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Axiom of Choice – 100$^{\text{th}}$ Next

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Abstract. In this survey, first, we give a short history oversee of the Axiom of Choice; and, second, in the connection with this we give oversee our main results which are new equivalents of the Axiom of Choice, i.e., Zorn's lemma. These statement are of fixed apex type and fixed point type theorems. The survey includes comments about these theorems and some historical facts.

1. History and annotations

We shall first discuss an assumption that appears to be independent of, and yet consistent with, the usual logical assumptions regarding classes and correspondences, but whose absolute validity has been seriously questioned by many authors. This is the so-called Axiom of Choice, which has excited more controversy than any other axiom of set theory since its formulation by Ernst Zermelo in 1908. In this sense, many results are known in the set theory.

In 1904, Zermelo[1] stated a principle of choice similar to: If $\mathcal{D}$ is a family of nonempty sets, there is a function f such that $f(A) \in A$ for every $A \in \mathcal{D}$; and proved that it implied the well-ordering theorem. In 1908 Zermelo proposed main version of the Axiom of Choice. This is the connection and with a conversations with Erhardt Schmidt.

Bertrand Russell in 1906 gave a principle analogous to preceding. He announced this principle as a possible substitute for Zermelo's but he believed that it was weaker. Zermelo, in 1908 stated and, proved that Russell's and his formulations of the axiom of choice are equivalent. *The name "axiom of choice" is due to Zermelo* in 1904.

Apparently, the first specific reference to the axiom of choice was given in a paper by G. Peano[2] in 1890. In proving an existence theorem for ordinary differential equations, he ran across a situation in which such a statement is needed. Beppo Levi in 1902, while discussing the statement that the union of a disjoint set S of nonempty sets has a cardinal number greater than or equal to the cardinal

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[1]Before 1904, when Zermelo published his proof that the axiom of choice implies the *well-ordering theorem*, the well-ordering theorem was considered as self-evident. Cantor and the others used it without hesitation.

[2]Giuseppe Peano: "But as one cannot apply infinitely many times an *arbitrary* rule by which one assigns to a class A an individual of this class, a determinate rule is stated here".

number of S, remarked that its proof depended on the possibility of selecting a single member from each element of S. Others, including Georg Cantor, had used the principle earlier, but did not mention it specifically.

In this time, the Axiom of Choice asserts that for every set S there is a function f which associates each nonempty subset A of S with a unique member $f(A)$ of A. From a psychological perspectie, one might express the Axiom by saying that on element is "chosen" from each subset A of S. However, if S is infinite, it is difficult to conceive how to make such choices – unless a rule is available to specify an element in each A.

David Hilbert, in 1926, once wrote that Zermelo's Axiom of Choice was the axiom "most attacked up to the present in the mathematical literature ..."; to this, Abraham Fraenkel later added that "the axiom of choice is probably the most interesting and, in spite of its late appearance, the most discussed axiom of mathematics, second only to Euclid's axiom of parallels which was introduced more than two thousand years ago".

The equivalence of the axiom of choice and the trichotomy was given by Hartogs in 1915. As in the case of the well-ordering theorem, the trichotomy was considered self-evident and was used without hesitation before 1915.

As mathematics developed futher there also developed a need for another nonconstructive proposition; a principle, which Kuratowski, Hausdorff, Zorn, and others, used to replace transfinite induction and the well-ordering theorem. It appears, at first glance, unrelated to the axiom of choice, but actually is equivalent to it.

This principle and principles similar to it are often referred to as forms of Zorn's lemma. In 1933 Artin and Chevalley first referred to the principle as Zorn's lemma.[3]

The history of maximal principles is quite tangled. The earliest reference to a maximal principle in the literature is in 1907 from Hausdorff.

In 1910 independently Janiszewski, Mazurkiewicz and Zoretti published a special case Hausdorff's principle in the form of a theorem in topology. In 1905 Lindelöf, in 1911 Brouwer, and in 1920 Sierpiński derivated some more general topological theorems from the well-ordering theorem.

In 1922 Kuratowski derived minimal principles equivalent to the preceding principles from the well-ordering theorem. Kuratowski in 1922 used a minimal principle to prove a theorem in analysis, as and R. L. Moore in 1932.

[3] **What were the beginnings of Zorn's principle**? According to his later reminiscences, he first formulated it at Hamburg in 1933, where Claude Chevalley and Emil Artin then took it up as well. Indeed, when Zorn applied it to obtain representatives from certain equivalence classes on a group, Artin recognized that Zorn's principle yields the Axiom of Choice. By late in 1934, Zorn's principle had found users in the United States who dubbed in Zorn's lemma. In October, when Zorn lectured on his principle to the American Mathematical Society in New York, Solomon Lefschetz recomended that Zorn publish his result. The paper appeared, the following year, in 1935.

It wasn't until 1935 that Max Zorn published his paper. He was the first one to use a maximal principle in algebra. He stated without proof that this the maximal principle is equivalent to the axiom of choice. For this proof Zorn credits Artin and Kuratowski.

In France, where the Axiom had been so poorly received three decades earlier, Zorn's friend Chevalley introduced the maximum principle to the Bourbakists[4] and after dedicing, Bourbaki stated Zorn's principle as a corollary.

In 1940, also influenced by Zorn, the Princeton topologist John Tukey deduced from the Axiom four variants of what he termed Zorn's lemma, and sketched a proof of their equivalence to the Axiom of Choice.

Nevertheless, there remained one final independent rediscovery, due to the German algebraist O. Teichmüller then in 1939 a Privatdozent at Berlin.[5] This principle is often referred to as form *of Teichmüller-Tukey lemma.*

The Serbian mathematician Djuro Kurepa found in 1952 a number of relations R such that the corresponding maximal principle was an equivalent.

In 1960 two American mathematicians, Herman and Jean Rubin, were prompted by Kurepa's research to consider maximal principles. In addition, H. Rubin found two statements which were equivalent to the Axiom of Choice in ZF, but were weaker in ZFU. In 1963 the Rubins published a book summarizing and completing much of the earlier work on equivalents.

On the other hand, in 1936 the American mathematician Marshall Stone, then at Harvard, contributed his influential findings on the representation of Boolean rings. Stone deduced a proposition equivalent to it and later known as the *Stone Representation Theorem.*

In 1939 A. Tarski was studying the number of prime ideals found in rings of sets. Later, in 1940, Birkohoff observed that his representation theorem for distributive lattices had been inspired by the researches of Tarski.

Probably the most well-known and important topological equivalent of the Axiom of Choice is the *Tychonoff Compactness Theorem* in 1935 from a maximal principle for which in 1955 J. Kelley proved the converse.

The second development occured on the frontier between algebra, analysis, and set theory: Stefan Banach's researches at Lwów on functional analysis.

In 1929 Banach established a fundamental result later known as the *Hahn-Banach theorem.* To obtain this result, Banach relied on the well-ordering theorem. In this sense, in analysis, the following facts are connection and hold: Krein-Milman theorem, Alaoglu's theorem, and Bell-Fremlin theorem, as and many others.

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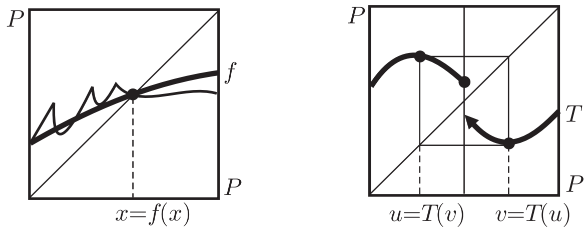
[4]A group of young French mathematicians wrote collectively under the pseudonym of *Nicolas Bourbaki*, from Dieudonné.

[5]For one of these his maximal principles Teichmüller credited to Erhard Schmidt, whose conversations with Zermelo in 1904 had assisted him in proving the well-ordering theorem.

2. The Axiom of Choice and Fixed Point Theory

In general, equivalents of the Axiom of Choice appear frequently in almost all branches of mathematics in a large variety of different forms.

The fixpoint problem for a given mapping $f|P$ is very easy to formulate: the question is whether some $\zeta \in P$ satisfies $f(\zeta) = \zeta$. Many problems are reducible to the existence of fixpoints of certain mappings. The question remains whether statement (of the Axiom of Choice type) could be equivalently expressed in the fixpoint language as well. The answer is affirmative.



An existence theorem asserts the existence of an object belonging to a certain set and possessing certain properties. Many existence theorems can be formulated so that the under lying set is a partially ordered set and the crucial property is maximality.

This principle and principles similar to it are often referred to as a form of Zorn's lemma. A strong form of Zorn's lemma is due to Bourbaki. In this paper we present a new strong form of Zorn's and Bourbaki's lemma. On the other hand, we notice that a statement on the existence of maximal elements (in certain partially ordered complete subsets of a normed linear space) played a central role in the proof of the fundamental statement of Bishop and Phelps on the density of the set of support points of a closed convex subset of a Banach space.

The transfinite induction argument is based on Zorn's lemma. This argument was later used in a different setting by Brøndsted and Rockafellar, Browder, Ekeland, Brøndsted and others. Recently Brézis and Browder proved a very general principle concerning order relations which unifies a number of diverse results in nonlinear functional analysis.[6]

[6]**Zermelo's Reply to His Critics**. During the summer of 1907 Zermelo took stock of the criticisms directed against both his Axiom and his proof of the well-ordering theorem. One in 1908 was a reply to his critics, and the other also in 1908 contained the first axiomatization of set theory. Zermelo's first article in 1908 began with a new demonstration of the well-ordering theorem.

From them he developed the properties of his θ-chains, which generalized Dedekind's earlier concept of chain. Zermelo corresponded with Jourdain in 1907, but apparently their letters focused on a generalization of König's theorem.

Although he had read Borel's article and the published correspondence between: Baire, Borel, Hadamard, and Lebesgue, he concetrated on refuting Peano with whom he had previously feuded over the equivalence theorem.

During 1906 he corresponded with Poincaré regarding his proof and his axiomatization of set theory. A letter, as well as three others from Poincaré, is kept in Zermelo's *Nachlass* at the

The principle extending the one of Ekeland and having a number of interesting applications to convex as well as nonconvex analysis has been to preformulate in some equivalent forms by Altman, Turinici and some others.

Let P be a partially ordered set. A self mapping f of P into itself is called an **increasing** (**decreasing**) mapping if $x \preccurlyeq y$ implies $f(x) \preccurlyeq f(y)$ $(f(y) \preccurlyeq f(x))$ for all $x, y \in P$.

The main purpose of the present paper is to establish a theorem on the existence of maximal elements in certain partially ordered sets.

On the other hand, in this paper we prove some new equivalents of the Axiom of Choice. These statements are fixed point type theorems and fixed apex type theorems. Also, this paper presents new characterizations of inductiveness of posets in terms of fixed apices and fixed points.

In addition, by $\{a_\alpha\}$ for $\alpha < \omega$, where ω is any (finite or transfinite) ordinal we shall denote the **sequence** (or **net**) whose constructive terms are $a_0, a_1, \ldots, a_\alpha, \ldots$ with $\alpha < \omega$; the set of all terms of this sequence will be denoted by $\{a_\alpha\}$. The sequence $\{a_\alpha\}$ is, of course, called **increasing** (**strictly increasing**) if $a_\alpha \preccurlyeq a_\beta$ $(a_\alpha \prec a_\beta)$ for any $\alpha < \beta < \omega$; analogously we define decreasing (strictly decreasing) sequence.

Call a poset P **inductive** (**chain complete**) when every nonempty chain in P has an upper bound (least upper bound, i.e., supremum) in P. Also, call a poset P **increasing inductive** (**increasing chain complete**) when every increasing sequence has an upper bound (supremum) in P.

In connection with the preceding, call a poset P **conditionally increasing inductive** (**conditionally increasing chain complete**) when every increasing countable sequence $\{x_n\}_{n\in\mathbb{N}}$ with upper bounds implies that every increasing sequence in the set $S(x_n) := \{x \in P \,|\, x_n \preccurlyeq x\}$ has an upper bound (supremum) in P. Call a poset P **countable increasing inductive** (**countable increasing chain complete**) when every increasing countable sequence has an upper bound (supremum) in P. Also, call a poset P **nontrivial** when in P there is at least one sequence $\{x_n\}_{n\in\mathbb{N}}$ with upper bounds.

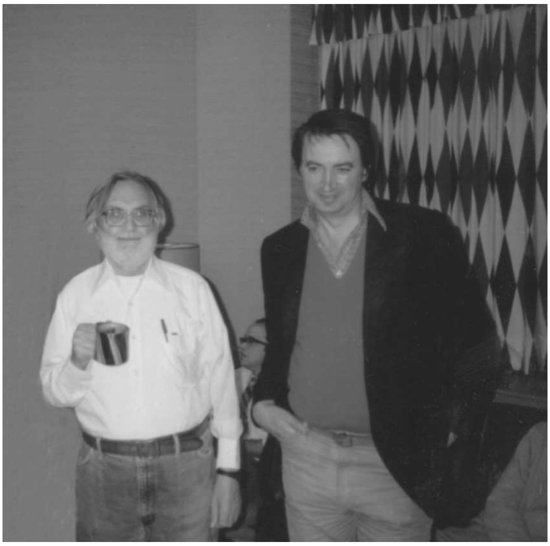
In our paper in 1988 we investigated the concept of fixed apices for a mapping f of a **poset** (:= partially ordered set) P into itself. A map f of a partially ordered set P to itself has a **fixed apex** $u \in P$ iff for $u \in P$ there is $v \in P$ such that $f(u) = v$ and $f(v) = u$. The points $u, v \in P$ are called **fixed apices** of f if $f(u) = v$ and $f(v) = u$.

We notice that fixed points are evidently fixed apices and the set of all fixed points can be a proper subset of the set of fixed apices. Also, a fixed point is evidently a fixed apex.

As mathematics developed further there also developed a need for another non-constructive proposition; a principle, which Kuratowski, Hausdorff, Zorn, Bourbaki, and others used to replace transfinite induction and the well-ordering theorem. It appears, at first glance, unrelated to the axiom of choice, but actually is equivalent to it.

This principle and principles similar to it are often referred to as forms of Zorn's lemma, i.e., if P is an inductive poset, then P has a maximal element. A strong form of Zorn's lemma due to Bourbaki states that if P is a **quasi inductive poset** (i.e., every nonempty well ordered chain in P has an upper bound), then P has a maximal element.

University of Freiburg in Breisgau. De facto, Z e r m e l o emerged as a realist in much situations, perhaps a Platonist!?



Max Zorn and Milan Tasković at
Indiana University (Bloomington – USA, November 1987)

In this part we present a new strong form of Zorn's and Bourbaki's lemma. Then our maximal principle is expressible in the following form.

Lemma 1. (Maximal Principle). *Let P be a nontrivial conditionally increasing inductive partially ordered set. Then P has a maximal element.*

In order to prove this principle we need the following essential lemmas and some further results.

We notice that the map P has a fixed apex if and only if $f^2 := f(f)$ has a fixed point.

Namely, if f has a fixed apex $u \in P$, then $u = f(v)$ and $v = f(u)$, so f^2 has a fixed point. On the other hand, if the equation $x = f^2(x)$ has a solution $\xi = f^2(\xi)$ for some $\xi \in P$, then f has fixed apices ξ, $f(\xi) \in P$, because $\xi = f^2(\xi)$ and $f(\xi) = f(\xi)$.

Let P be a partially ordered set and f a mapping from P into P. For any $f : P \to P$ it is natural to consider the following set

$$\mathrm{Sub}\, f(P) := f(P) \cup \{a \in P \,|a = \mathrm{ub}C \ \text{ for some chain } C \text{ in } f(P)\},$$

where $\mathrm{ub}C$ is an upper bound of C.

Lemma 2. (Fixed Apices Lemma). *Let P be a conditionally increasing inductive poset, and f a mapping from P into P such that*

$$\text{(M)} \qquad x \preccurlyeq f^2(x) \quad \textit{for all} \quad x \in \mathrm{Sub}\, f(P).$$

If there is a countable increasing sequence in P with an upper bound $b \in \mathrm{Sub}\, f(P)$, then map f has a fixed apex.

Proof. From the conditions of this statement, let $\{x_n\}_{n\in\mathbb{N}}$ be a countable increasing sequence in P with an upper bound $b \in \mathrm{Sub}(P)$.

Since P is a conditionally increasing inductive poset we obtain that every increasing sequence in the set $S(x_n)$ has an upper bound in P. Let $\mathcal{T}$ be the system of chains of the set $S(x_n)$.

The family $\mathcal{T}$ is an inductive partially ordered set. By Zorn's lemma there exists a maximal element $z \in P$, i.e., $\operatorname{Sub} f(P)$ has a maximal element $z \in \operatorname{Sub} f(P)$. From condition (M) we have $z \preccurlyeq f^2(z)$ and, because z is a maximal element of set $\operatorname{Sub} f(P)$, $f^2(z) \preccurlyeq z$. Hence, $f^2(z) = z$, so, from the preceding remark, f has a fixed apex.

The following result of Bourbaki allows us to prove the basic fixpoint statement for complete posets.

Lemma 3. (Bourbaki in 1950). *Let P be a chain complete poset and $f : P \to P$ a map such that $x \preccurlyeq f(x)$ for all $x \in P$. Then f has a fixed point.*

Our next statement extends Lemma 3 of Bourbaki to conditionally increasing inductive posets.

Lemma 4. (Fixed Point Lemma). *Let P be a conditionally increasing inductive poset, and f a mapping from P into P such that*

$$\text{(T)} \qquad x \preccurlyeq f(x) \quad \textit{for all} \quad x \in \operatorname{Sub} f(P).$$

If there is a countable increasing sequence in P with an upper bound $b \in \operatorname{Sub} f(P)$, then map f has a fixed point.

Proof. From the conditions of this statement, let $\{x_n\}_{n \in \mathbb{N}}$ be a countable increasing sequence in P, and its an upper bound $b \in \operatorname{Sub} f(P)$ exists.

Since P is a conditionally increasing inductive poset we obtain that every increasing sequence in the set $S(x_n)$ has an upper bound in P. Let F be the system of chains of set $S(x_n)$ The family F is an inductive poset. By Zorn' lemma there exists a maximal elements $z \in P$, i.e., $\operatorname{Sub} f(P)$ has a maximal elements $z \in \operatorname{Sub} f(P)$. From (T) we have $z \preccurlyeq f(z)$ and because z is a maximal element of set $\operatorname{Sub} f(P)$, $f(z) \preccurlyeq z$. Hence, $f(z) = z$, i.e., f has a fixed point.

In connection with the preceding, P is said to have the **conditionally fixed point property** if every map f of P into itself, with condition (T) and with condition that there is a countable increasing sequence with an upper bound $b \in \operatorname{Sub} f(P)$, has a fixed point.

Analogously, P is said to have **conditionally fixed apex property** if every map f of P into itself, with condition (M) and with condition that there is a countable increasing sequence with an upper bound $b \in \operatorname{Sub} f(P)$, has a fixed apex.

With the help of Lemmas 2 and 4, in this part, we present some characterization of inductiveness and conditionally increasing inductiveness of posets in terms of fixed apices.

Theorem 1. *Let P be a partially ordered set. Then the following statements are equivalent:*

(a) *P is conditionally increasing inductive,*

(b) *P has the conditionally fixed apex property,*
(c) *P has the conditionally fixed point property.*

Proof. Lemma 2 implies that (b) is a consequence of (a). On the other hand, from (T) we have the following inequality $f(x) \preccurlyeq f^2(x)$, i.e., $x \preccurlyeq f(x) \preccurlyeq f^2(x)$ for all $x \in \operatorname{Sub} f(P)$. Since (M) holds, and since there is a countable increasing sequence with an upper bound $b \in \operatorname{Sub} f(P)$, it follows from (b) that f has a fixed apex $u \in \operatorname{Sub} f(P)$, i.e., $u = f(v)$ and $v = f(u)$. From inequality (T) we have the following inequalities $u \preccurlyeq f(u) = v \preccurlyeq f(v) = u$, i.e., $u = v = f(u)$. Thus f has a fixed point, so (c) holds.

We need only show that (c) implies (a). Suppose that the poset P is not conditionally increasing inductive. Then there exists a chain C in $S(x_n)$ that does not have an upper bound. Let U be a chain cofinal with C such that

$$U := \{x \in C \,|x_0 \preccurlyeq x\}, x_0 = \text{ a fixed element of } C = \min U.$$

Thus all the elements of U can be arranged in a sequence, i.e., one can show that there exists increasing sequence $\{x_\alpha\}$ in U such that its upper bound does not exist, and $\{x_\alpha\}$ is strictly increasing, and for each $t \in U$, there exists $\alpha(t)$ such that $\alpha(t) < \alpha$ implies $t \preccurlyeq x_\alpha$. We define a mapping f from P into itself by

$$f(x) = \begin{cases} x_\beta \text{ if } x = x_\alpha \in U, \\ x_0 := \min U, \text{ if } x \notin U, \end{cases} \tag{1}$$

where $x_\alpha \preccurlyeq x_\beta (x_\alpha \neq x_\beta)$ for any $\alpha < \beta < \omega$ and where ω is any (finite or transfinite) ordinal. Thus we have defined a function f on P to P. Now, for any $x \in U$ ($\supset \operatorname{Sub} f(P)$) we have $x \preccurlyeq f^2(x)$, i.e., $x := x_\alpha \preccurlyeq x_\gamma = f(x_\beta) = f(f(x_\alpha)) = f^2(x_\alpha) = f^2(x)$ for $\alpha < \beta < \gamma < \omega$, so f statisfies (M).

Also, there is a countable increasing sequence of form $\{y_n := f(x_n) | x_n \notin U, n \in \mathbb{N}\}$ with the upper bound $b := x_0 \in \operatorname{Sub} f(P)$, and f does not have a fixed apex. This completes the proof.

Proof of Lemma 1. Since P is a nontrivial conditionally increasing inductive poset, we obtain that every increasing sequence in $S(x_n)$ has an upper bound in P. Then the system of chains of the set $S(x_n)$ is an inductive poset. Thus by Zorn's lemma there exists a maximal element in P. This completes the proof.

Annotation. We the notice, as an immediate application of the preceding Theorem 1, we have directly that Zorn's lemma and our Maximal Principle (Lemma 1) are equivalent.

3. New forms of the Axiom of Choice

We shall now discuss an assumption that appears to be independent of, and yet consistent with (see Gödel in 1940) the usual logical assumptions regarding classes and correspondences, but whose absolute validity has been seriously questioned by many authors.[7] This is the so called Axiom of Choice, which has excited more controversy than any other axiom of set theory since its formulation by Zermelo

[7]**Poincaré's opinion.** In this time, Richard's paradox also influenced Henri Poincaré, at that time the patriarch of French mathematics and an astute critic of research in the foundations of mathematics.

in 1908. In this sense, many results are known, in the set theory (see the references). In this part we prove some new equivalents of the Axiom of Choice. These statements are of fixed point type theorems and fixed apex type theorems.

Let P be a partially ordered set and f a mapping from P into P. For any $f : P \to P$ it is natural to consider the following set

$$\mathrm{Sub} \uparrow f(P) := f(P) \cup \{a \in P \,|\, a = \mathrm{ub}C \ \text{ for some increasing net } C \text{ in } f(P)\},$$

where ub$C$ is an upper bound of C.

We are now in a position to formulate our following statement.

Theorem 2. (Axiom of Choice). *Let $(P, \preccurlyeq)$ be a partially ordered set. Then the following statements are equivalent:*

(ZL) (Zorn's lemma). *Let P be an inductive poset. Then P has a maximal element.*

(MP) (Maximal Principle, Lemma 1). *Let P be a nontrivial conditionally increasing inductive poset. Then P has a maximal element.*

(FA) *Let P be a conditionally increasing inductive poset, and f a mapping from P into P such that*

$$\text{(A)} \qquad x \preccurlyeq f^2(x) \quad \text{for all} \quad x \in \mathrm{Sub} \uparrow f(P).$$

If there is a countable increasing sequence in P with an upper bound $b \in \mathrm{Sub} \uparrow f(P)$, then f has a fixed apex.

(FP) *Let P be a conditionally increasing inductive poset and f a mapping from P into P such that*

$$\text{(N)} \qquad x \preccurlyeq f(x) \quad \text{for all} \quad x \in \mathrm{Sub} \uparrow f(P).$$

If there is a countable increasing sequence in P with an upper bound $b \in \mathrm{Sub} \uparrow f(P)$, then f has a fixed point.

(ZT) (Zermelo). *Let P be a chain complete poset and f a mapping from P into itself such that*

(a) *there is an element $\theta \in P$ with $\theta \preccurlyeq x$ for all $x \in P$,*

(b) *$x \preccurlyeq f(x)$ for all $x \in P$,*

(c) *if $x, y \in P$ and $x \preccurlyeq y \preccurlyeq f(x)$ then either $x = y$ or $f(x) \preccurlyeq f(y)$.*

Then there is an element $\xi \in P$ with $f(\xi) = \xi$.

Proof. From Theorem 1, (ZL) is equivalent to (MP). Thus, we need only show that (MP) implies (FA) implies (FP) implies (ZT) implies (ZL).

(MP) implies (FA). By the Maximal Principle there exists a maximal element $z \in P$, i.e., $\mathrm{Sub} \uparrow f(P)$ has a maximal element $z \in \mathrm{Sub} \uparrow f(P)$. From condition (A) we have $z \preccurlyeq f^2(z)$ and, because z is maximal element of the set $\mathrm{Sub} \uparrow f(P)$, $f^2(z) \preccurlyeq z$. Hence, $f^2(z) = z$, so, from the preceding remarks f has a fixed apex.

Poincaré's articles of 1905–1906 attacking such research, particularly Russell's and Hilbert's, included an analysis of Zermelo's proof. In contrast to Russell, Poincaré believed that any attempt to prove or disprove Zermelo's Axiom from other postulates was illusory:

"The axioms in question [the Axiom of Choice and Russell's Multiplicative Axiom] will always be propositions which some will admit as "self-evident" and which others will doubt.

Each person will believe only his intuition. Yet there is one point in which everyone will agree: The Axiom is "self-evident" for finite classes.

But if it is unprovable for infinite classes, it is doubtless unprovable also for finite classes, which are not yet distinguished from the former at this stage in the theory.

Thus it [the Axiom of Choice] is a synthetic *a priori* judgment without which the "theory of cardinals" would be impossible, for finite as well as infinite numbers."

(FA) implies (FP). From (N) we have the following inequality $f(x) \preccurlyeq f^2(x)$, i.e., $x \preccurlyeq f(x) \preccurlyeq f^2(x)$ for all $x \in \mathrm{Sub} \uparrow f(P)$. Since (A) holds, it follows from (FA) that f has a fixed apex $u \in \mathrm{Sub} \uparrow f(P)$, then $u = f(v)$ and $v = f(u)$. From inequality (N) we have the following inequalities $u \preccurlyeq f(u) = v \preccurlyeq f(v) = u$, i.e., $u = v = f(u)$. Thus f has a fixed point, i.e., (FP) holds.

(FP) implies (ZT). Since P is chain complete, and hence $\mathrm{Sub} \uparrow f(P)$ is a conditionally increasing inductive poset. Applying (FP) to the set $\mathrm{Sub} \uparrow f(P)$, we obtain that f has a fixed point.

(ZT) implies (ZL). Suppose that the result (ZL) is false. Then for each $x \in P$ there exists $y \in P$ with $x \preccurlyeq y$ and $x \neq y$. Let $\mathcal{T}_0$ be the family of all nonempty chains of P and let $\mathcal{T} = \mathcal{T}_0 \cup \{\varnothing\}$. The family $\mathcal{T}$ is partially ordered by the inclusion relation between subset of P. For each $A \in \mathcal{T}_0$ the set

$$U_A = \{x \in P : x \quad \text{is an upper bound for } A \text{ and } x \notin A\}$$

is nonempty because, if x is an upper bound for A and $y \in P$ is such that $x \preccurlyeq y$ and $x \neq y$, then $y \in U_A$. Let $U_\varnothing = \{x_0\}$, where x_0 is an arbitrary element of P. Let g be a mapping with domain $X := \{U_A : A \in \mathcal{T}\}$, and now, we define a mapping g from X into itself by $g(x) = x$, i.e., g is the identity mapping. For each $A \in \mathcal{T}$ let $f(A) = A \cup \{g(U_A)\}$. By definition of g and U_A we have $a \preccurlyeq g(U_A)$ for all $a \in A$ and all $A \in \mathcal{T}_0$. It is now clear that $f(A) \in \mathcal{T}$ for all $A \in \mathcal{T}$ and hence f maps $\mathcal{T}$ into itself.

We shall prove that $\mathcal{T}$, partially ordered by inclusion, and f satisfy the conditions of (ZT). First we observe that $\varnothing \in \mathcal{T}$ and $\varnothing \in A$ for all $A \in \mathcal{T}$ so $\mathcal{T}$ satisfies condition (a) of (ZT). Next let $\mathcal{R}$ be a nonempty subfamily of $\mathcal{T}$ such that $\mathcal{R}$ is chain ordered by inclusion and let $A = \cup_{B \in \mathcal{R}} B$. Let $a, b \in A$. There are sets $C, D \in \mathcal{R}$ with $a \in C$ and $b \in D$. Since $\mathcal{R}$ is a chain ordered by inclusion either $C \subset D$ or $D \subset C$ and in either case we see that there is one set in $\mathcal{R}$ which contains both a and b. Since each set in $\mathcal{R}$ is a chain ordered subset of P it follows that either $a \preccurlyeq b$ or $b \preccurlyeq a$. This proves that $A \in \mathcal{T}$ and it is then easy to see that $A = \sup \mathcal{R}$. Thus $\mathcal{T}$ satisfies the condition of chain completeness of (ZT). By definition of f we have $A \subset f(A)$. Also condition (b) of (ZT) is satisfied. Also, it follows immediately that condition (c) of (ZT) is satisfied.

We can now conclude from (ZT) that there is a set $A_0 \in \mathcal{T}$ with $f(A_0) = A_0$. Thus we have a contradiction. The proof is now complete.

4. CONDITIONALLY CHAIN COMPLETENESS

In this section, we present some new characterizations of conditionally increasing chain completeness.

Let P be a partially ordered set and f a mapping from P into P. For any $f : P \to P$ it is natural to consider the following set

$$\uparrow \overline{f(P)} := f(P) \cup \{a \in P \,|a = \sup C \quad \text{for some increasing net } C \text{ in } f(P)\},$$

where $\sup C$ is a least upper bound of C.

In connection with the preceding, analogous to (FA) and (FP) of Theorem 2, we have the following immediate consequences:

Lemma 5. *Let P be a conditionally increasing chain complete poset and f a mapping from P into itself such that*

(SA) $$x \preccurlyeq f^2(x) \quad \textit{for all} \quad x \in \uparrow \overline{f(P)}.$$

If there is a countable increasing sequence in P with an upper bound $b \in \uparrow \overline{f(P)}$, then f has a fixed apex.

Lemma 6. *Let P be a conditionally increasing chain complete poset and f a mapping from P into itself such that*

$$\text{(SN)} \qquad x \preccurlyeq f(x) \quad \text{for all} \quad x \in \uparrow \overline{f(P)}.$$

If there is a countable increasing sequence in P with an upper bound $b \in \uparrow \overline{f(P)}$, then f has a fixed point.

Proofs. P is a conditionally increasing chain complete poset, and hence $\uparrow \overline{f(P)}$ is a conditionally increasing inductive poset, as well. Applying (FA) and (FP) of Theorem 2 to the set $\uparrow \overline{f(P)}$, we obtain that f has a fixed apex or fixed point.

In connection with the preceding, P is said to have the **s-conditionally fixed point property** if every map f of P into itself with condition (SN) and with condition that there is a countable increasing sequence with an upper bound $b \in \uparrow \overline{f(P)}$, has a fixed point. Analogously, P is said to have the **s-conditionally fixed apex property** if every map f of P into itself with condition (SA) and with condition that there is a countable increasing sequence with an upper bound $b \in \uparrow \overline{f(P)}$, has a fixed apex.

The following our result allows us to prove the basic fact for teasing from chain completeness to inductiveness.

Lemma 7. (Tasković in 1978, Lemma of noninductiveness). *Let P be a set totally ordered by the order relation $\preccurlyeq$. If the nonempty part A of P does not have a supremum, then there is, by relation $\preccurlyeq$, a well ordered subset B of the set A which does not have an upper bound in A.*

A brief proof of this statement based on Zermelo's theorem may be found at Tasković in 1978. A brief second proof of this fact may be found at Tasković in 1988.

From the preceding statements and Lemma 7, we are now in a position to formulate the following our statement.

Theorem 3. (Conditionally Chain Completeness). *Let P be a partially ordered set, then the following statements are equivalent:*

(a) *P is conditionally increasing chain complete.*

(b) *P has the s-conditionally fixed apex property.*

(c) *P has the s-conditionally fixed point property.*

Proof. Lemma 5 implies that (b) is a consequence of (a). On the other hand, from (SN) we have the following inequality $f(x) \preccurlyeq f^2(x)$, i.e., $x \preccurlyeq f(x) \preccurlyeq f^2(x)$ for all $x \in \uparrow \overline{f(P)}$. Since (SA) holds, it follows from (b) that (c) holds. Thus, we need only show that (c) implies (a).

Suppose that the poset P is not increasing chain complete. Then there exists an increasing net (chain) A in P that does not have a least upper bound. We define a mapping f from P into itself by (1). Then, from Lemma 7, f is well defined, and for any $x \in U(\supset\uparrow \overline{f(P)})$, we have $x \preccurlyeq f(x)$, i.e., $x := x_\alpha \preccurlyeq x_\beta = f(x_\alpha) = f(x)$ for $\alpha < \beta < \omega$, and where ω is any (finite or transfinite) ordinal. Thus, condition (SN) holds for all $x \in\uparrow \overline{f(P)}$, and f does not have a fixed point. Also, there is a countable increasing sequence of form $\{y_n := f(x_n)\,|x_n \notin U, n \in \mathbb{N}\}$ with the upper bound $b := x_0 \in\uparrow \overline{f(P)}$. This completes the proof.

5. Locally Forms of Conditionally Chain Completeness

In this section we present some further facts of conditionally increasing chain completeness of posets in terms of fixed apices and fixed points as some locally forms of the preceding conditions (M) and (T). We say that a mapping $f : P \to P$ has a **fork** if f satisfies the following inequalities

$$a \preccurlyeq f(a) \preccurlyeq f(b) \preccurlyeq b \quad \text{for some} \quad a, b \in P.$$

Also, a mapping $f : P \to P$ has a **left fork** if $a \preccurlyeq f(a)$ for some $a \in P$. Dually, f has a **right fork** if $f(b) \preccurlyeq b$ for some $b \in P$.

On the other hand, we say that a mapping $f : P \to P$ has a **befork** if f satisfies the following inequalities

$$a \preccurlyeq f^2(a) \preccurlyeq f^2(b) \preccurlyeq b \quad \text{for some} \quad a, b \in P.$$

A mapping $f : P \to P$ has a **left befork** if $a \preccurlyeq f^2(a)$ for some $a \in P$. Dually, f has a **right befork** if $f^2(b) \preccurlyeq b$ for some $b \in P$.

In this section the objects of forks are with central position in our considerations. We begin with the following essential lemmas.

Lemma 8. (Locally Fixed Apices Lemma). *Let P be a conditionally increasing chain complete poset and f an increasing mapping from P into P such that*

$$\text{(LM)} \qquad a \preccurlyeq f(a) \quad \textit{for some} \quad a \in P.$$

If there is a countable increasing sequence in P with the upper bound $a \in P$, then f has a fixed apex.

Proof. From the conditions of this statement, let $\{x_n\}_{n\in\mathbb{N}}$ be a countable increasing sequence in P with the upper bound $a \in P$. Since P is a conditionally increasing chain complete poset we obtain that every increasing sequence in the set $S(x_n)$ has a least upper bound in P.

Because $a \preccurlyeq f^2(a)$ and f is isotone, we find $f^2(a) \preccurlyeq f^4(a)$, and inductively, that $f^{2n}(a) \preccurlyeq f^{2n+2}(a)$ for each $n \in \mathbb{N} \cup \{0\}$, and some $a \in P$. Thus, $\{f^{2n}(a)|n \in \mathbb{N} \cup \{0\}\}$ is an increasing sequence in $S(x_n)$ so a least upper bound of $\{f^{2n}(a)|n \in \mathbb{N} \cup \{0\}\}$ exists. The system of chains C for which

$$\text{(SC)} \qquad x \in C \quad \text{implies} \quad f^2(x) \in C \quad \text{and} \quad x \preccurlyeq f^2(x)$$

contains the nonempty chain $\{f^{2n}(a)|n \in \mathbb{N} \cup \{0\}\}$, and therefore contains a maximal chain M by Zorn's lemma. By assumption $\mu = \sup M \in P$ exists, where $\sup M$ is a least upper bound of M. Since M satisfies (SC), we have $x \preccurlyeq f^2(x) \preccurlyeq f^2(\mu)$ for all $x \in M$, so that $\mu \preccurlyeq f^2(\mu)$.

On the other hand, if $\mu \notin M$, then the chain $M \cup \{f^{2n}(\mu) | n \in \mathbb{N} \cup \{0\}\}$ properly contains M, and satisfies (SC) in contradiction to the maximality of M. Therefore, $\mu \in M$ and also $f^2(\mu) \in M$, hence $f^2(\mu) \preccurlyeq \mu$. This makes μ a fixed point of f^2, i.e., $f^2(\mu) = \mu$, so that from the former remark, f has a fixed apex.

The following result allows us to prove the basic fixpoint theorem for chain complete posets (see: Abian-Brown, Amann, De Marr, Höft, Smithson).

Lemma 9. *Let P be a chain complete poset and f an increasing mapping from P into P such that $a \preccurlyeq f(a)$ for some $a \in P$. Then f has a fixed point.*

Our next statement extends Lemma 9 to conditionally increasing chain complete posets.

Lemma 10. (Locally Fixed Point Lemma). *Let P be a conditionally increasing chain complete poset and f an increasing mapping from P into P such that*

(LT) $$a \preccurlyeq f(a) \quad \textit{for some} \quad a \in P.$$

If there is a countable increasing sequence in P with the upper bound $a \in P$, then f has a fixed point.

A brief proof of this statement based on Zorn's lemma may be found at Tasković in 1993. The proof is analogous to the proof of Lemma 8.

Theorem 4. (Conditionally Chain Completeness). *Let P be a partially ordered set. If P is a conditionally increasing chain complete poset, then the following statements holds:*

(a) *Every increasing mapping f from P into P, such that the following sequence of the form*

(CS) $$a \preccurlyeq f(a) \preccurlyeq f^2(a) \preccurlyeq \cdots \preccurlyeq f^n(a) \preccurlyeq \cdots \quad \textit{for some} \quad a \in P,$$

has an upper bound, has a fixed point.

(b) *Every increasing mapping f from countable increasing inductive poset P into P satisfying the condition*

(LT) $$a \preccurlyeq f(a) \quad \textit{for some} \quad a \in P,$$

has a fixed point.

In connection with this statement, we note, that conditions (a) and (b), in Theorem 4, have the following equivalent forms:

(a') *Every increasing mapping f from P into P, such that the following sequence of the form*

$$a \preccurlyeq f^2(a) \preccurlyeq \cdots \preccurlyeq f^{2n}(a) \preccurlyeq \cdots \quad \textit{for some} \quad a \in P,$$

has an upper bound, has a fixed apex.

(b') *Every increasing mapping f from countable increasing inductive poset P into P satisfying the condition*

(LM) $$a \preccurlyeq f^2(a) \quad \textit{for some} \quad a \in P,$$

has a fixed apex.

Proof of Theorem 4. Evident, (b) is a consequence of (a). Further, we show that (a) holds. First, $\{f^n(a)|n \in \mathbb{N} \cup \{0\}\}$ is an increasing countable sequence in P, and with an upper bound denoted by m. Since P is a conditionally increasing chain complete poset we obtain that every increasing sequence in the set $S(f^n(a))$ has a least upper bound in P. The system of chains C for which

$$\text{(SC')} \qquad x \in C \quad \text{implies} \quad f(x) \in C \quad \text{and} \quad x \preccurlyeq f(x),$$

contains the nonempty chain $\{f^n(m)|n \in \mathbb{N} \cup \{0\}\}$ with supremum and therefore contains a maximal chain M by Zorn's lemma. Further, this is analogous to the proof of Lemma 8. Thus, f has a fixed point in P, i.e., (a) holds.

We notice, (b) is a consequence of (a), because the condition (LT) implies the condition (CS). The proof is complete.

As an immediate application of the our former results (equivalents of inductiveness, increasing inductiveness and quasi inductiveness) in 1992, we have that the following statements are equivalents: Zorn's lemma and Bourbaki's lemma.

We note, also in 1992, based on the above facts (equivalents of chain completeness and increasing chain completeness) and in connection with the some our former results we have the following statement.

Theorem 5. (Local forms, Tasković in 1992). *Let $(P, \preccurlyeq)$ be a partially ordered set. Then the following statements are equivalent:*

(a) *Zorn's lemma.*

(b) *Let P be an increasing chain complete poset, and f an increasing mapping from P into P such that*

$$\text{(LM)} \qquad a \preccurlyeq f^2(a) \quad \text{for some} \quad a \in P.$$

Then f has a fixed apex.

(c) *Let P be an increasing chain complete poset, and f an increasing mapping from P into P such that*

$$\text{(LT)} \qquad a \preccurlyeq f(a) \quad \text{for some} \quad a \in P.$$

Then f has a fixed point.

Proof. (a) implies (b). Let us consider the subset A of X given by $A := \{x \in X : x \preccurlyeq f^2(x)\}$. By the hypothesis we see that A is a nonempty poset. Since $x \preccurlyeq f^2(x)$ implies $f(x) \preccurlyeq f^2(f(x))$ we see that f maps A into A. Next, let C be a nonempty increasing ordered chain of A and ξ a least upper bound of C. Also, since $x \preccurlyeq \xi$ for every $x \in C$ it follows that $f^2(x) \preccurlyeq f^2(\xi)$, for every $x \in C$. However, since C is a subset of A, it follows that $x \preccurlyeq f^2(x) \preccurlyeq f^2(\xi)$ for every $x \in C$. So $f^2(\xi)$ is an upper bound of C, and consequently $\xi \preccurlyeq f^2(\xi)$. From the above it follows that A is a nonempty poset with the property that each nonempty increasing chain of A has an upper bound, i.e., A is an increasing inductive set, and f maps A into A and thus, by (FA) of Theorem 2 we see that f has a fixed apex as desired.

On the other hand, the proofs of (b) implies (c) and (c) implies (a), are analogous to the proofs of the preceding statements (FA) implies (FP) and (ZT) implies (ZL) of Theorem 2.

We note, a brief proof of this statement may be found at Tasković in 1988. The proof is analogous to the preceding proof.

6. Completeness of lattices

An order-preserving (isotone or increasing) map f of a partially ordered set P to itself has a fixed point if there is an element $p \in P$ such that $f(p) = p$. P is said to have the **fixed point property** if every isotone map of P into itself has a fixed point. The first of the fixed point theorems for partially ordered sets goes back to Tarski and Knaster in 1927, who proved that the lattice of all subsets of a set has the fixed point property. In the mid-1955's Tarski published a far-reaching generalization: *Every complete lattice has the fixed point property.*

Tarski raised the question whether the converse of this result also holds. Davis in a companion paper proved the converse: *Every lattice with the fixed point property is complete.*

In this section, we present some new characterizations of completeness of posets, i.e., lattices.

Let P be a partially ordered set and f a mapping from P into P. For any $f : P \to$P it is natural to consider the following set, which is dually with $\uparrow \overline{f(P)}$, i.e.,

$$\downarrow \underline{f(P)} := f(P) \cup \{a \in P | a = \inf C, \text{ for some decreasing sequence } C \text{ in } f(P)\},$$

where $\inf C$ is a greatest lower bound of C.

In connection with the preceding, analogous to the Lemmas 2 and 4 we have the following immediate consequences:

Lemma 11. *Let L be a complete lattice and f a mapping from L into itself such that*

$$\text{(SA)} \qquad x \preccurlyeq f^2(x) \quad \textit{for all} \quad x \in \uparrow \overline{f(L)}.$$

Then f has a fixed apex.

Lemma 12. *Let L be a complete lattice and f a mapping from L into itself such that*

$$\text{(SN)} \qquad x \preccurlyeq f(x) \quad \textit{for all} \quad x \in \uparrow \overline{f(L)}.$$

Then f has a fixed point.

Proofs. L is a complete lattice, and hence $\uparrow \overline{f(L)}$ is an inductive lattice, also. Applying the Lemmas 2 and 4 to the set $\uparrow \overline{f(L)}$, we obtain that f has a fixed apex or a fixed point.

On the other hand, we are now in a position to formulate our the following statements, which are dually of the preceding conditions.

Lemma 13. (Dually of (SA)). *Let L be a complete lattice and f a mapping from L into itself such that*

(DSA) $$f^2(x) \preccurlyeq x \quad \textit{for all} \quad x \in\downarrow \underline{f(L)}.$$

Then f has a fixed apex.

Lemma 14. (Dually of (SN)). *Let L be a complete lattice and f a mapping from L into itself such that*

(DSN) $$f(x) \preccurlyeq x \quad \textit{for all} \quad x \in\downarrow \underline{f(L)}.$$

Then f has a fixed point.

Proofs are analogous to the proof of the preceding Lemmas 11 and 12. With the help of Lemmas 11 and 13 we now obtain the main result of this section.

Theorem 6. *For a lattice L to be complete it is necessary and sufficient that every mapping f on L to L with the condition (SA) or (DSA) have a fixed apex.*

Proof. Since the condition of the theorem is known to be necessary for the completeness of a lattice, from Lemma 11 and Lemma 13 we have only to show that it is sufficient. In other words, we have to show that, under the assumption that the lattice L is not complete, there exists mapping f on L to L without fixed apices and with the condition (SA) or (DSA).

Suppose that the lattice L is not complete. We first notice that there exists at least one subset of L without a least upper bound (for otherwise the lattice would be complete). Hence we can find a chain A of L with the following properties: least upper bound, i.e., supremum of A, does not exist. From Lemma 7, let U be a chain cofinal with A such that

$$U := \{x \in A | x_0 \preccurlyeq x\}, \quad x_0 = \text{ a fixed element of } A = \min U.$$

Thus all the elements of U can be arranged in a sequence, i.e., one can show that there exists increasing sequence $\{x_\alpha\}$ in U such that: $\{x_\alpha\}$ is strictly increasing and for each $t \in U$, there exist $\alpha(t)$ such that $\alpha(t) < t$ implies $t \preccurlyeq x_\alpha$, and upper bound of $\{x_\alpha\}$ does not exist. We define a mapping f from L into itself according to the following presciption

$$f(x) = \begin{cases} x_\beta, \text{ if } x = x_\alpha \in U, \\ x_0 := \min U, \text{ if } x \notin U, \end{cases}$$

where $x_\alpha \preccurlyeq x_\beta$ $(x_\alpha \neq x_\beta)$ for $\alpha < \beta < \omega$, and where ω any (finite or transfinite) ordinal. Thus we have defined a function f on L to L. Now, for any $x \in U$ $(\supset\uparrow \overline{f(L)})$ either $x \preccurlyeq f^2(x)$, i.e., $x := x_\alpha \preccurlyeq x_\gamma = f(x_\beta) = f(f(x_\alpha)) = f^2(x_\alpha) = f^2(x)$ for $\alpha < \beta < \gamma < \omega$; so f satisfied the condition (SA), which does not have a fixed apex.

In the second case, we first notice that there exists at least one subset of L without a greatest lower bound (for otherwise the lattice would be complete). Hence we can find a chain B of L with the following properties: greatest lower bound, i.e., infimum of B, does not exist. From Lemma 7 let V be a chain cofinal with B such that

$$V := \big\{x \in B | x \preccurlyeq x_0\big\}, \quad x_0 = \text{ a fixed element of } B = \max V.$$

Thus all the elements of V can be arranged in a sequence, i.e., one can show that there exists decreasing sequence $\{x_\beta\}$ in V such that: $\{x_\beta\}$ is strictly decreasing and for each $t \in V$, there exists $\beta(t)$ such that $\beta(t) < t$ implies $x_\beta \preccurlyeq t$ and lower bound of $\{x_\beta\}$ does not exist. We define a mapping f from L into itself according to the following presciption

(2) $$f(x) = \begin{cases} x_\beta, \text{ if } x = x_\alpha \in V, \\ x_0 := \max V, \text{ if } x \notin V, \end{cases}$$

where $x_\beta \preccurlyeq x_\alpha (x_\alpha \neq x_\beta)$ for $\alpha < \beta < \omega$, and where ω any (finite or transfinite) ordinal. Thus we have defined a function f on L to L. Now, for some $x \in V$ $(\supset\downarrow \underline{f(L)})$ either $f^2(x) \preccurlyeq x$, i.e., $f^2(x) = f^2(x_\alpha) = f(x_\beta) = x_\gamma \preccurlyeq x_\alpha := x$ for $\alpha < \beta < \gamma < \omega$; so f satisfied the condition (DSA), which does not have a fixed apex.

Thus the function $f : L \to L$, is with the condition (SA) or (DSA) and does not have fixed apices. This completes the proof.

In connection with the preceding, L is said to have the ***d*-increasing fixed point property** if every map f of L into itself with the condition (SN) or (DSN) has a fixed point. Analogously, L is said to have the ***d*-increasing fixed apex property** if every map f of L into itself with the condition (SA) or (DSA) has a fixed apex.

We are now in a position to formulate the following statement, from the preceding facts.

Theorem 7. *Let L be a lattice, then the following statements are equivalent:*
(a) *L is a complete lattice.*
(b) *L has the d-increasing fixed point property.*
(c) *L has the d-increasing fixed apex property.*

Proof. From Lemmas 12 and 14, (b) is a consequence of (a). Lemmas 11 and 13 implies that (c) is a consequence of (a), i.e., (b). Thus, we need only show that (c) implies (a).

Suppose that the lattice L is not complete. Then there exists a sequence (chain) A in L, that does not have a least upper bound. We define a mapping f from L into itself with (2) from Lemma 7. Then f is well defined and for any $x \in U$ $(\supset\uparrow \overline{f(L)})$ we have $x \preccurlyeq f^2(x)$, i.e., $x := x_\alpha \preccurlyeq x_\gamma = f(x_\beta) = f(f(x_\alpha)) = f^2(x_\alpha) = f^2(x)$ for $\alpha < \beta < \omega$. Now, the condition (SA) holds, for $x \in\uparrow \overline{f(L)}$. Thus, f satisfies the condition (SA), and does not have a fixed apex.

Dually, when there exists a sequence (chain) B in L, that does not have a greatest lower bound, we define a mapping f from L into itself with (2) from Lemma 7. Then f is well defined and thus, the condition (DSA) holds for all $x \in\downarrow \underline{f(L)}$. Thus, f satisfies the condition (DSA), and does not have a fixed apex. This completes the proof of this preceding statement.

We are now in a position to formulate the following result, from the preceding statements.

Theorem 8. *Let L be a lattice. Then the following statements are equivalent:*
(a) *L is a complete lattice,*
(b) *L has the d-increasing fixed apex property,*
(c) *L has the d-increasing fixed point property,*
(d) (Tarski, Davis). *L has the fixed point property.*

7. CHAIN COMPLETE POSETS

Let P be a partially ordered set. In this section we consider a concept of upper increasing mappings of a poset into itself. A self mapping f of P into itself is called a **local upper increasing mapping** if $x \preccurlyeq f(x)$ implies $f(x) \preccurlyeq f^2(x)$ for all $x \in P$. Also, a self mapping f of P into itself is called a **local quasi upper increasing mapping** if $x \preccurlyeq f^2(x)$ implies $f(x) \preccurlyeq f^3(x)$ for all $x \in P$. Analogously, we define lower increasing and quasi lower increasing mappings of a poset P into itself.

We say that a mapping f of poset P into itself has a **sup-conditionally upper fork** if existing of supremum (denoted by η) of the set $P(\preccurlyeq f) := \{x \in P | x \preccurlyeq f(x)\}$ implies that $\eta \preccurlyeq f(\eta)$. On the other hand, a mapping f of poset P into itself has a **sup-conditionally upper befork** if existing of supremum (denoted by μ) of the set $P(\preccurlyeq f^2) := \{x \in P | x \preccurlyeq f^2(x)\}$ implies that $\mu \preccurlyeq f^2(\mu)$.

In order to prove characterizations of increasing chain completeness we need the following essential lemmas and some further results.

Lemma 15. *Let P be an increasing chain complete poset, and f a local quasi upper increasing mapping from P into P such that*

$$\text{(LM)} \qquad a \preccurlyeq f^2(a) \quad \text{for some} \quad a \in P.$$

If map f has the sup-conditionally upper befork, then f has a fixed apex.

Proof. Because $a \preccurlyeq f^2(a)$ and f is local quasi upper isotone, we find $f^2(a) \preccurlyeq f^4(a)$, and inductively, that $f^{2n}(a) \preccurlyeq f^{2n+2}(a)$ for each $n \in \mathbb{N} \cup \{0\}$, and some $a \in P$. Thus, $\{f^{2n}(a) | n \in \mathbb{N} \cup \{0\}\}$ is an increasing sequence in P so a least upper bound of $\{f^{2n}(a) | n \in \mathbb{N} \cup \{0\}\}$ exists. The system of chains C for which

$$\text{(SC)} \qquad x \in C \quad \text{implies} \quad f^2(x) \in C \quad \text{and} \quad x \preccurlyeq f^2(x)$$

contains the nonempty chain $\{f^{2n}(a) | n \in \mathbb{N} \cup \{0\}\}$, and therefore contains a maximal chain M by Zorn's lemma. By assumption $\mu = \sup M \in P$ exists, where $\sup M$ is a least upper bound of M.

Since M satisfies (SC), we have $x \preccurlyeq f^2(x)$ for all $x \in M$. Since f has the sup-conditionally upper befork we have $\mu \preccurlyeq f^2(\mu)$, so that $\{f^{2n}(\mu) | n \in \mathbb{N} \cup \{0\}\}$ is an increasing sequence. Thus $x \preccurlyeq \mu \preccurlyeq f^2(\mu) \preccurlyeq \cdots \preccurlyeq f^{2n}(\mu) \preccurlyeq \cdots$ for all $x \in M$.

On the other hand, if $\mu \notin M$, then the chain $M \cup \{f^{2n}(\mu) | n \in \mathbb{N} \cup \{0\}\}$ properly contains M, and satisfies (SC) in contradiction to the maximality of M. Therefore, $\mu \in M$ and also $f^2(\mu) \in M$, hence $f^2(\mu) \preccurlyeq \mu$. This makes μ a fixed point of f^2, i.e., $f^2(\mu) = \mu$, so from the preceding remark, f has a fixed apex.

With the help of Lemma 15, in this part, we present a characterization of increasing chain completeness of posets in terms of fixed apices.

Theorem 9. *A poset P is increasing chain complete if and only if every local quasi upper increasing mapping f on P to P with the sup-conditionally upper befork that satisfies (LM) has a fixed apex.*

Proof. Since by Lemma 15 the condition of this statement is known to be necessary for the increasing chain completeness of a poset, we have only to show that it is sufficient. Suppose that the poset P is not increasing chain complete.

We first notice that there exists an increasing sequence $\{x_\alpha\} := C$ in P that does not have a least upper bound. From Lemma 7, let U be a cofinal chain in C such that $U := \{x \in C | x_0 \preccurlyeq x\}$ where x_0 is a fixed element in C and $x_0 \in U$. Thus all the elements of U can be arranged in a sequence, i.e., one can show that there exists an increasing sequence $\{x_\alpha\}$ in U such that it does not possess an upper bound, and $\{x_\alpha\}$ is strictly increasing, and for each $t \in U$, there exists $\alpha(t)$ such that $\alpha(t) < t$ implies $t \preccurlyeq x_\alpha$. We define a mapping f from P into itself by

$$f(x) = \begin{cases} x_\beta, \text{ if } x = x_\alpha \in U, \\ \min U := x_0, \quad x \notin U, \end{cases}$$

where $x_\alpha \preccurlyeq x_\beta (x_\alpha \neq x_\beta)$ for $\alpha < \beta < \omega$, and where ω is any (finite or transfinite) ordinal. Thus, we have defined a local quasi upper increasing function f on P to P. Now, for some $a \in U \subset P$ we have $a \preccurlyeq f^2(a)$, i.e., $a = x_\alpha \preccurlyeq x_\gamma = f(x_\beta) = f(f(x_\alpha)) = f^2(a)$, for $\alpha < \beta < \gamma < \omega$; so f satisfies (LM) and does not have a fixed apex. Also, f is with the sup-conditionally upper befork. This completes the proof.

Our next statement extends Lemma 9 to increasing chain complete posets for local upper increasing mappings.

Lemma 16. *Let P be an increasing chain complete poset and f a local upper increasing mapping from P into P such that*

(LT) $$a \preccurlyeq f(a) \quad \textit{for some} \quad a \in P.$$

If map f has the sup-conditionally upper fork, then f has a fixed point.

A brief proof of this statement based on Zorn's lemma may be found at Tasković in 1993. The proof is analogous to the proof of Lemma 15.

In connection with the preceding, P is said to have the **local fixed point property** if every local upper increasing map f of P into itself with the sup-conditionally upper fork satisfying condition (LT) has a fixed point. Analogously, P is said to have the **local fixed apex property** if every local quasi upper increasing map f of P into itself with the sup-conditionally upper befork satisfying condition (LM) has a fixed apex.

We are now in a position to formulate our main general statement of this part.

Theorem 10. *Let P be a partially ordered set. Then the following statements are equivalent:*

(a) *P is a chain complete poset,*

(b) *P is an increasing chain complete poset,*

(c) *P has the local fixed point property,*

(d) *P has the local fixed apex property.*

Proof. From Theorem 8, (b) is equivalent to (d). Lemma 16 implies that (c) is a consequence of (b). Also, (b) is a consequence of (a), and (d) is equivalent to (c). Thus, we need only show that (d) implies (a).

Suppose that the poset P is not chain complete, then there exists a chain C in P that does not have a least upper bound. From Lemma 7, let U be a chain cofinal with C without upper bound. We define a local quasi upper increasing mapping f from P into itself by (1). Then f is well defined and for some $a \in U(\subset P)$ we have $a \preccurlyeq f^2(a)$, i.e., $a := x_\alpha \preccurlyeq x_\gamma = f(x_\beta) = f(f(x_\alpha)) = f^2(a)$, for $\alpha < \beta < \omega$. Thus, condition (LM) holds, and f does not have a fixed apex. Also, f is with the sup-conditionally upper befork. This completes the proof.

8. FURTHER ON AXIOM OF CHOICE

In this part we prove some new variants of the Axiom of Choice. These statements are facts of fixed point and fixed apex type. We are now in a position to formulate our the following statement.

Theorem 11. (Axiom of Choice). *Let P be a partially ordered set, with an ordered relation $\preccurlyeq$. Then the following statements are equivalent:*

(ZL) (Zorn's lemma). *Let P be an inductive partially ordered set. Then P has a maximal element.*

(FA) *Let P be an increasing chain complete poset, and f a local quasi upper increasing mapping from P into P such that*

$$\text{(LM)} \qquad a \preccurlyeq f^2(a) \quad \textit{for some} \quad a \in P.$$

If map f has the sup-conditionally upper befork, then f has a fixed apex.

(FP) *Let P be an increasing chain complete poset, and f a local upper increasing mapping from P into P such that*

$$\text{(LT)} \qquad a \preccurlyeq f(a) \quad \textit{for some} \quad a \in P.$$

If map f has the sup-conditionally upper fork, then f has a fixed point.

Proof. From Lemmas 15 and 16, (FA) and (FP) are consequences of (ZL). Thus, we need only show that (FA), as and (FP), implies (ZL).

Suppose that the result (ZL) is false. Then for each $x \in P$ there exists $y \in P$ with $x \preccurlyeq y$ and $x \neq y$. Let $\mathcal{T}_0$ be the family of all nonempty chains of P and let $\mathcal{T} = \mathcal{T}_0 \cup \{\varnothing\}$. The familly $\mathcal{T}$ is partially ordered by the inclusion relation $\subset$ between subset of P. For each $A \in \mathcal{T}_0$ the set

$$U_A = \big\{x \in P : x \quad \text{is an upper bound for } A \text{ and } x \notin A\big\}$$

is nonempty because, if x an upper bound for A and $y \in P$ is such that $x \preccurlyeq y$ and $x \neq y$, then $y \in U_A$. Let $U_\varnothing = \{x_0\}$, where x_0 is an arbitrary element of P. Let g be a mapping with domain $X := \{U_A : A \in \mathcal{T}\}$, and now, we define a mapping g from X into itself by $g(x) = x$, i.e., g is the identity mapping. For each $A \in \mathcal{T}$ let $f(A) = A \cup \{g(U_A)\}$. It is now clear that $f(A) \in \mathcal{T}$ for all $A \in \mathcal{T}$ and hence f maps $\mathcal{T}$ into itself.

We shall prove that $\mathcal{T}$, partially ordered by inclusion, and f satisfy the conditions of (FA) and (FP). Let $\mathcal{R}$ be a nonempty subfamily of $\mathcal{T}$ such that $\mathcal{R}$ is chain ordered by inclusion and let $A = \cup_{B \in \mathcal{R}} B$. Let $a, b \in A$. There are sets $C, D \in \mathcal{R}$

with $a \in C$ and $b \in D$. Since $\mathcal{R}$ is a chain ordered by inclusion either $C \subset D$ or $D \subset C$ and in either case we see that there is one set in $\mathcal{R}$ which contains both a and b. Since each set in $\mathcal{R}$ is a chain ordered subset of P it follows that either $a \preccurlyeq b$ or $b \preccurlyeq a$. This proves that $A \in \mathcal{T}$ and it is then easy to see that $A = \sup \mathcal{R}$. Thus $\mathcal{T}$ satisfies the condition of increasing chain completeness of (FA) and (FP). By definition of f we have $A \subset f(A)$. Thus conditions (LM) and (LT) are satisfied. Also, it follows immediately that conditions that f has a sup-conditionally upper fork and befork are satisfied.

Since $A \subset f(A)$ for all $A \in \mathcal{T}$, f is a local upper and a local quasi upper increasing mapping. We can now conclude from (FP) or from (FA) that there is a set $A_0 \in \mathcal{T}$ with $f(A_0) = A_0$ or with $f^2(A_0) = A_0$. Thus we have contradiction. The proof is now complete.

Annotation. We notice, in connection with this statements that in 1975 and in 1978 we first time considered and proved the following result.

Theorem 12. (Tasković in 1975, 1978 and in 1986 from Math. Japonica). *Let P be a partially ordered set with an ordered relation $\preccurlyeq$. Then the following statements are equivalent:*

(ZL) (Zorn's lemma). *Let P be an inductive poset. Then P has a maximal element.*

(TM) (Tasković in 1975 and 1978). *Let $\mathcal{F}$ be a family of mappings of a partially ordered set P into itself such that*

$$x \preccurlyeq f(x) \quad (f(x) \preccurlyeq x) \quad \text{for every } f \in \mathcal{F}$$

and for every $x \in P$. If each chain in P has an upper bound (lower bound), then the family $\mathcal{F}$ has a common fixed point for all $f \in \mathcal{F}$.

(ZT) (Zermelo in 1904). *Let P be a chain complete poset an f a mapping from P into itself such that:*

(a) *there is an element $\theta \in P$ with $\theta \preccurlyeq x$ for every $x \in P$,*

(b) *$x \preccurlyeq f(x)$ for every $x \in X$,*

(c) *if $x, y \in P$ and $x \preccurlyeq y \preccurlyeq f(x)$ then either $x = y$ or $f(x) \preccurlyeq f(y)$.*

Then there is an element $\zeta \in P$ such that $f(\zeta) = \zeta$.

A brief proof of this statement may be found in 1986 from: Tasković (Math. Japonica, **31** (1986), 979-991).

The case (TM) of this statement indipendently and different is proved in 1976 from S. Kasahara.

But, we notice that the case (ZT) of Zermelo is given in 1904 (and in 1908) without proof.

First time the explicit proof that (ZT) is an equivalent of Zorn's lemma (i.e., the Axiom of Choice) is given in 1978 from: Tasković (*Banach's mappings of fixed point on spaces and ordered sets*, These, University of Beograd, 1978, p.p. 151; and Math. Balkanica, **9** (1979), p.p. 130.).

We notice that in the brief proof of Theorem 12 (the case that (ZT) implies Zorn's lemma) in 1978 there exists a typographycal error (that is $f(U_A) \in U_A$ but need be $f(U_A) = U_A$)!

On the other hand, in 1988 and in 1992 the following result whose applications in nonlinear functional analysis and fixed point theory are considered, see: Tasković (*The axiom of choice, fixed point theorems, and inductive ordered sets*, Proc. Amer. Math. Soc., **116** (1992), 897-904) and P. Howard - J.E. Rubin (*Consequences of the Axiom of Choice*, Math. Surveys and Monographs, Vol. **59**, 1998, Amer. Math. Soc. Providence, Rhode Island, USA, 432 pages.).

Theorem 13. (Tasković in 1988 and in 1992). *Let P be a partially ordered set with an ordered relation $\preccurlyeq$. Then the following statements are equivalent:*

(a) (Zorn's lemma). *Let P be an inductive poset. Then P has a maximal element.*

(b) (Fixed Apices Lemma). *Let P be an inductive (chain-complete) poset, and f a mapping from P into P such that*

$$x \preccurlyeq f^2(x) \quad \textit{for all} \quad x \in \operatorname{Sub} f(P),$$

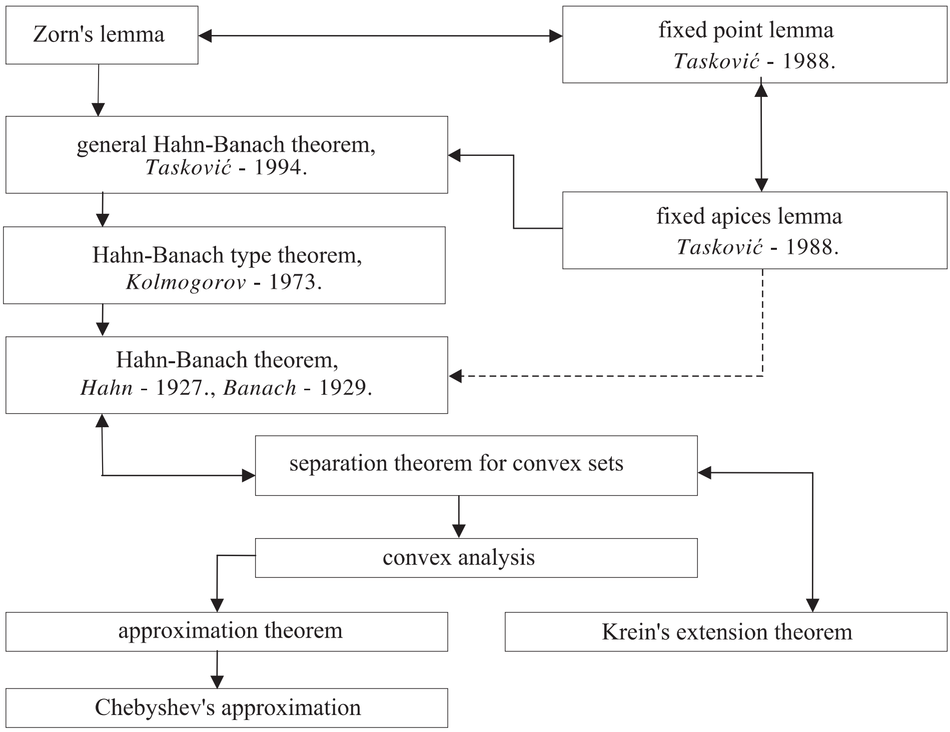
then f has a fixed apex.

(c) (Fixed Point Lemma). *Let P be an inductive (chain-complete) poset, and f a mapping from P into P such that*

$$x \preccurlyeq f(x) \quad \textit{for all} \quad x \in \operatorname{Sub} f(P),$$

then f has a fixed point.

In 1992 consideration is also given to the existence of simultaneous fixed points and fixed apexes of families of mappings. Applications in nonlinear functional analysis in 1992 are considered.



In connection with this, let (X, ρ) be a metric space and $G : X \to \mathbb{R}^0_+ := [0, +\infty)$ be a given function. Define a relation $\boxed{\preccurlyeq}$ on X as **Brøndsted ordering** by the following condition:

(B) $\quad a \boxed{\preccurlyeq} b \quad$ if and only if $\quad \rho[a, b] \leq G(a) - G(b)$ for all $a, b \in X$.

On the other hand, define a relation $\preccurlyeq_{G,\rho}$ on the metric space X as **Tasković ordering** by the following condition:

$$\text{(Ta)} \qquad a \preccurlyeq_{G,\rho} b \quad \text{if and only if} \quad \rho[a,b] \leq G(b) - G(a)$$

for all $a, b, \in X$.

We notice that the ordering (B) is not dually, in comparable, with the ordering (Ta)!

In further, as immediate applications of the preceding Theorem 13 we have the following results.

In this sense as an immediate consequence of Fixed Apices Lemma we obtain the following result.

Theorem 14. (Analytic Principles of Choice, Tasković in 1988). *Let P be a partially ordered set with an ordered $\preccurlyeq$. Then the following statements are equivalent:*

(a) (Fixed Apices Lemma, Tasković in 1988). *If P is an inductive set and $f : P \to P$ such that*

$$x \preccurlyeq f^2(x) \quad \textit{for all} \quad x \in \operatorname{Sub} f(P),$$

then f has a fixed apex.

(b) (Tasković in 1993). *If (X, ρ) is a complete metric space and $G : X \to \mathbb{R}^0_+$ is a lower semicontinuous function, then in the Brøndsted ordering every $f : X \to X$ satisfying $x \boxed{\preccurlyeq} f^2(x)$ for every $x \in X$ has a fixed apex.*

(c) (Tasković in 1993). *If (X, ρ) is a complete metric space and $G : X \to \mathbb{R}$ is a bounded above and upper semicontinuous function, then in the ordering $\preccurlyeq_{G,\rho}$ every $f : X \to X$ satisfying $x \preccurlyeq_{G,\rho} f^2(x)$ for every $x \in X$ has a fixed apex.*

Short proof. Define relations $\boxed{\preccurlyeq}$ and $\preccurlyeq_{G,\rho}$ on X by (B) and (Ta), and applying Fixed Apices Lemma in two direction, directly, we obtain this statement.

On the other hand, in the preceding sense, as an immediate consequence of Fixed Point Lemma we obtain the following result.

Theorem 15. (Analytic forms of Axiom of Choice, Tasković in 1988). *Let P be a partially ordered set with an ordered $\preccurlyeq$. Then the following statements are equivalent:*

(d) (Fixed Point Lemma, Tasković in 1988). *If P is an inductive set and $f : P \to P$ such that*

$$x \preccurlyeq f(x) \quad \textit{for all} \quad x \in \operatorname{Sub} f(P),$$

then f has a fixed point.

(e) (Caristi in 1976, Mańka in 1988). *If (X, ρ) is a complete metric space and $G : X \to \mathbb{R}^0_+$ is a lower semicontinuous function, then in the Brøndsted ordering every $f : X \to X$ satisfying $x \boxed{\preccurlyeq} f(x)$ for every $x \in X$ has a fixed point.*

(h) (Tasković in 1986 and in 1988). *If (X, ρ) is a complete metric space and $G : X \to \mathbb{R}$ is a bounded above and upper semicontinuous function, then in the ordering $\preccurlyeq_{G,\rho}$ every $f : X \to X$ satisfying $x \preccurlyeq_{G,\rho} f(x)$ for every $x \in X$ has a fixed point.*

Short proof. Define relations $\boxed{\preccurlyeq}$ and $\preccurlyeq_{G,\rho}$ on X by (B) and (Ta), and applying Fixed Point Lemma in two direction, we obtain immediately this statement.

Annotation. In connection with the preceding facts see and papers: Kirk in 1976, H. Höft and P. Howard in 1994, Abian in 1985, Turinici in 1984, Brunner in 1987, Makowski and Wiśniewski in 1969, Baker in 1964, Kenyon in 1963, Dj. Kurepa in 1952, and Brøndsted in 1979.

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