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A Common Fixed Point Theorem On Transversal Upper Intervally Spaces

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Dedicated to professor M. Tasković on his 60^{th} *birthday*

Abstract. This paper is to present a common fixed point theorem for family of commuting mappings defined on transversal upper intervally spaces. This result extends results of M. Tasković [5].

1. Definitions and previous results

Definition of transversal intervally spaces was given by M. Tasković (see [5]).

Definition 1.1. Let X be a nonempty set. The symmetric function $\rho : X \times X \to [a,b] \subset \mathbf{R}_+^0$ for $a < b$, is called an **upper intervally transversal** on X if there is a function $g : [a,b] \times [a,b] \to [a,b]$ such that

$$\rho(x,y) \leq \max\Big\{\rho(x,z), \rho(z,y), g(\rho(x,z), \rho(z,y))\Big\}$$

for all $x, y, z \in X$. A **transversal upper intervally space** is a set X together with a given upper intervally transversal on X. The function g is called **upper bisection function**.

Definition 1.2. A mapping $M : \mathbf{R} \to [a,b] \subset \mathbf{R}_+^0$ for $a < b$ is called an **upper distribution function** if it is nonincreasing, left-continuous with $\inf\limits_{x \in \mathbf{R}} M_{u,v}(x) = a$ and $\sup\limits_{x \in \mathbf{R}} M_{u,v}(x) = b$. We will denote by $\mathcal{D}$ the set of all upper distribution functions.

Definition 1.3. A **transversal upper intervally T-space** is a pair (X, ρ), where X is a transversal upper intervally space and where the upper intervally transversal is defined with $\rho[u,v] = M_{u,v}(x)$ satisfying $M_{u,v} = M_{v,u}, \quad M_{u,v}(c) = b$ for some $c \in \mathbf{R}$, and

$$M_{u,v}(x) = a \quad \text{for} \quad x > c \quad \text{if and only if} \quad u = v.$$

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Examples can be found in [5].

Definition 1.4. (a) A sequence $(p_n)_{n\in\mathbf{N}}$ in (X,ρ) converges to a point $p \in X$ if for some $c \in \mathbf{R}$ and for every $\mu > c$ and every $\sigma > 0$, there exists a natural $\mathcal{M}(\mu,\sigma)$, such that $M_{p,p_n}(\mu) < a + \sigma$, whenever $n \geq \mathcal{M}(\mu,\sigma)$.

(b) The sequence $(p_n)_{n\in\mathbf{N}}$ will be called fundamental in (X,ρ) if for some $c \in \mathbf{R}$ and each $\mu > c$ and every $\sigma > 0$, there exists a natural $\mathcal{M}(\mu,\sigma)$, such that $M_{p_n,p_m}(\mu) < a + \sigma$, whenever $n, m \geq \mathcal{M}(\mu,\sigma)$. A transversal upper intervally T-space will be called complete if each fundamental sequence in X converges to an element in X.

Definition 1.5. A mapping T of a transversal upper intervally T-space (X,ρ) into itself will be called a **intervally upper contraction** iff there exists a nondecreasing function $\varphi : [c,+\infty) \to [c,+\infty)$ for some $c \in \mathbf{R}$ such that

$$(As) \qquad \lim_{n\to\infty} \varphi^n(t) = +\infty, \quad \text{for every} \quad t > c,$$

satisfying the condition:

$$(Pc) \qquad M_{Tu,Tv}(x) \leq \max\Big\{M_{u,v}(\varphi(x)), M_{u,Tu}(\varphi(x)), M_{v,Tv}(\varphi(x)), M_{v,Tu}(\varphi(x)), M_{u,Tv}(\varphi(x))\Big\}$$

for all $u, v \in X$ and for every $x > c$.

M. Tasković has proven the next theorem (see [5]).

Theorem 1.1. *Let (X,ρ) be a complete transversal upper intervally T-space, where the upper transverse $\rho[u,v] = M_{u,v}(x)$ and the upper bisection function $g : [a,b] \times [a,b] \to [a,b]$ is nondecreasing such that $g(t,t) \leq t$ for all $t \in [a,b]$. If T is any intervally upper contraction mapping of X into itself, then there is a unique point $p \in X$ such that $Tp = p$. Moreover, $T^n q \to p$ for each $q \in X$.*

2. Main result

Theorem 2.1. *Let (X,ρ) be a complete transversal upper intervally T-space where the upper intervally transversal is defined with $\rho[u,v] = M_{u,v}(x)$ and the upper intervally bisection function $g : [a,b] \times [a,b] \to [a,b]$ is nondecreasing such that $g(t,t) \leq t$ for every $t \in [a,b]$. Let (T_n), for $n \in \mathbf{N}$ be a sequence of mappings from X into itself and $S : X \to X$ be a continuous bijective function commuting with each of T_n, satisfying condition $T_n(X) \subseteq S(X)$, for all $n \in \mathbf{N}$. Let exists a nondecreasing function $\varphi : [c,+\infty) \to [c,+\infty)$, for some $c \in \mathbf{R}$ such that condition (As) holds. If for all points $u, v \in X$ and all mappings T_i and T_j the inequality*

$$(Pcg) \qquad M^2_{T_iu,T_jv}(x) \leq \max\Big\{M^2_{Su,Sv}(\varphi(x)), M^2_{Su,T_iu}(\varphi(x)), M^2_{Sv,T_jv}(\varphi(x)), M_{Su,T_jv}(\varphi(x))M_{Sv,T_iu}(\varphi(x)), M_{Su,T_jv}(\varphi(x))M_{Su,T_iu}(\varphi(x))\Big\},$$

holds for every $x > c$, then there is a unique common fixed point $p \in X$ for S and all of mappings T_n.

Proof. Let u_0 be an arbitrary point from X. Let us define sequence (u_n), for $n \in \mathbf{N}$ as follows:

$$u_n = S^{-1}(T_n(u_{n-1})), \quad \text{for} \quad n \in \mathbf{N} \tag{1}$$

We show that the sequence $v_n = S(u_n) = T_n(u_{n-1})$, for $n \in \mathbf{N}$ is fundamental in X.

From condition (Pcg) and for all $a > c$ the next inequalities follow:

$$\begin{aligned}
(2) \qquad & M^2_{Su_{n-1},Su_n}(\mu) = M^2_{T_{n-1}u_{n-2},T_nu_{n-1}}(\mu) \leq \\
& \max\Big\{M^2_{Su_{n-2},T_{n-1}u_{n-2}}(\varphi(\mu)), M^2_{Su_{n-1},T_nu_{n-1}}(\varphi(\mu)), M^2_{Su_{n-2},Su_{n-1}}(\varphi(\mu)), \\
& M_{Su_{n-2},T_nu_{n-1}}(\varphi(\mu))M_{Su_{n-1},T_{n-1}u_{n-2}}(\varphi(\mu)), \\
& M_{Su_{n-2},T_nu_{n-1}}(\varphi(\mu))M_{Su_{n-2},T_{n-1}u_{n-2}}(\varphi(\mu))\Big\} = \\
& \max\Big\{M^2_{Su_{n-2},Su_{n-1}}(\varphi(\mu)), M^2_{Su_{n-1},Su_n}(\varphi(\mu)), \\
& M_{Su_{n-2},Su_n}(\varphi(\mu))M_{Su_{n-1},Su_{n-1}}(\varphi(\mu)), \\
& M_{Su_{n-2},Su_n}(\varphi(\mu))M_{Su_{n-2},Su_{n-1}}(\varphi(\mu))\Big\}.
\end{aligned}$$

Since the space is a transversal upper intervally space then for every $x \geq c$ the following inequalities hold:

$$\begin{aligned}
(*) \qquad M_{u,v}(x) &\leq \max\Big\{M_{u,w}(x), M_{w,v}(x), g(M_{u,w}(x), M_{w,v}(x)\Big\} \\
&\leq \max\Big\{M_{u,w}(x), M_{w,v}(x)\Big\},
\end{aligned}$$

because $g(u,v) \leq g(\max\{u,v\}, \max\{u,v\}) \leq \max\{u,v\}$. From previous follows that

$$M_{Su_{n-2},Su_n}(\varphi(\mu)) \leq \max\{M_{Su_{n-2},Su_{n-1}}(\varphi(\mu)), M_{Su_{n-1},Su_n}(\varphi(\mu))\}. \tag{3}$$

Then, from inequality (3) and the fact that values of upper distribution functions are in interval $[a,b]$ next inequalities follow:

$$\begin{aligned}
(4) \quad & M_{Su_{n-2},Su_n}(\varphi(\mu))M_{Su_{n-1},Su_{n-1}}(\varphi(\mu)) = M_{Su_{n-2},Su_n}(\varphi(\mu)) \leq \\
& \max\Big\{M_{Su_{n-2},Su_{n-1}}(\varphi(\mu)), M_{Su_{n-1},Su_n}(\varphi(\mu))\Big\} \leq \\
& \max\Big\{M^2_{Su_{n-2},Su_{n-1}}(\varphi(\mu)), M^2_{Su_{n-1},Su_n}(\varphi(\mu))\Big\}.
\end{aligned}$$

$$\begin{aligned}
(5) \quad & M_{Su_{n-2},Su_n}(\varphi(\mu))M_{Su_{n-2},Su_{n-1}}(\varphi(\mu)) \leq \\
& \max\Big\{M^2_{Su_{n-2},Su_{n-1}}(\varphi(\mu)), M_{Su_{n-1},Su_n}(\varphi(\mu))M_{Su_{n-2},Su_{n-1}}(\varphi(\mu))\Big\}.
\end{aligned}$$

From the fact that $\max\{u^2, v^2, uv\} = \max\{u^2, v^2\}$, for all $u, v \in [a, b]$, inequalities (2), (4) and (5) imply:

$$(6) \qquad M^2_{Su_{n-1},Su_n}(\mu) \le \max\left\{M^2_{Su_{n-2},Su_{n-1}}(\varphi(\mu)), M^2_{Su_{n-1},Su_n}(\varphi(\mu))\right\}.$$

From last follows:

$$(7) \qquad M_{Su_{n-1},Su_n}(\mu) \le \max\left\{M_{Su_{n-2},Su_{n-1}}(\varphi(\mu)), M_{Su_{n-1},Su_n}(\varphi(\mu))\right\}.$$

Since φ is a nondecreasing function and $\varphi(\mu) > c$, $\varphi(\mu) > \mu$ for every $\mu > c$ it follows by induction that for every $k \in \mathbf{N}$ the following inequality holds:

$$(8) \qquad M_{Su_{n-1},Su_n}(\mu) \le \max\left\{M_{Su_{n-2},Su_{n-1}}(\varphi(\mu)), M_{Su_{n-1},Su_n}(\varphi^k(\mu))\right\},$$

and when $k \to +\infty$ we get that for every $n \in \mathbf{N}$:

$$(9) \qquad M_{Su_{n-1},Su_n}(\mu) \le M_{Su_{n-2},Su_{n-1}}(\varphi(\mu)).$$

By induction we can prove the inequality (10) for the sequence $\{v_n\}$.

$$(10) \qquad M_{v_{n-1},v_n}(\mu) \le M_{v_0,v_1}(\varphi^{n-1}(\mu)).$$

From $(*)$, and last inequality, for $m > n$ and arbitrary $\mu > c$, follows:

$$M_{v_n,v_m}(\mu) \le \max\left\{M_{v_n,v_{n+1}}(\mu), \dots, M_{v_{m-1},v_m}(\mu)\right\} \le \max\left\{M_{v_0,v_1}(\varphi^n(\mu)), \dots, M_{v_0,v_1}(\varphi^{m-1}(\mu))\right\} = M_{v_0,v_1}(\varphi^n(\mu)).$$

From (As) we conclude that exists a natural $\mathcal{M}(\mu,\sigma)$ such that $M_{v_0,v_1}(\varphi^{M(\mu,\sigma)}(\mu)) < a + \sigma$. We can take that $n, m \ge \mathcal{M}(\mu,\sigma)$ and we conclude that v_n is a fundamental sequence in (X, ρ). Since the space is complete, then there is a point $p \in X$ such that $v_n \to p$.

We shall prove that p is a common fixed point for S and T_n. Since S commutates with each of T_n, then from (1) and the fact that $T_nSu_{n-1} = ST_nu_{n-1} = SSu_n$ follows:

$$\begin{aligned}
M^2_{SSu_n,T_kp}(\mu) &= M^2_{ST_nu_{n-1},T_kp}(\mu) = M^2_{T_nSu_{n-1},T_kp}(\mu) \le \\
&\max\Big\{M^2_{SSu_{n-1},Sp}(\varphi(\mu)), M^2_{SSu_{n-1},T_nSu_{n-1}}(\varphi(\mu)), M^2_{Sp,T_kp}(\varphi(\mu)), \\
&M_{SSu_{n-1},T_kp}(\varphi(\mu))M_{Sp,T_nSu_{n-1}}(\varphi(\mu)), M_{SSu_{n-1},T_kp}(\varphi(\mu))M_{SSu_{n-1},T_nSu_{n-1}}\Big\} = \\
&\max\Big\{M^2_{SSu_{n-1},Sp}(\varphi(\mu)), M^2_{SSu_{n-1},SSu_n}(\varphi(\mu)), M^2_{Sp,T_kp}(\varphi(\mu)), \\
&M_{SSu_{n-1},T_kp}(\varphi(\mu))M_{Sp,SSu_n}(\varphi(\mu)), M_{SSu_{n-1},T_kp}(\varphi(\mu))M_{SSu_{n-1},SSu_n}\Big\}.
\end{aligned}$$

From continuity of S and because $Su_n \to p$ when $n \to +\infty$, we get that for every $k \in \mathbf{N}$ follows:

$$(11)\quad \begin{aligned} M^2_{Sp,T_kp}(\mu) \leq & \max\Big\{M^2_{Sp,Sp}(\varphi(\mu)), M^2_{Sp,Sp}(\varphi(\mu)), M^2_{Sp,T_kp}(\varphi(\mu)), \\ & M_{Sp,T_kp}(\varphi(\mu))M_{Sp,Sp}(\varphi(\mu)), M_{Sp,T_kp}(\varphi(\mu))M_{Sp,Sp}(\varphi(\mu))\Big\} \\ = & M^2_{Sp,T_kp}(\varphi(\mu)). \end{aligned}$$

Because all of the functions in last inequality are nonincreasing we conclude that for each $m \in \mathbf{N}$ the inequality $M_{Sp,T_kp}(\mu) \leq M_{Sp,T_kp}(\varphi^m(\mu))$ holds. When $m \to +\infty$, for every $\mu > c$, we obtain $M_{Sp,T_kp}(\mu) = a$. From this, for every $k \in \mathbf{N}$ we obtain $(**)$ $S(p) = T_k(p)$. In following text we shall show that p is a common fixed point for all of mappings T_n.
From inequality:

$$(12)\quad \begin{aligned} & M^2_{Su_n,T_kp}(\mu) = M^2_{T_nu_{n-1},T_kp}(\mu) \leq \\ & \max\Big\{M^2_{Su_{n-1},Sp}(\varphi(\mu)), M^2_{Su_{n-1},Su_n}(\varphi(\mu)), M^2_{Sp,T_kp}(\varphi(\mu)), \\ & M_{Su_{n-1},T_kp}(\varphi(\mu))M_{Sp,Su_n}(\varphi(\mu)), M_{Su_{n-1},T_kp}(\varphi(\mu))M_{Su_{n-1},Su_n}(\varphi(\mu))\Big\}, \end{aligned}$$

when $n \to +\infty$, because $(**)$ holds, we conclude that:

$$(13)\quad \begin{aligned} M^2_{p,T_kp}(\mu) \leq \max\Big\{ & M^2_{p,T_kp}(\varphi(\mu)), M^2_{p,p}(\varphi(\mu)), M^2_{T_kp,T_kp}(\varphi(\mu)), \\ & M_{p,T_kp}(\varphi(\mu))F_{T_kp,p}(\varphi(\mu)), M_{p,T_kp}(\varphi(\mu))M_{p,p}(\varphi(\mu))\Big\}, \end{aligned}$$

From last, we obtain that for each $\mu > c$ holds the following:

$$(14)\qquad M_{p,T_kp}(\mu) \leq M_{p,T_kp}(\varphi(\mu)).$$

Next, we obtain that for every $m \in \mathbf{N}$ follows $M_{p,T_kp}(\mu) \leq M_{p,T_kp}(\varphi^m(\mu))$, and when $m \to +\infty$, we conclude that for every $\mu > c$ the fact $M_{p,T_kp}(\mu) = a$ holds, and it implies that for each $k \in \mathbf{N}$ we get $p = T_kp = Sp$.
Let us prove uniqueness of common fixed point p. Suppose that there is another common fixed point $q \neq p$. From

$$(15)\quad \begin{aligned} M^2_{p,q}(\mu) = M^2_{T_ip,T_jq}(\mu) \leq \max\Big\{ & M^2_{Sp,Sq}(\varphi(\mu)), M^2_{Sp,p}(\varphi(\mu)), M^2_{Sq,q}(\varphi(\mu)), \\ & M_{Sp,q}(\varphi(\mu))M_{Sq,p}(\varphi(\mu)), M_{Sp,q}(\varphi(\mu))M_{Sp,p}(\varphi(\mu))\Big\} = M^2_{p,q}(\varphi(\mu)). \end{aligned}$$

follows that for every $\mu > c$ holds that $M_{p,q}(\mu) \leq M_{p,q}(\varphi(\mu))$, and so, for every $m \in \mathbf{N}$, we obtain that $M_{p,q}(\mu) \leq M_{p,q}(\varphi^m(\mu))$, and when $m \to +\infty$, we conclude that for every $\mu > c$ holds the fact $M_{p,q}(\mu) = a$. From conditions for distribution functions we get that $p = q$. This completes the proof. □

Comment. It is easy to prove that from condition (Pcg) for $T = T_i = T_j$ and $S = I$, where I is an identical mapping, follows:

$$M^2_{Tu,Tv}(x) \le \max\Big\{M^2_{u,v}(\varphi(x)), M^2_{u,Tu}(\varphi(x)), M^2_{v,Tv}(\varphi(x)),$$
$$M_{u,Tv}(\varphi(x))M_{v,Tu}(\varphi(x)), M_{u,Tv}(\varphi(x))M_{u,Tu}(\varphi(x))\Big\} \le$$
$$\max\Big\{M^2_{u,v}(\varphi(x)), M^2_{u,Tu}(\varphi(x)), M^2_{v,Tv}(\varphi(x))$$
$$M^2_{u,Tv}(\varphi(x)), M^2_{v,Tu}(\varphi(x))\Big\},$$

and we can conclude that mappings satisfying condition (Pcg) are intervally upper contractions.

From these conclusions follows that Theorem 2 is an extension of Theorem 1.

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