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Two Characterizations of (n,m)–groups for $n \geq 3m$

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Abstract. In this paper two characterization of (n,m)–groups for $n \geq 3m$ are proved. (The case $m=1$ is proved in [5].)

1. Preliminaries

Definition 1.1 ([1])**.** Let $n \geq 2$ and let $(Q;A)$ be an n–groupoid. We say that $(Q;A)$ is a Dörnte n–group [briefly: n–group] iff is an n–semigroup and n–quasigroup as well (See, also [7]).

Definition 1.2 ([2])**.** Let $n \geq m+1$ and $(Q;A)$ be an (n,m)–groupoid $(A : Q^n \to Q^m)$. We say that $(Q;A)$ is an (n,m)–group iff the following statements hold:

(i) For every $i,j \in \{1,\dots,n-m+1\}$, $i<j$, the following law holds

$$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-m}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-m})$$

[$< i,j >$–associative law*]*; and

(ii) For every $i \in \{1,\dots,n-m+1\}$ and for every $a_1^n \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds

$$A(a_1^{i-1}, x_1^m, a_i^{n-m}) = a_{n-m+1}^n.$$

For $m=1$ $(Q;A)$ is an n–group. Cf. [7].

Definition 1.3 ([4])**.** Let $n \geq 2m$ and let $(Q;A)$ be an (n,m)–groupoid. Let also $\mathbf{e}$ be a mapping of the set Q^{n-2m} into the set Q^m. Then, we say that $\mathbf{e}$ is an $\{1, n-m+1\}$–**neutral operation** of the (n,m)–groupoid $(Q;A)$ iff for every sequence a_1^{n-2m} over Q and for every $x_1^m \in Q^m$ the following equalities hold:

$$A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = x_1^m$$

and

$$A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m.$$

For $m=1$ $\mathbf{e}$ is a $\{1,n\}$–neutral operation of the n–groupoid $(Q;A)$. Cf. Chapter II in [7].

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2. AUXILIARY PROPOSITIONS

Proposition 2.1 ([6]). *Let $n \geq 2m$ and let (Q, A) be an (n,m)–groupoid. Further on, let the following statements hold:*

(a) *The $< 1, n-m+1 >$–associative law*[1] *holds in $(Q; A)$;*
(b) *For every $a_1^n \in Q$, there is* ***at least one*** *$x_1^m \in Q^m$ such that the equality $A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n$ holds;*
(c) *For every $a_1^n \in Q$, there is* ***at least one*** *$y_1^m \in Q^m$ such that the equality $A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n$ holds. Then $(Q; A)$ has a $\{1, n-m+1\}$–neutral operation.*

For $m = 1$: Prop.2.5-II in [7].

In this paper, among others, the following $< i, j >$–associative laws have the prominence:

(1L) $$A(A(x_1^n), x_{n+1}^{2n-m}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-m})$$

and

(1R) $$A(x_1^{n-m-1}, A(x_{n-m}^{2n-m-1}), x_{2n-m}) = A(x_1^{n-m}, A(x_{n-m+1}^{2n-m})).$$

Proposition 2.2 ([6]). *Let $n > m+1$ and let $(Q; A)$ be an (n,m)–groupoid. Also, let:*

(α) *the (1L) [(1R)] law holds in $(Q; A)$; and*
(β) *for every $x_1^m, y_1^m, a_1^{n-m} \in Q$ the following implication holds*

$$A(x_1^m, a_1^{n-m}) = A(y_1^m, a_1^{n-m}) \Rightarrow x_1^m = y_1^m$$

$$[A(a_1^{n-m}, x_1^m) = A(a_1^{n-m}, y_1^m) \Rightarrow x_1^m = y_1^m].$$

Then, $(Q; A)$ is an (n,m)–semigroup [cf. (i) in Def. 1.2].

Proposition 2.3 ([3]). *Let $(Q; A)$ be an (n,m)–groupoid and $n \geq m+2$. Also, let the following statements hold:*

(1) *$(Q; A)$ is an (n,m)–semigroup (cf. (i) in Def. 1.2);*
(2) *For every $a_1^n \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds*
$$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n;$$
(3) *For every $a_1^n \in Q$ there is exactly one $y_1^m \in Q^m$ such that the following equality holds*
$$A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n.$$

Then, $(Q; A)$ is an (n,m)–group.

Sketch of the proof. a) $A(a, a_1^{i-1}, x_1^m, a_i^{n-m-2}, b) = A(a, a_1^{i-1}, y_1^m, a_i^{n-m-2}, b) \overset{\ddagger}{\Rightarrow}$

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[1] $A(A(x_1^n), x_{n+1}^{2n-m}) = A(x_1^{n-m}, A(x_{n-m+1}^{2n-m}))$.
‡ $i \in \{1, \ldots, n-m-1\}$

$$A(c_{i+1}^{n-m}, A(a, a_1^{i-1}, x_1^m, a_i^{n-m-2}, b), c_1^i) =$$
$$= A(c_{i+1}^{n-m}, A(a, a_1^{i-1}, y_1^m, a_i^{n-m-2}, b), c_1^i) \overset{(1)}{\Rightarrow}$$
$$= A(A(c_{i+1}^{n-m}, a, a_1^{i-1}, x_1^m), a_i^{n-m-2}, b, c_1^i) =$$
$$= A(A(c_{i+1}^{n-m}, a, a_1^{i-1}, y_1^m), a_i^{n-m-2}, b, c_1^i) \overset{(3)}{\Rightarrow}$$
$$= A(c_{i+1}^{n-m}, a, a_1^{i-1}, x_1^m) = A(c_{i+1}^{n-m}, a, a_1^{i-1}, y_1^m) \overset{(2)}{\Rightarrow}$$
$$= x_1^m = y_1^m.$$

b) $A(a, a_i^{n-m-2}, x_1^m, a_1^{i-1}, b) = A(a, a_i^{n-m-2}, y_1^m, a_1^{i-1}, b) \Rightarrow$
$$A(c_1^i, A(a, a_i^{n-m-2}, x_1^m, a_1^{i-1}, b), c_{i+1}^{n-m}) =$$
$$= A(c_1^i, A(a, a_i^{n-m-2}, y_1^m, a_1^{i-1}, b), c_{i+1}^{n-m}) \overset{(1)}{\Rightarrow}$$
$$= A(c_1^i, a, a_i^{n-m-2}, A(x_1^m, a_1^{i-1}, b, c_{i+1}^{n-m})) =$$
$$= A(c_1^i, a, a_i^{n-m-2}, A(y_1^m, a_1^{i-1}, b, c_{i+1}^{n-m})) \overset{(2)}{\Rightarrow}$$
$$= A(x_1^m, a_1^{i-1}, b, c_{i+1}^{n-m}) = A(y_1^m, a_1^{i-1}, b, c_{i+1}^{n-m}) \overset{(3)}{\Rightarrow}$$
$$= x_1^m = y_1^m.$$

c) $A(a, a_1^{i-1}, x_1^m, a_i^{n-m-2}, b) = b_1^m \overset{b)}{\Leftrightarrow}$
$$A(c_{i+1}^{n-m}, A(a, a_1^{i-1}, x_1^m, a_i^{n-m-2}, b), c_1^i) = A(c_{i+1}^{n-m}, b_1^m, c_1^i) \overset{(1)}{\Leftrightarrow}$$
$$A(A(c_{i+1}^{n-m}, a, a_1^{i-1}, x_1^m), a_i^{n-m-2}, b, c_1^i) = A(c_{i+1}^{n-m}, b_1^m, c_1^i).$$

□

3. Results

Theorem 3.1. *Let $n \geq 3m$ and let $(Q; A)$ be an (n, m)–groupoid. Then, $(Q; A)$ is an (n, m)–group iff there is a mapping $\mathbf{e}$ of the set Q^{n-2m} into the set Q^m such that the laws*

(1L) $$A(A(x_1^n), x_{n+1}^{2n-m}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-m})$$

(1Lm) $$A(A(a_1^m, b_1^{n-m}), c_1^m, d_1^{n-2m}) = A(a_1^m, A(b_1^{n-m}, c_1^m), d_1^{n-2m}),$$

(2L) $$A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m$$

and

(2R) $$A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = x_1^m$$

hold in the algebra $(Q; A, \mathbf{e})$.

Remark 3.1. For $m = 1$: (1L)=(1Lm).

Proof. a) $\Rightarrow$ Let $(Q; A)$ be an (n, m)–group. Then, by Proposition 2.1, there is an algebra $(Q; A, \mathbf{e})$ of the type $< (n, m), (n-2m, m) >$ in which the laws (1L), (1Lm), (2L) and (2R) hold.

b) $\Leftarrow$ Let $(Q; A, \mathbf{e})$ be an algebra of the type $< (n, m), (n-2m, m) >$ in witch the laws (1L), (1Lm), (2L) and (2R) are satisfied. Firstly, we prove that under the assumptions the following statements hold:

1° For every $x_1^m, y_1^m, b_1^m \in Q^m$ and for every sequence a_1^{n-2m} over Q the following implication holds

$$A(x_1^m, b_1^m, a_1^{n-2m}) = A(y_1^m, b_1^m, a_1^{n-2m}) \Rightarrow x_1^m = y_1^m;$$

2° $(Q;A)$ is an (n,m)–semigroup;

3° For every $x_1^m, y_1^m, b_1^m \in Q^m$ and for every sequence a_1^{n-2m} over Q the following implication holds

$$A(a_1^{n-2m}, b_1^m, x_1^m) = A(a_1^{n-2m}, b_1^m, y_1^m) \Rightarrow x_1^m = y_1^m;$$

4° For every $a_1^n \in Q$ there is exactly one sequence x_1^m over Q and exactly one sequence y_1^m over Q such that the following equalities hold

$$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n \quad \text{and}$$
$$A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n.$$

Sketch of the proof of 1°.

$$\begin{aligned}
&A(x_1^m, b_1^m, a_1^{n-2m}) = A(y_1^m, b_1^m, a_1^{n-2m}) \Rightarrow \\
&A(A(x_1^m, b_1^m, a_1^{n-2m}), \mathbf{e}(a_1^{n-2m}), c_1^{n-3m}, \mathbf{e}(b_1^m, c_1^{n-3m})) = \\
&A(A(y_1^m, b_1^m, a_1^{n-2m}), \mathbf{e}(a_1^{n-2m}), c_1^{n-3m}, \mathbf{e}(b_1^m, c_1^{n-3m})) \overset{(1Lm)}{\Longrightarrow} \\
&A(x_1^m, A(b_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})), c_1^{n-3m}, \mathbf{e}(b_1^m, c_1^{n-3m})) = \\
&A(y_1^m, A(b_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})), c_1^{n-3m}, \mathbf{e}(b_1^m, c_1^{n-3m})) \overset{(2R)}{\Longrightarrow} \\
&A(x_1^m, b_1^m, c_1^{n-3m}, \mathbf{e}(b_1^m, m, c_1^{n-3m})) = \\
&A(y_1^m, b_1^m, c_1^{n-3m}, \mathbf{e}(b_1^m, c_1^{n-3m})) \quad \overset{(2R)}{\Longrightarrow} \quad x_1^m = y_1^m.
\end{aligned}$$

The proof of the statement 2°. By 1°, (1L) and by Prop. 2.2.

Sketch of the proof of 3°.

$$\begin{aligned}
&A(a_1^{n-2m}, b_1^m, x_1^m) = A(a_1^{n-2m}, b_1^m, y_1^m) \Rightarrow \\
&A(\mathbf{e}(c_1^{n-3m}, b_1^m), c_1^{n-3m}, \mathbf{e}(a_1^{n-2m}), A(a_1^{n-2m}, b_1^m, x_1^m)) = \\
&A(\mathbf{e}(c_1^{n-3m}, b_1^m), c_1^{n-3m}, \mathbf{e}(a_1^{n-2m}), A(a_1^{n-2m}, b_1^m, y_1^m)) \overset{2°}{\Longrightarrow} \\
&A(\mathbf{e}(c_1^{n-3m}, b_1^m), c_1^{n-3m}, A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, b_1^m), x_1^m) = \\
&A(\mathbf{e}(c_1^{n-3m}, b_1^m), c_1^{n-3m}, A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, b_1^m), y_1^m) \overset{(2L)}{\Longrightarrow} \\
&A(\mathbf{e}(c_1^{n-3m}, b_1^m), c_1^{n-3m}, b_1^m, x_1^m) = \\
&A(\mathbf{e}(c_1^{n-3m}, b_1^m), c_1^{n-3m}, b_1^m, y_1^m) \quad \overset{(2L)}{\Longrightarrow} \quad x_1^m = y_1^m.
\end{aligned}$$

Sketch of the proof of 4°.

a) $A(a_1^{n-2m}, b_1^m, x_1^m) = d_1^m \overset{3°}{\Longleftrightarrow}$

$$A(\mathbf{e}(c_1^{n-3m}, b_1^m), c_1^{n-3m}, \mathbf{e}(a_1^{n-2m}), A(a_1^{n-2m}, b_1^m, x_1^m)) =$$

$$A(\mathbf{e}(c_1^{n-3m}, b_1^m), c_1^{n-3m}, \mathbf{e}(a_1^{n-2m}), d_1^m) \overset{2^\circ,(2L)}{\Longleftrightarrow}$$
$$x_1^m = A(\mathbf{e}(c_1^{n-3m}, b_1^m), c_1^{n-3m}, \mathbf{e}(a_1^{n-2m}), d_1^m).$$

b) $A(y_1^m, b_1^m, a_1^{n-2m}) = d_1^m \overset{1^\circ}{\Leftrightarrow}$

$$A(A(y_1^m, b_1^m, a_1^{n-2m}), \mathbf{e}(a_1^{n-2m}), c_1^{n-3m}, \mathbf{e}(b_1^m, c_1^{n-3m})) =$$
$$A(d_1^m, \mathbf{e}(a_1^{n-2m}), c_1^{n-3m}, \mathbf{e}(b_1^m, c_1^{n-3m})) \overset{2^\circ,(2R)}{\Longleftrightarrow}$$
$$y_1^m = A(d_1^m, \mathbf{e}(a_1^{n-2m}), c_1^{n-3m}, \mathbf{e}(b_1^m, c_1^{n-3m})).$$

Finally, by 2°, 4° and by Prop. 2.3, we conclude that $(Q; A)$ is an (n, m)–group. □

Similarly, one could prove also the following proposition:

Theorem 3.2. *Let $n \geq 3m$ and let $(Q; A)$ be an (n, m)–groupoid. Then, $(Q; A)$ is an (n, m)–group iff there is a mapping $\mathbf{e}$ of the set Q^{n-2m} into the set Q^m such that the laws*

(1R) $$A(x_1^{n-m-1}, A(x_{n-m}^{2n-m-1}), x_{2n-m}) = A(x_1^{n-m}, A(x_{n-m+1}^{2n-m})),$$

(1Rm) $$A(a_1^{n-2m}, A(b_1^m, c_1^{n-m}), d_1^m) = A(a_1^{n-2m}, b_1^m, A(c_1^{n-m}, d_1^m)),$$

(2L) $$A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m$$

(2R) $$A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = x_1^m$$

hold in the algebra $(Q; A, \mathbf{e})$.

Remark 3.2. For $m = 1$: (1R)=(1Rm).

Remark 3.3. $m = 1$ theorem 3.1 and theorem 3.2 are proved in [5]. Cf. theorem 2.2-IX and theorem 2.3-IX in [7].

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