Mathematica Moravica
Vol. **8–1** (2004), 63–71

On Hyperquasigroups

Janez Ušan and Radoslav Galić

Abstract. In the paper we define and study hyperquasigroups of the rang $m(\in N)$.

1. Preliminaries

Definition 1.1. Let Q be a non-empty set and $P(Q)$ its power set. Let A be a mapping of the set Q^2 into the set $P(Q)$. Then:

a) we say that the mapping A is a **hyperoperation in** Q; and
b) we say that the ordered pair $(Q;A)$ is a **hypergroupoid**.

Definition 1.2. Let $(Q;A)$ be a hypergroupoid. Also, let:

$$\mathbf{A}(X,Y) \stackrel{def}{=} \begin{cases} \bigcup\limits_{(x,y)\in X\times Y} A(x,y); & X\neq\emptyset, Y\neq\emptyset \\ \emptyset; & X=\emptyset \ or\ Y=\emptyset \end{cases}$$

for all $X,Y\in P(Q)$. Then, we say that the groupoid $(P(Q);\mathbf{A})$ is a **associated** (or coresponds) to the hypergroupoid $(Q;A)$. (For example: Table 1 and Table 2.)

A	1	2
1	$\{1\}$	$\{2\}$
2	$\{2\}$	$\{1,2\}$

Table 1

$\mathbf{A}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\emptyset$
$\{1\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\emptyset$
$\{2\}$	$\{2\}$	$\{1,2\}$	$\{1,2\}$	$\emptyset$
$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\emptyset$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Table 2

Remark 1.1. a) If for all $x,y\in Q$ $A(x,y)\in\mathbf{Q}$, where $\mathbf{Q}\stackrel{def}{=}\{\{x\}|x\in Q\}$, then $(\mathbf{Q},\mathbf{A})$ is a groupoid.

b) If $(Q,\mathcal{A})$ is a groupoid and $A(x,y)\stackrel{def}{=}\{\mathcal{A}(x,y)\}$, then $(Q;A)$ is a hypergroupoid.

2000 *Mathematics Subject Classification.* 20N99, 20N05.
Key words and phrases. Quasigroups, hypergroupoids, hyperquasigroups.

c) Let $\mathcal{A} : D \to Q, D \subseteq Q^2$. /i.e. $(Q;\mathcal{A})$ is a partial groupoid/ and let

$$A(x,y) \stackrel{def}{=} \begin{cases} \{\mathcal{A}(x,y);\}; & (x,y) \in D \\ \emptyset; & (x,y) \in Q^2 \setminus D \end{cases}$$

for all $x, y \in Q$. [1] Then $(Q;A)$ is a hypergroupoid.

d) Let ρ be a 3-ary relation in Q. Also, let

$$A(x,y) \ni z \stackrel{def}{\Longleftrightarrow} (x,y,z) \in \rho /(x,y,z) \in \rho \stackrel{def}{\Longleftrightarrow} A(x,y) \ni z/$$

for all $x, y, z \in Q$. Then $(Q;A)$ is a hypergroupoid /then ρ is a 3-ary realtion in Q/.

Definition 1.3. Let $(Q;A)$ be a hypergroupoid and let for all $(x,y) \in Q^2$ $A(x,y) \neq \emptyset$. Then, we say that (Q,A) is a **hypergroup** iff the following statements hold:

(a) $\mathbf{A}(\mathbf{A}(\{x\},\{y\}),\{z\}) = \mathbf{A}(\{x\},\mathbf{A}(\{y\}),\{z\}))$ for each $x, y, z \in Q$; and

(b) For every $a, b \in Q$ there is at least one $x \in Q$ and at least one $y \in Q$ such that the following formulas hold

$$A(a,x) \ni b \qquad \text{and} \qquad A(y,a) \ni b.$$

Remark 1.2. A notion of a **hypergroup** was introduced by F. Marty in [1] as a generalization of the notion of a group. Cf. [2].

2. Parastrophic hyperoperations

Proposition 2.1. *Let $(Q;A)$ be a hypergroupoid and let α be a permutation of the set $\{1,2,3\}$. Also, let*

$$A^\alpha(x_1,x_2) \ni x_3 \stackrel{def}{\Leftrightarrow} A(x_{\alpha(1)},x_{\alpha(2)}) \ni x_{\alpha(3)}$$

for all $x_1, x_2, x_3 \in Q$. Then (Q,A^α) is a hypergroupoid.

Proof. By Def. 1.1. □

Definition 2.1. Let $(Q;A)$ be a hypergroupoid and let α be a permutation of the set $\{1,2,3\}$. Also, let

$$A^\alpha(x_1,x_2) \ni x_3 \stackrel{def}{\Leftrightarrow} A(x_{\alpha(1)},x_{\alpha(2)}) \ni x_{\alpha(3)}$$

[1]If $D = \emptyset$, then $A(x,y) = \emptyset$ for all $(x,y) \in Q^2$.

for all $x_1, x_2, x_3 \in Q$. Then, we shall say that the hyperoperation A^α is a $\alpha-$**parastrophic hyperoperation to the hyperoperation** A. Let:

$$\overset{I}{A} \overset{def}{=} A[I = \{(x,x)|x \in \{1,2,3\}\}],$$

(a)
$$\begin{aligned}
{}^{-1}A(x_1,x_2) \ni x_3 &\overset{def}{\Leftrightarrow} A(x_3,x_2) \ni x_1,\\
A^{-1}(x_1,x_2) \ni x_3 &\overset{def}{\Leftrightarrow} A(x_1,x_3) \ni x_2,\\
A^*(x_1,x_2) \ni x_3 &\overset{def}{\Leftrightarrow} A(x_2,x_1) \ni x_3 \ \ [A^*(x_1,x_2) = A(x_2,x_1)],\\
{}^{-1}(A^*)(x_1,x_2) \ni x_3 &\overset{def}{\Leftrightarrow} A(x_3,x_1) \ni x_2 \text{ and}\\
(A^*)^{-1}(x_1,x_2) \ni x_3 &\overset{def}{\Leftrightarrow} A(x_2,x_3) \ni x_1
\end{aligned}$$

for all $x_1, x_2, x_3 \in Q$. (See, also [4] or [6])

Proposition 2.2. *Let $(Q; A)$ be a hypergroupoid and let α be a permutation of the set $\{1,2,3\}$. Then*

$$\{A^\alpha, {}^{-1}(A^\alpha), (A^\alpha)^{-1}, (A^\alpha)^*, {}^{-1}((A^\alpha)^*), ((A^\alpha)^*)^{-1}\} =$$
$$\{A, {}^{-1}A, A^{-1}, A^*, {}^{-1}(A^*), (A^*)^{-1}\}.$$

Proof. By (a) and since $(\{1,2,3\}!, \circ)$ is a group. □

3. HYPERGROUPOIDS WITH DIVISIONS

Definition 3.1. We shall say that a hypergroupoid $(Q; A)$ is a **hypergroupoid with divisions** iff the following statemets hold:

(1) For all $a, b \in Q$ $\quad A(a,b) \neq \emptyset$,
(2) For all $a, b \in Q$ $\quad {}^{-1}A(a,b) \neq \emptyset$,[2]
(3) For all $a, b \in Q$ $A^{-1}(a,b) \neq \emptyset$.[3]

Cf.1.4 and 1.5.

Proposition 3.1. *Let $(Q; A)$ be a hypergroupoid with division. Then, for all $\alpha \in \{1,2,3\}!$ the hypergroupoid (Q, A^α) is a hypergroupoid with divisions.*

Proof. a) By Def. 2.2, (a) from 2, we obtain

$$\begin{aligned}
A^*(x_1,x_2) &= A(x_2,x_1),\\
{}^{-1}(A^*)(x_1,x_2) &= A^{-1}(x_2,x_1) \quad \text{and}\\
(A^*)^{-1}(x_1,x_2) &= {}^{-1}A(x_2,x_1)
\end{aligned}$$

for all $x_1, x_2 \in Q$. Whence, by (1)-(3) from Def. 3.1, we conclude the statements

(4) For all $a, b \in Q$ $\quad A^*(a,b) \neq \emptyset$,

[2] For all $a, b \in Q$ there is at least one $x \in Q$ such that the following formula holds $A(x,a) \ni b$.
[3] For every $a, b \in Q$ there is at least one $y \in Q$ such that the following formula holds $A(a,y) \ni b$.

(5) For all $a, b \in Q \quad {}^{-1}(A^*)(a, b) \neq \emptyset$ and
(6) For all $a, b \in Q \ (A^*)^{-1}(a, b) \neq \emptyset$
hold.

b) By a) and by Prop. 2.3, we conclude that the proposition is satisfied.

□

4. Hypercancelation hypergroupoids

Definition 4.1. We shall say that a hypergroupoid $(Q; A)$ is a **hypercancellation hypergroupoid** iff there are $p, q, r \in N \cup \{0\}$ such that the following statements hold:

(i) $(\exists (a_1, a_2) \in Q^2)|A(a_1^2)| = p$ and $(\forall (x, y) \in Q^2)|A(x, y)| \leq p$;
(ii) $(\exists (b_1, b_2) \in Q^2)|{}^{-1}A(b_1^2)| = q$ and $(\forall (x, y) \in Q^2)|{}^{-1}A(x, y)| \leq q$; and
(iii) $(\exists (c_1, c_2) \in Q^2)|A^{-1}(c_1^2)| = r$ and $(\forall (x, y) \in Q^2)|A^{-1}(x, y)| \leq r$.

Moreover, we shall say that a hypercancellation hypergroupoid $(Q; A)$ is a hypercancelation hypergroupoid of the type $< p, q, r >$.

Example 4.1. Let $(Q; \mathcal{A})$ be a cancellation groupoid. Then $(Q; A)$, where

$$A(x, y) \stackrel{def}{=} \{\mathcal{A}(x, y)\}$$

for all $(x, y) \in Q^2$, is a hypercancellation hypergroupoid of the type $< 1, 1, 1 >$.

Example 4.2. Let Q be a non-empty set and let $A(x, y) = \emptyset$ for all $(x, y) \in Q^2$. Then:

a) ${}^{-1}A(x, y) = \emptyset$ for all $(x, y) \in Q^2$;
b) $A^{-1}(x, y) = \emptyset$ for all $(x, y) \in Q^2$; and
c) (Q, A) is a hypercancellation hypergroupoid of the type $< 0, 0, 0 >$.

Example 4.3. Let Z be the set of all integers and let

$$\mathcal{A}(2k - 1, 2t - 1) = \mathcal{A}(2k - 1, 2t) = \mathcal{A}(2k, 2t - 1) = \mathcal{A}(2k, 2t) = k + t$$

for all $k, t \in Z$ Also, let

$$A(x, y) \stackrel{def}{=} \{\mathcal{A}(x, y)\}$$

for all $x, y \in Z$. Then $(Q; A)$ is a hypercancellation hypergroupoid of the type $< 1, 2, 2 >$.

Theorem 4.1. *Let $(Q; A)$ be a hypercancellation hypergroupoid of the type $< p, q, r >$. Then ${}^{-1}A, A^{-1}, A^*, {}^{-1}(A^*)$ and $(A^*)^{-1}$ are hypercancellation hyperoperations, respectively, of the type $< q, p, r >, < r, q, p >, < p, r, q >, < r, p, q >$ and $< q, r, p >$*

Sketch of a part of the proof.

$$\begin{aligned} &{}^{-1}({}^{-1}A) = A, \\ &({}^{-1}A)^{-1}(x, y) \ni z && \Leftrightarrow {}^{-1}A(x, z) \ni y \\ &&& \Leftrightarrow A(y, z) \ni x \\ &&& \Leftrightarrow A^{-1}(y, x) \ni z. \end{aligned}$$

So, $(Q; {}^{-1}A)$ is a hypercancellation hypergroupoid of the type $< q, p, r >$. □

Theorem 4.2. *Let $(Q; A)$ be a finite hypergroupoid with divisions. Then $(Q; A)$ is a hypercancellation hypergroupoid.*

Proof. By Def. 1.1., Def. 3.1, Def. 4.1 and by $|Q| \in N$. □

Theorem 4.3. *There is a hypergroupoid $(Q; A)$ such that the following statements hold:*

a) (Q, A) *is a hypergroup; and*
b) $(Q; A)$ *is not a hypercancellation hypergroupoid.*

Proof. Let $(Q; \cdot)$ be an infinite group. Also, let

$$A(x, y) \overset{def}{=} \{x \cdot k \cdot y | k \in \overline{Q} \wedge \overline{Q} \subseteq Q \wedge |\overline{Q}| \geq |N|\}$$

for all $x, y \in Q$. Then $(Q; A)$ is a hypergroup. Moreover, $(Q; A)$ is not a hypercancellation hypergroupoid. □

5. Hyperquasigroups

Definition 5.1. We shall say that a hypergroupoid $(Q; A)$ is a **hyperquasigroup** iff $(Q; A)$ is a hypergroupoid with divisions and hypercancelation hypergroupoid as well[4].

Remark 5.1. Since each hyperquasigroup $(Q; A)$ is a hypercancellation hypergroupoid of the type $< p, q, r >$, we shall say that it is a hyperquasigroup of the **type** $< p, q, r >$. Moreover, we called m is a **rang** hyperquasigroup $(Q; A)$ of the type $< p, q, r >$ iff $m = \max\{p, q, r\}$. Together, by Th. 4.4, for all $\alpha \in \{1, 2, 3\}!$ rang $(Q; A^\alpha) = m$.

Remark 5.2. Groupoid $(Z; \mathcal{A})$ from Example 4.3 is a groupoid with divisions. However, $(Z, \mathcal{A})$ is not a cancelation groupoid, i.e. $(Z; \mathcal{A})$ is not a quasigroup.

By Th. 4.5 and by Def. 5.1, we obtain:

Theorem 5.1. *Every finite hypergroupoid with divisions is a hyperquasigroup.*

By Th. 4.6, Def. 1.4 and by Def. 5.1, we have:

Theorem 5.2. *There are hypergroups with are not hyperquasigroups.*

By Def. 1.4, Def. 3.1 and by Th. 5.4, we obtain:

Theorem 5.3. *Every finite hypergroup is a hyperquasigroup.*

[4]See, also [3]

6. 3− < m > −NET

Definition 6.1. Let $\Im$ be a non–empty set and let $\mathcal{L}$ be a non-empty subset of the set $\Im$. We shall say that the elements of $\Im$ are **points** and the elements of $\mathcal{L}$ are **lines**. Also, let $\{L_1, L_2, L_3\} \overset{def}{=} \mathcal{L}/\sim$, where $\sim$ is a equivalence relation on the set $\mathcal{L}$. Then, we shall say that the object $(\Im; L_1, L_2, L_3)$ is a **3-< m >-net**, $m \in N$, iff the following conditions hold:

M1. Each point belongs to exactly one line of each equaivalence (paralel) class [L_1, L_2 and L_3].

$\mathbf{M2_1}$. If L_a and L_b are different classes of $\{L_1, L_2, L_3\}$, then for all $l_1 \in L_a$ and $l_2 \in L_b$ the following formula holds $|l_1 \cap l_2| \leq m$.

$\mathbf{M2_2}$. If L_a and L_b are different classes of $\{L_1, L_2, L_3\}$, then for all $l_1 \in L_a$ and $l_2 \in L_b$ the following formula holds $|l_1 \cap l_2| \geq 1$.

$\mathbf{M2_3}$. There is at least one $(l_1, l_2) \in L_a \times L_b$, where $a, b \in \{1, 2, 3\}$ and $a \neq b$, such that the following equality holds $|l_1 \cap l_2| = m$.

M3. $|L_1| = |L_2| = |L_3|$.

M4. Let L_a, L_b and L_c be arbitrary mutually different classes of $\{L_1, L_2, L_3\}$. Then, for all $A, B \in \Im$ and for every $l_1 \in L_a, l_2 \in L_b$ and $l, \bar{l} \in L_c$ the following condition holds

$$A \neq B \wedge A \in l_1 \cap l_2 \wedge B \in l_1 \cap l_2 \wedge A \in l \wedge B \in \bar{l} \Rightarrow l \neq \bar{l}.$$[5]

Remark 6.1. 3− < 1 > −net is a 3-net. Cf. [4, 5, 6].

Remark 6.2. a) The conditions **M1**–**M4** (for $m = 2$) hold in the object represented to Diagram 1.

b) The conditions **M1**–**M3** (for $m = 2$) hold in the object represented to Diagram 2. However, condition **M4** does not hold.

7. HYPERQUASIGROUPS AND 3− < m > −NETS

Theorem 7.1. *Let $(Q; A)$ be a hyperquasigroup, $|Q| \geq 2$ and let $Rang(Q; A) = m[\in N]$. Also, let*

(t) $\Im \overset{def}{=} \{(x, y, z) | x \in Q \wedge y \in Q \wedge A(x, y) \ni z\}$,

(c_1) $L_1 \overset{def}{=} \{(x, y, z) | y \in Q \wedge z \in Q \wedge {}^{-1}A(z, y) \ni x | x \in Q\}$

(c_2) $L_2 \overset{def}{=} \{(x, y, z) | x \in Q \wedge z \in Q \wedge A^{-1}(x, z) \ni y | y \in Q\}$ *and*

(c_3) $L_3 \overset{def}{=} \{(x, y, z) | x \in Q \wedge y \in Q \wedge A(x, y) \ni z | z \in Q\}$.

Then $(\Im; L_1, L_2, L_3)$ is a 3-< m >-net.

Proof. 1) By $(c_1) - (c_3)$ and by Def. 5.1, we have: $L_1 \cap L_2 = L_1 \cap L_3 = L_2 \cap L_3 = \emptyset$ and $|L_1| = |L_2| = |L_3| = |Q|$ [**M3**].

2) By $(t), (c_1) - (c_3)$ and by Def. 5.1, we obtain **M1**.

3) By $Rang(Q; A) = m$, Rem. 5.2 and by $(c_1) - (c_3,)$ we have $\mathbf{M2_1}$ and $\mathbf{M2_3}$.

4) By $(c_1) - (c_3)$ and by Def. 5.1 [Def. 3.1], we obtain $\mathbf{M2_2}$.

4) By $(x,y,z_1) \neq (x,y,z_2) \Longleftrightarrow z_1 \neq z_2$,
$(x,y_1,z) \neq (x,y_2,z) \Longleftrightarrow y_1 \neq y_2$ and
$(x_1,y,z) \neq (x_2,y,z) \Longleftrightarrow x_1 \neq x_2$,
we have, also, **M4**.

Remark: For $m = 1$ see Rem. 6.2. □

Theorem 7.2. *Let $(\Im; L_1, L_2, L_3)$ be a $3- < m > -$net. Also let*

(l_1) $L_1 = \{< 1, x > | x \in Q\}$,
(l_2) $L_2 = \{< 2, x > | x \in Q\}$ *and*
(l_3) $L_3 = \{< 3, x > | x \in Q\}$,

where Q is an arbitrary set such that $|Q| = |L_1| (= |L_2| = |L_3|)$. Finally, let

(h) $A(x,y) \ni z \overset{def}{\Leftrightarrow} (\exists T \in \Im) T \in < 1, x > \cap < 2, y > \cap < 3, z >$

for all $x, y, z \in Q$. Then, $(Q; A)$ is a hyperquasigroup of the rang m.

Proof. 1) By **M1** and by (h), we conclude that A is a hyperoperation in Q.

2) By **M2₂**, **M3** and by 1), we have $(Q; A)$ is a hypergroupoid with divisions.

3) By **M2₁**, **M2₃**, by Def.4.1 and by 1), we conclude that $(Q; A)$ is a hypercancelative hypergroupoid.

4) By **M2₁**, **M2₃**, 2), 3), Def. 5.1 and by Rem. 5.2, we conclude that $(Q; A)$ is a hyperquasigroup of the rang m.

Remark: By **M4** and **M1**, $F : \Im \to \{(x,y,z) | (h)\}$ is a bijection. See, also Rem. 6.3, Th.7.1 and Th.7.2. □

Theorem 7.3. *Let $(Q; A)$ be a hyperquasigroup. Then, there are permutations α, β, γ of Q such that the following formula holds*

$$(\exists e \in Q)(\exists x \in Q)(L(e,x) \ni x \wedge L(x,e) \ni x),$$

where

$$L(x,y) \overset{def}{=} \gamma A(\alpha(x), \beta(y))$$

and

$$\varphi\{a_1, \ldots, a_s\} \overset{def}{=} \{\varphi(a_1), \ldots, \varphi(a_s)\}, \qquad a_1^s \in Q, \quad \varphi \in Q!, \quad 1 \leq s \leq m.^6$$

Proof. By Def.6.1, Th. 7.1 and by Th. 7.2. See Diag. 3;

$$1 \leq | < 1, e > \cap < 2, x > | \leq m, \qquad 1 \leq | < 3, x > \cap < 2, e > | \leq m.$$

See, also [5], p.p. 13–16. □

8. REMARKS

Remark 8.1. Hypergroupoid $(Q; A)$ defined by Table 3 is a hyperquasigroup. Hypergroupoid $(Q; \overline{A})$, where $\overline{A}(x,y) = A(x,y)$ for all $(x,y) \in Q^2 \setminus \{(3,3)\}$ and $\overline{A}(3,3) = \{1\} [\overline{A}(3,3) \subseteq A(3,3) \wedge \overline{A}(3,3) \neq A(3,3)]$, is too a hyperquasigroup.

Remark 8.2. Hipergroupoid defined by Table 4 is a hyperquasigroup. For all $(Q; \overline{B})$, where $\overline{B}(x,y) = B(x,y)$ for all $(x,y) \in Q^2 \setminus \{(1,3),(3,2)\}, \overline{B}(1,3) \subseteq B(1,3), \overline{B}(3,2) \subseteq B(3,2)$ and $|\overline{B}(1,3)| = |\overline{B}(3,2)| = 1$, is not a hyperquasigroup.

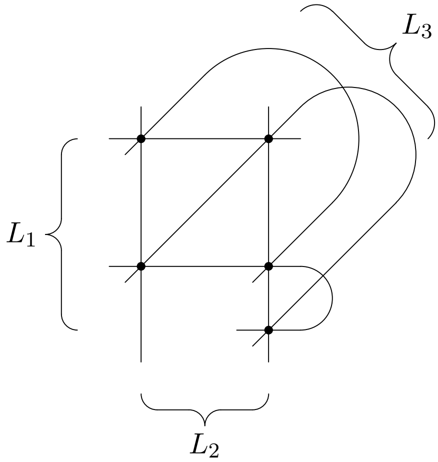

Diagram 1.

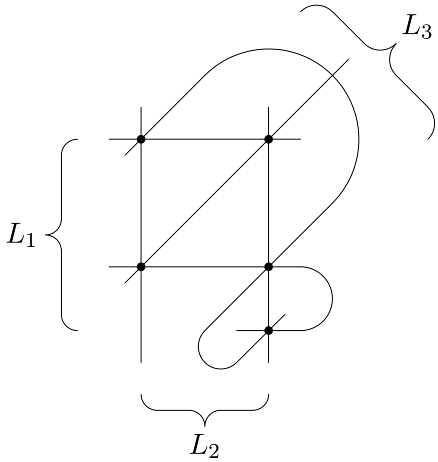

Diagram 2.

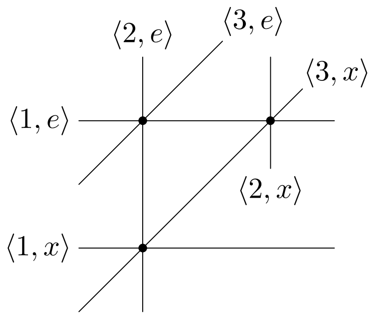

Diagram 3.

A	1	2	3
1	$\{1\}$	$\{2\}$	$\{3\}$
2	$\{3\}$	$\{1\}$	$\{2\}$
3	$\{2\}$	$\{3\}$	$\{1,2\}$

Table 3

B	1	2	3
1	$\{1\}$	$\{1\}$	$\{2,3\}$
2	$\{2\}$	$\{1\}$	$\{3\}$
3	$\{3\}$	$\{2,3\}$	$\{1\}$

Table 4

Remark 8.3. In the searching for notion of hyperquasigroup there has been in aspect, also, the following:

1) There is groupoid with divisions which is not a quasigroup; and
2) Every semigroup with divisions is a quasigroup (group). [We say that a groupoid $(Q;\cdot)$ is a groupoid with divisions iff for every $a, b \in Q$ there exist at least one $x \in Q$ and at least one $y \in Q$ such that the following equalities hold $a \cdot x = b$ and $y \cdot a = b$.]

REFERENCES

[1] F. Marty, *Sur une généralization de la notion de groupe*, Huitiéme congres de mathématiciens Scandinaves, Stockholm, (1934), 45–49.

[2] J. Mittas, *Hypergroupes canoniques valués et hypervalués*, Math. Balkanica **1**(1971), 181-185.

[3] G. Tallini, *Geometric Hyperquasigroups and Line Spaces*, Acta Universitatis Carolinae – Mathematica et Physica, Vol. 25, No. **1**(1984), 69–73.

[4] V. D. Belousov, *Foundation of the theory of quasigroups and loops*, (Russian), "Nauka", Moscow 1967.

[5] V. D. Belousov, *Algebraic nets and quasigroups*, (Russian), Stiinta, Kishinev 1971.

[6] J. Dénes and A. D. Keedwell, *Latin Squares – New Developments in the Theory and Applications*, North–Holland 1991.

[7] J. Ušan, *k-$<2>$-nets*, (Russian), Rev. of Research, Fac. of Sci. Univ. of Novi Sad, Math. Ser., 16, **2**(1986), 173–196.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF NOVI SAD
TRG D. OBRADOVIĆA 4
21000 NOVI SAD,
SERBIA AND MONTENEGRO

FACULTY OF ELECTRICAL ENGINEERING
UNIVERSITY OF OSIJEK
KNEZA TRPIMIRA 2B
HR - 31000 OSIJEK
CROATIA