = (G|\mathbb{R}_+^0)^{-1}$.

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[2] $BC(\mathbb{R}_+^0)$ denoted the set of all functions which are difference of two convex functions on $\mathbb{R}_+^0$.

The function G^{-1}, defined in the preceding sense, is continuous, difference is two convex functions, i.e., $G^{-1} \in BC(\mathbb{R}^0_+)$, and strictly decreasing on $\mathbb{R}^0_+$.

If to make corresponding replacements variables, from the inequalities (7)-(10), for the function G^{-1} we obtain the following inequalities:

$$G^{-1}(as) \leq aG^{-1}(s) \quad \text{for} \quad s \geq 0 \quad \text{and} \quad 0 \leq a \leq 1. \tag{11}$$

$$G^{-1}(as) \geq aG^{-1}(s) \quad \text{for} \quad s \geq 0 \quad \text{and} \quad a \geq 1. \tag{12}$$

$$\frac{t}{G^{-1}(t)} < \frac{s}{G^{-1}(s)} \quad \text{for} \quad 0 \leq s < t. \tag{13}$$

$$G^{-1}(s+t) > G^{-1}(s) + G^{-1}(t) \quad \text{for} \quad s, t > 0. \tag{14}$$

We notice that equalities hold in (7), (8), (11) and (12) if and only if the point (s, a) is on edge region in which the preceding inequalities hold.

5. Complementary M-functions

For a M-function hers the **complementary function** (in the sense of inequalities (4)) is the function G^* in the following form:

$$G^*(t) = \int_0^{|t|} \psi^*(s)ds.$$

From the some former facts, complementary mapping is an idempotent mapping of the class $d(M)$ into itself. From Proposition 1 we have that

$$A(s) = aG(bs) \quad \text{implies} \quad A^*(s) = aG^*\left(\frac{t}{ab}\right),$$

where $a, b > 0$ are constants.

As an immediate consequence of the former inequality (1) is the following statement.

Theorem 5.1. *Let the function $G \in d(M)$. Then for mutually complementary functions G and G^* the following inequality holds in the form*

$$(s - \psi^{-1}(0))t \leq G(s) - G^*(t) \tag{15}$$

for $s, t \geq 0$, where equality holds in this case if and only if $\psi(s) = 0$.

From the preceding Theorem 1 as an immediate consequence we have the **quasilinear representation** of an arbitrary complementary function in the form

$$G^*(t) = \min_{s \geq 0} \Big(G(s) - [s - \psi^{-1}(0)]t\Big). \tag{16}$$

6. THE COMPARISON OF M-FUNCTIONS

In the sequel, an essential role will be played by the *rapidity of growth* of the values of an M-function as $n \to \infty$. In connection with this, for the function $G \in d(M)$ we shall say that before the function $R \in d(M)$, in note $G \prec R$ or $G(s) \prec R(s)$, if there exists constant $K > 0$ such that

$$G^{-1}(s) \leq R^{-1}(Ks) \quad \text{for enough large } s. \tag{17}$$

Otherwise, we shall say that the M-functions G and R are **equivalent** in write $G \sim R$ or $G(s) \sim R(s)$ if $G \prec R$ and $R \prec G$, i.e., if there exist constants K, $k > 0$ such that

$$R^{-1}(ks) \leq G^{-1}(s) \leq R^{-1}(Ks) \quad \text{for enough large } s. \tag{18}$$

We notice that the binar relation $\prec$ defined a quasiorder in $d(M)$, which means that $\sim$ is an equivalente relation in $d(M)$ agreed with $\sim$; but the relation $\leq$ defined is on the quotient set $d(M)/\sim = \{[G] : G \in d(M)\}$ with

$$[G] \leq [R] \quad \text{if and only if} \quad G \prec R,$$

and she is a partial order relation on $d(M)/\sim$. In connection with this, from (7) and (8), directly calculating we obtain

$$G^{-1}\big(s \min\{a, 1\}\big) \leq aG^{-1}(s) \leq G^{-1}\big(s \max\{a, 1\}\big)$$

for $a > 1$ and for $s \geq 0$; hence, from the preceding facts, we have $G \sim aG$. Since $G(s) \sim G(bs)$ for $b > 0$, thus we obtain

$$G(s) \sim aG(bs) \quad \text{for all} \quad a, b > 0.$$

Some annotations. We notice that in (17) we can suppose that there exists constant $K \geq 1$ instead $K > 0$. Indeed, if holds (17) for $s \geq z$ and if $r < z$, then for $r \leq s \leq z$ the following inequality holds

$$G^{-1}(s) \leq R^{-1}(K_0 s), \quad \text{for} \quad K_0 = \max\left\{1, \sup_{r \leq s \leq z}\left(\frac{G^{-1}(s)}{R^{-1}(s)}\right)\right\};$$

and thus, for $K_1 = \max\{K, K_0\}$, we obtain that is $G^{-1}(s) \leq R^{-1}(K_1 s)$ for $s \geq r$.

From this, for an arbitrary constant $r > 0$, we have that fact $G \prec R$ is an equivalent with the fact: that there exists constant $K \geq 1$ such that

$$G^{-1}(s) \leq R^{-1}(Ks) \quad \text{for every} \quad s \geq r.$$

In connection with this, also we notice that the fact $G \prec R$ is equivalent with the fact that there exist constants K_0, $K > 0$ such that

$$G^{-1}(s) \leq K_0 + R^{-1}(Ks) \quad \text{for every} \quad s \geq 0. \tag{19}$$

Indeed, let $G^{-1}(s) \leq R^{-1}(Ks)$ for every $s \geq 0$. Thus we have $G^{-1}(s) \leq K_0 + R^{-1}(Ks)$ for some $K_0 > 0$, i.e., holds (19). Reversed, if holds (19) and if

$K_0 = R^{-1}(K_1)$ for $K_1 > 0$, from (14) we have

$$G^{-1}(s) \leq R^{-1}(K_1) + R^{-1}(Ks) \leq R^{-1}(K_1 s) + R^{-1}(Ks) \leq$$
$$\leq R^{-1}(K_1 s + Ks) = R^{-1}((K_1 + K)s)$$

for $s \geq 1$. This means that $G \prec R$ is equivalent with inequality (19).

On the other hand, from properties of M-functions and the order relation $\prec$, directly follow that $G \prec R$ is equivalent with the fact:

$$\inf_{b \geq 1} \left(\limsup_{s \to \infty} \frac{G^{-1}(s)}{R^{-1}(bs)} \right) = 0.$$

In connection with the preceding partial ordering, let $G \,\square\!\!\!\!\prec\, R$ for $G, R \in d(M)$ have the following means: that there exists constant $K > 0$ such that

$$G^{-1}(s) \leq K R^{-1}(s) \quad \text{for enough large } s;$$

and let $G \,\square\!\!\!\!\sim\, R$ for $G, R \in d(M)$ have the following means $G \,\square\!\!\!\!\prec\, R$ and $R \,\square\!\!\!\!\prec\, G$, i.e., this means that there exist two constants K, $k > 0$ such that

$$k R^{-1}(s) \leq G^{-1}(s) \leq K R^{-1}(s) \quad \text{for enough large } s.$$

It is easily seen that $\square\!\!\!\!\prec$ is a quasiorder, till $\square\!\!\!\!\sim$ is an equivalence relation in the $d(M)$-class. The fact $G \,\square\!\!\!\!\prec\, R$ is equivalent with the fact: that there exist two constants K_0, $K > 0$ such that

$$G^{-1}(s) \leq K_0 + K R^{-1}(s) \quad \text{for every} \quad s \geq 0.$$

The $d(\Delta_2)$-condition. The following condition in order that an $d(M)$-class be linear (i.e., identical with the $d(L^\psi)$-space) is essential.

We say that the M-function G satisfies the **$d(\Delta_2)$-condition** for large values of s if there exists constant $K > 0$ such that

$$G^{-1}(2s) \leq K G^{-1}(s) \quad \text{for enough large } s. \tag{20}$$

If M-function G satisfies (20) for every $s \geq 0$, then we say that G satisfies $d(\Delta_2)$-condition for every $s \geq 0$. The class of all M-functions which satisfies $d(\Delta_2)$-condition denoted by $d(M_2)$. In this context, the mappings p_∞, $q_\infty : d(M) \to [0, 1]$ we define by:

$$p_\infty(G) = \liminf_{s \to \infty} \frac{s\psi(s)}{G(s)} \quad \text{and} \quad q_\infty(G) = \limsup_{s \to \infty} \frac{s\psi(s)}{G(s)};$$

for which we have $p_\infty(bG(as)) = p_\infty(G)$ and $q_\infty(bG(as)) = q_\infty(G)$ for arbitrary parametars $a, b > 0$. Since $s \mapsto s^{-r}G(s)$ for $s \geq t$ is a nonincreasing function if and only if

$$\frac{s\psi(s)}{G(s)} \leq r \quad \text{for every} \quad s \geq t,$$

and since $s \mapsto s^{-r}G(s)$ for $s \geq t$ is a nondecreasing function if and only if

$$\frac{s\psi(s)}{G(s)} \geq r \quad \text{for every} \quad s \geq t,$$

thus, we obtain directly the following equalities for functions $p_\infty(G)$ and $q_\infty(G)$ in the following forms:

$$p_\infty(G) = \sup\big\{r \in (0,1) : s^{-r}G(s) \quad \text{increasing for} \quad s \geq t\big\}$$

and

$$q_\infty(G) = \inf\big\{r \in (0,1) : s^{-r}G(s) \quad \text{decreasing for} \quad s \geq t\big\}.$$

Otherwise, from the preceding facts and geometric reasons we can considered the mappings $\mathcal{L}_\infty(G)$, $\mathcal{Z}_\infty(G) : d(M) \to [1,\infty)$ defined by

$$\mathcal{L}_\infty(G) = \liminf_{s\to\infty} \frac{s\psi(0)}{G(s)}, \quad \text{and} \quad \mathcal{Z}_\infty(G) = \limsup_{s\to\infty} \frac{s\psi(0)}{G(s)};$$

for which, also, $\mathcal{L}_\infty(bG(as)) = \mathcal{L}_\infty(G)$ and $\mathcal{Z}_\infty(bG(as)) = \mathcal{Z}_\infty(G)$ for arbitrary parameters $a, b > 0$. Also, the mappings $p, q : d(M) \to [0,1]$ we define by:

$$p(G) = \inf_{s>0} \frac{s\psi(s)}{G(s)}, \quad \text{and} \quad q(G) = \sup_{s>0} \frac{s\psi(s)}{G(s)};$$

for which the following inequalities hold: $p(G) \leq p_\infty(G) \leq q_\infty(G) \leq q(G) \leq 1$. Also, the mappings $\mathcal{L}(G)$, $\mathcal{Z}(G) : d(M) \to [1,\infty)$ we define by

$$\mathcal{L}(G) = \inf_{s>0} \frac{s\psi(0)}{G(s)}, \quad \text{and} \quad \mathcal{Z}(G) = \sup_{s>0} \frac{s\psi(0)}{G(s)};$$

and thus holds: $1 \leq \mathcal{L}(G) \leq \mathcal{L}_\infty(G) \leq \mathcal{Z}_\infty(G) \leq \mathcal{Z}(G)$. Also, for the preceding mappings hold and many other properties, see: Tasković [24].

Annotations. From the preceding considers we see that for further work can be essential the following condition: there exists a constant $K > 0$ such that

$$G\left(\frac{s}{2}\right) \leq KG(s) \quad \text{for enough large } s.$$

Also, from the preceding considers and the properties of the p and q functions, for an arbitrary function $G \in d(M)$ holds the following inequalities:

$$\frac{1}{q}s\psi(s) \leq G(s) \leq \frac{1}{p}s\psi(s), \tag{21}$$

$$\min\big\{a^p, a^q\big\}G(s) \leq G(as) \leq \max\big\{a^p, a^q\big\}G(s), \tag{22}$$

$$G\Big(\min\big\{a^{1/p}, a^{1/q}\big\}s\Big) \leq aG(s) \leq G\Big(\max\big\{a^{1/p}, a^{1/q}\big\}s\Big), \tag{23}$$

$$\frac{p}{q}\min\big\{a^{p-1}, a^{q-1}\big\} \leq \frac{\psi(as)}{\psi(s)} \leq \frac{q}{p}\max\big\{a^{p-1}, a^{q-1}\big\}. \tag{24}$$

7. THE FUNCTIONS ρ_G

Let S be a closed interval on real line and $\mathcal{M} = \mathcal{M}(S,\mu)$ is a set of all extension real μ-measurable functions on S. For an M-function G we define on the quotient set $A = \mathcal{M}/\!\equiv$ the function $\rho_G : A \to \mathbb{R}^* := \mathbb{R} \cup \{\pm\infty\}$ with

$$\rho_G(x) = \int_S G(x(s))ds,$$

at to what ρ_{G^*} (or ρ^*_G) suitable the complementary function. This definition is correct because

$$x \equiv y \quad \text{implies} \quad \rho_G(x) = \rho_G(y); \tag{25}$$

and, from the fact that G is an even function arise that ρ_G is even, i.e., $\rho_G(|x|) = \rho(x)$. Also, $\rho_G(u) = 0$ if and only if $u \equiv 0$, and

$$\inf\{u, v\} \equiv 0 \quad \text{implies} \quad \rho_G(u+v) = \rho_G(u) + \rho_G(v); \tag{26}$$

and, $u \leq v$ implies $\rho_G(u) \leq \rho_G(v)$, and

$$\rho_G(au + bv) \leq a\rho_G(u) + b\rho_G(v) \quad \text{for} \quad a, b > 0 \quad \text{and} \quad a + b = 1;$$

at to what for the function $G \in d(M)$ holds the following inequality

$$\int \big[u(v - \psi^{-1}(0))\big] ds \leq \rho_G(u) - \rho^*_G(u).$$

It follows from this inequality and from Levi's theorem that holds **quasilinear representation** in the following form:

$$\rho_G(u) = \sup_{\rho^*_G(v) < \infty} \left(\int \big[u(v - \psi^{-1}(0))\big] ds + \rho^*_G(v) \right).$$

We set $H = \{s \in S : u(s) \geq v(s)\}$ and $H_1 = S \backslash H$. Then we have $\sup\{u, v\} = u\chi_H + v\chi_{H_1}$, and thus from (26) we obtain $\rho_G(\sup\{u, v\}) \leq \rho_G(u) + \rho_G(v)$, i.e., by induction, for $n \in \mathbb{N}$, the following inequality holds:

$$\rho_G\big(\sup\{u_1, \ldots, u_n\}\big) \leq \rho_G(u_1) + \cdots + \rho_G(u_n).$$

We notice that some properties of the M-functions have influence and on the some properties of ρ_G functions. In this sense hold the following facts:

(a) The fact $G \boxminus R$ is equivalent with the fact: that there exist constants K_0, $K > 0$, such that holds the following inequality

$$\rho_{G^{-1}}(u) \leq K_0 + K\rho_{R^{-1}}(u) \quad \text{for every} \quad u \in A^+,$$

where A^+ is positive order cone with the ordering $\leq$ in A, i.e, where the equality holds $A^+ = \{[u] \in A : [u] \geq [0]\}$.

(b) The fact $G \prec R$ is equivalent with the fact: that there exist constants K_0, $K > 0$ such that holds the following inequality

$$\rho_{G^{-1}}(u) \leq K_0 + \rho_{R^{-1}}(Ku) \quad \text{for every} \quad u \in A^+. \tag{27}$$

Proof. Let $G \prec R$. If in inequality (19) put $u(s)$ in the place s and afterwards take the intergral over S, directly we obtain

$$\rho_{G^{-1}}(u) \leq K_0\mu(S) + \rho_{R^{-1}}(Ku),$$

i.e., (27) holds. Reversed, to serve for the formula $\rho_{G^{-1}}(b\chi_H) = G^{-1}(b)\mu(H)$ for $b \in H$ and $H \subset S$, from (27) for $u = u_t = t\chi_S$ ($t \geq 0$), we obtain

$$G^{-1}(t) \leq \frac{K_0}{\mu(S)} + R^{-1}(Kt),$$

whence, again from inequality (19), arise the fact $G \prec R$. The proof is complete.

8. BOUNDNESS OF SET ON ρ_G

The set $X \subset A$ is called the G**-bounded** iff $\sup \rho_G(X) < +\infty$. In this sense holds the following statement.

Theorem 8.1. *Let holds the preceding designate. Then the following are mutually equivalent facts:*
(a) $G \boxed{\prec} R$.
(b) $\rho_R^{-1}(\mathbb{R}_+^0) \subset \rho_G^{-1}(\mathbb{R}_+^0)$.
(c) $\sup \rho_G((\rho_G^{-1}([0,a])) < +\infty$ *for some* $a > 0$.
(d) *Every* R*-bounded set is* G*-bounded.*

For the proof of this statement we utilize Levi's theorem and some properties of G and ρ_G functions.

We set $d(L_G^*(S)) = d(L_G^*) = \{x \in A : \rho_G(x) < +\infty\}$. If utilize only linearity of the space L, we can further consider structure of $d(L_G^*)$ as a subset of linear space L. In this sense, we set

$$d(L_G(S)) = d(L_G) = \big\{x \in A : ax \in d(L_G^*) \quad \text{for some} \quad a > 0\big\},$$

and $d(L_G^+) = \{u \in d(L_G) : u \geq 0\}$. For the linear space $d(L_G)$ we say that **determinate** with the M-function G. Then hold the following facts: $d(L_G^*) \subset L = L_1(S)$, $d(L_G) \subset L$, and:
(a) $d(L_R^*) \subset d(L_G^*)$ *if and only if* $G \boxed{\prec} R$.
(b) $d(L_R) \subset d(L_G)$ *if and only if* $G \prec R$.

From this facts arise that the inequalites hold $d(L_G^*) \subset d(L_R^*)$ and $d(L_G) \subset d(L_R)$ if and only if $G \boxed{\sim} R$ and $G \sim R$, respectively.

This facts exhibit that all functions of one $\sim$-class determined only one linear space $d(L_G)$, where the mapping of $[G]$ and $d(L_G)$ is bijective. If in the set

$$\mathcal{D} = \big\{d(L_G) : G \in d(M)\big\}$$

defined the order relation with "$\subset$", then from the preceding facts the sets $(\mathcal{D}, \subset)$ and $(d(M)/\!\sim, \leq)$ are antimorphisms. In this sense, the mapping $\mathcal{U} : [G] \to d(L_G)$ is antimorphism, and $(\mathcal{D}, \subset)$ is a lattice.

In connection with the former facts, for the function $G \in d(M)$ we define, first, an upper norm, in denoted $|| \cdot ||_{(G)} : d(L_G) \to \mathbb{R}_+^0$ by

$$||x||_{(G)} = \inf \left\{a > 0 : \rho_G\left(\frac{x}{a}\right) \leq 1\right\};$$

and, then we can broaden this norm to a limitid upper norm as a function, denoted this extension by $|| \cdot ||_{(G)}^*$. In this sense, a hers **quasilinear representation** is in the following form:

$$||x||_{(G)}^* = \sup_{\rho_G^*(v)<\infty} \left(\frac{\int u(v - \psi^{-1}(0))ds}{1 - \rho_{(G)}^*(v)}\right).$$

Theorem 8.2. *If the mapping $T : A \to A$ satisfy for every $x \in A$ and for every $a \in \mathbb{R}$ the following conditions: $|T(ax)| = |a||Tx|$ and*

$$\rho_{G^{-1}}(Tx) \leq K_0 + \rho_{R^{-1}}(Kx),$$

where $K > 0$ and $K_0 \geq 0$ are constants, then for every $x \in A$ holds the following inequality in the form

$$||Tx||^*_{(G^{-1})} \leq K(K_0+1)||x||^*_{(R^{-1})}. \tag{28}$$

Proof. Since $T(0) = 0$, we have that (28) holds for $||x||^*_{(R^{-1})} = 0$. This inequality holds and for $||x||^*_{(R^{-1})} = +\infty$. We set $0 < a = ||x||^*_{(R^{-1})} < +\infty$. Then we have

$$\rho_{G^{-1}}\left(\frac{Tx}{(K_0+1)Ka}\right) \leq \frac{1}{K_0+1}\rho_{G^{-1}}\left(\frac{Tx}{Ka}\right) =$$
$$= \frac{1}{K_0+1}\rho_{G^{-1}}\left(T\left(\frac{x}{Ka}\right)\right) \leq \frac{1}{K_0+1}\left[K_0 + \rho_{R^{-1}}\left(\frac{x}{a}\right)\right] \leq 1,$$

whence, from definition of norm $||\cdot||^*_{(G^{-1})}$, we obtain the inequality in the form (28). The proof is complete.

Theorem 8.3. *The fact $G \prec R$ is equivalent with the fact: that there exists a constant $K > 0$ such that holds the following inequality in the form*

$$||x||^*_{(G^{-1})} \leq K||x||^*_{(R^{-1})} \quad \textit{for every} \quad x \in A. \tag{29}$$

Proof. If $G \prec R$, then from (27) and Theorem 3 for $Tx = x$ we obtain inequality (29). Reversed, let (29) holds. Since $||x||^*_{(G^{-1})} < +\infty$ if and only if $x \in d(L_{G^{-1}})$, from (29) arise $d(L_{R^{-1}}) \subset d(L_{G^{-1}})$; and thus $G \prec R$ from a former fact. The proof is complete.

Applying Theorem 4, and from some former facts, directly we obtain the following statements:

(a) If $G \prec R$, then $d(L_{R^{-1}}) \subset d(L_{G^{-1}})$ and there exists a constant $K > 0$ such that holds the following inequality in the form:

$$||x||_{(G^{-1})} \leq K||x||_{(R^{-1})} \quad \text{for every} \quad x \in d(L_{R^{-1}}). \tag{30}$$

(b) If $G \sim R$, then $d(L_{G^{-1}}) = d(L_{R^{-1}})$ and there exist constants K_0, $K > 0$ such that for every $x \in d(L_{R^{-1}})$ hold the following inequalities in the form

$$K_0||x||_{(R^{-1})} \leq ||x||_{(G^{-1})} \leq K||x||_{(R^{-1})}. \tag{31}$$

Annotations. We notice that topology on $d(L_{(R^{-1})})$ is redefined of they which make topology in $d(L_{(G^{-1})})$. This means that the order relation in the class of M-functions have and algebric and topological consequences.

From inequality (30) arise that the operator $T : d(L_{R^{-1}}) \to d(L_{G^{-1}})$ defined by $Tx = x$ is continuous.

On the other hand, inequality (31) demonstrate that the equivalent M-functions determined linear homeomorphic spaces, where the identical mapping is a linear homeomorphism.

In connection with this, we define the following two norms. For an M-function G we define the extension limitid upper norm $||\cdot||_G^*: A \to [0,+\infty]$ by

$$||u||_G^* = \inf\left\{\int_u u(v-\psi^{-1}(0))ds : ||v||_{(G)} \le 1\right\}$$

for $u \in A^+$ and $||x||_G^* = |||x|||_G^*$ for $x \in A$. On the other hand, for an M-function G we define and the extension limited upper norm $||\cdot||_{d(G)}^*: A \to [0,+\infty]$ by

$$||u||_{d(G)}^* = \inf\left\{\frac{\int u(v-\psi^{-1}(0))ds}{1+\rho_G(v)} : \rho_G(v) < +\infty\right\}$$

for $a \in A^*$ and $||x||_{d(G)}^* = |||x|||_{d(G)}^*$ for $x \in A$. For the preceding two norms hold the following inequalities

$$||u||_{d(G)}^* \le ||u||_G^* \le 2||u||_{d(G)}^*. \tag{32}$$

From inequalities (32) arise $||x||_G^* < +\infty$ if and only if $x \in d(L_G)$. In the preceding context, for an M-function G we define the norm $||\cdot||_G^0 : A \to [0,+\infty]$ by

$$||u||_G^0 = \sup\left\{\int u(v-\psi^{-1}(0))ds : ||v||_{(G^*)} \le \frac{1}{2}\right\}$$

for $u \in A^+$ and $||x||_G^0 = |||x|||_G^0$ for $x \in A$. Also, a limited upper norm $||\cdot||_{d(G)}^0 : A \to [0,+\infty]$ we define by

$$||u||_{d(G)}^0 = \sup\left\{\frac{\int u(v-\psi^{-1}(0))ds}{1-\rho_G^*} : \rho_G^*(v) < \frac{1}{2}\right\}$$

for $x \in A^+$ and $||x||_{d(G)}^0 = |||x|||_{d(G)}^0$ for $x \in A$. For this two norms hold the following inequalities in the form

$$\frac{1}{2}||u||_{d(G)}^0 \le ||u||_G^0 \le ||u||_{d(G)}^0.$$

Annotations. In connection with the preceding spaces we can to speak on *ordinary convergence* in the sense that $\lim_{n\to\infty} x_n(s) = x(s)$ for $x_n, x \in A$ $(n \in \mathbb{N})$; and we can to speak on *convergence in middle* (of index G) in the sense that

$$x_n \to_G x \quad \text{if and onli if} \quad \rho_G(x_n - x) \to 0;$$

or we can to speak on *convergence in the space* $d(L_G)$ as on convergence via norms in a given spaces.

An inequality. *Let G be a M-function and we set $p = p(G)$ and $q = q(G)$. Then holds the following inequality*

$$||u(\lambda s)||_{(G)} \le \max\left\{\lambda^{-1/p}, \lambda^{-1/q}\right\}||u(s)||_{(G)}$$

for every $u \in d(L_G([0, +\infty)))$ *and for arbitrary fixed* $0 \le \lambda < +\infty$. (For the proof of this inequality, from (23), see: Tasković [24]).

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