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NEW RECURRENT FORMULAE OF $P(n)$ AND $\tau(n)$ FUNCTIONS

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Abstract. In this paper a new recurrent formulae of partition function $P(n)$ and Ramanujan's tau function $\tau(n)$ are given.

1. Introduction

The Partition function $P(n)$ (sequence A000041 in [3]) and Ramanujan's $\tau(n)$ function (sequence A000594 in [3]) are defined by the generating functions:

$$\sum_{n=0}^{\infty} P(n)x^n = \prod_{n=1}^{\infty}(1-x^n)^{-1} = \left(\frac{2x^{\frac{1}{8}}}{\theta_1'(0,\sqrt{x})}\right)^{\frac{1}{3}}, \tag{1}$$

where $\theta_1'(0,\sqrt{x})$ is the derivative of the Jacobi theta function of the first kind given by (see [5])$\theta_1(z,x) = 2x^{\frac{1}{4}} \sum\limits_{n=0}^{\infty}(-1)^n x^{n(n+1)} \sin[(2n+1)z]$, and

$$\sum_{n=1}^{\infty} \tau(n)x^n = x\prod_{n=1}^{\infty}(1-x^n)^{24} = x(1-3x+5x^3-7x^6+\cdots)^8. \tag{2}$$

$P(n)$ satisfies the following recurrence formula

$$P(n) = \frac{1}{n}\sum_{k=0}^{n-1}\sigma(n-k)P(k), \tag{3}$$

where $\sigma(n)$ is the divisor function defined by

$$\sigma(n) = \sum_{d|n} d. \tag{4}$$

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Ramanujan gave the recurrence formula as follows

$$(5)\quad \begin{aligned}(n-1)\tau(n) = &\sum_{m=1}^{s_n}(-1)^{m+1}(2m+1)\\ &\times\left[n-1-\frac{9}{2}m(m+1)\right]\tau\left(n-\frac{1}{2}m(m+1)\right)\end{aligned}$$

where

$$(6)\qquad s_n = \left[\frac{1}{2}(\sqrt{8n+1}-1)\right]$$

and $[x]$ denotes the integer part of x.

In this paper the recurrent formulae for $\tau(n)$ will be given, using integer sequences $\{a_n(k)\}$ and $\{b_n(k)\}$. Also, the analogous formulae for (3) and (5) will be given (formulae (11) and (10) respectively).

2. Statement of results and proof

Definicija 2.1. *For $n \in \mathbb{N}$ the integer sequence $\{b_n(k)\}$ is defined by*

$$\begin{aligned}b_n(0) &= \sum_{r=0}^{n-1}\left[\sum_{k=0}^{r}\omega(k)\omega(r-k)\right]\cdot\left[\sum_{k=0}^{n-1-r}\omega(k)\omega(n-1-r-k)\right],\\ b_n(k) &= b_{n-k}(0)b_{k+1}(0) \qquad 1\le k<n,\\ b_n(k) &= 0 \qquad k \ge n,\end{aligned}$$

where the sequence $\omega(k)$ is given by the formula

$$(7)\qquad \omega(k) = \begin{cases}(-1)^{\frac{1}{2}(\sqrt{8k+1}-1)}\sqrt{8k+1}, & k=\dfrac{m(m+1)}{2};\quad m\in\mathbb{N}_0\\ 0, & k\neq\dfrac{m(m+1)}{2};\quad m\in\mathbb{N}_0.\end{cases}$$

Using Definition 1. we have

$$b_n(k) = b_n(n-1-k) \qquad (k<n),$$

$$b_n(k) = \frac{b_{n-k}(0)}{b_{n-k-r}(0)}\, b_{n-r}(k) \qquad (r-1<k<n-r).$$

Lemma 1. *For $n\in\mathbb{N}_0$ we have*

$$\tau(n+1) = \sum_{k=0}^{n} b_{n+1}(k)$$

Table 1 The numbers $b_n(k)$ for $n = 1, 2, \ldots, 7$ and $k = 0, 1, \ldots, 6$

$n \setminus k$	0	1	2	3	4	5	6
1	1	0	0	0	0	0	0
2	−12	−12	0	0	0	0	0
3	54	144	54	0	0	0	0
4	−88	−648	−648	−88	0	0	0
5	−99	1056	2916	1056	−99	0	0
6	540	1188	−4752	−4752	1188	540	0
7	−418	−6480	−5346	7744	−5346	−6480	−418

Proof. Denote the polynomial by

$$Q_k(x) = \sum_{n=0}^{k-1} (-1)^n (2n+1) x^{\frac{n(n+1)}{2}}. \tag{8}$$

Applying the relations (2), (7) and (8) we have

$$\sum_{n=1}^{\infty} \tau(n) x^n = x[Q_\infty(x)]^8 = x\left[\sum_{n=0}^{\infty} \omega(n) x^n\right]^8. \tag{9}$$

Using the Cauchy multiplication of power series and because

$$\limsup_{n\to\infty} \mid \sqrt[n]{\omega(n)} \mid = 1$$

we have for $\mid x \mid < 1$

$$\sum_{n=1}^{\infty} \tau(n) x^n = x\left[\left[\left[\sum_{n=0}^{\infty} \omega(n) x^n\right]^2\right]^2\right]^2 =$$

$$= \sum_{n=0}^{\infty} \sum_{s=0}^{n} \left[\sum_{r=0}^{s} \left(\sum_{k=0}^{r} \omega(k)\omega(r-k)\right)\left(\sum_{k=0}^{s-r} \omega(k)\omega(s-r-k)\right)\right]$$
$$\times \left[\sum_{r=0}^{n-s} \left(\sum_{k=0}^{r} \omega(k)\omega(r-k)\right)\left(\sum_{k=0}^{n-s-r} \omega(k)\omega(n-s-r-k)\right)\right] x^{n+1}$$

Hence]

$$\tau(n+1)=$$
$$=\sum_{s=0}^{n}\left[\sum_{r=0}^{s}\left(\sum_{k=0}^{r}\omega(k)\omega(r-k)\right)\left(\sum_{k=0}^{s-r}\omega(k)\omega(s-r-k)\right)\right]$$
$$\times\left[\sum_{r=0}^{n-s}\left(\sum_{k=0}^{r}\omega(k)\omega(r-k)\right)\left(\sum_{k=0}^{n-s-r}\omega(k)\omega(n-s-r-k)\right)\right].$$

Because

$$b_{s+1}(0)=\sum_{r=0}^{s}\left(\sum_{k=0}^{r}\omega(k)\omega(r-k)\right)\left(\sum_{k=0}^{s-r}\omega(k)\omega(s-r-k)\right)$$

and

$$b_{n+1-s}(0)=\sum_{r=0}^{n-s}\left(\sum_{k=0}^{r}\omega(k)\omega(r-k)\right)\left(\sum_{k=0}^{n-s-r}\omega(k)\omega(n-s-r-k)\right)$$

we have $\tau(n+1)=\sum\limits_{k=0}^{n}b_{n+1}(k)$.

The following statements are similarly shown:

Lemma 2.

$$P(n)=-\frac{1}{24n}\sum_{k=0}^{n-1}(n+23k)\tau(n+1-k)P(k)$$

$$\tau(n)=-\frac{1}{n-1}\sum_{k=1}^{n-1}(24n-23k-1)P(n-k)\tau(k) \tag{10}$$

$$P(n)=\frac{1}{n}\sum_{m=0}^{s_n}(-1)^{m+1}(2m+1)[n-\frac{1}{3}m(m+1)]P(n-\frac{1}{2}m(m+1))$$

where s_n is defined by (6) .

Lemma 3. *For $n\in\mathbb{N}$ we have*

$$\begin{aligned}\tau(n)&=-\frac{24}{n-1}\sum_{k=1}^{n-1}\sigma(n-k)\tau(k),\\ \sigma(n)&=n-\sum_{\substack{d|n\\ d\neq n}}\mu\left(\frac{n}{d}\right)\sigma(d),\end{aligned}\tag{11}$$

where $\sigma(n)$ is defined by (4) and $\mu(n)$ is the Möbius function.

Proof. If $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$ are integer sequences and are related by (see [4])

$$1 + \sum_{n=1}^{\infty} \beta_n x^n = \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^{\alpha_n}}$$

then $\{\beta_n\}$ is said to be the Euler transform of $\{\alpha_n\}$:

$$\beta_1 = \gamma_1, \qquad \beta_n = \frac{1}{n}\left[\gamma_n + \sum_{k=1}^{n-1} \gamma_k \beta_{n-k}\right],$$

where $\gamma_n = \sum_{d|n} d\alpha_d$. The inverse transform gives $\alpha_n = \dfrac{1}{n} \sum_{d|n} \mu\left(\dfrac{n}{d}\right) \gamma_d$. For $\alpha_n = -24$, $\beta_n = \tau(n+1)$ and $\gamma_n = -24\sigma(n)$ the result of lemma is obtained.

Since $x \prod\limits_{k=1}^{n-1} (x^k)^{24} = x^{12n(n-1)+1}$, the following definition is reasonable:

Definicija 2.2. *For $n \in \mathbb{N}$ the integer sequence $\{a_n(k)\}_{k=0}^{\infty}$ is defined by*

$$x \prod_{k=1}^{n-1} (1-x^k)^{24} = \sum_{k=1}^{12n(n-1)+1} a_n(k) x^k, \qquad 0 < k \le 12n(n-1)+1$$

$$a_n(k) = 0, \qquad \textit{otherwise.}$$

Table 2 The numbers $a_n(k)$ for $n = 1, 2, \ldots, 6$ and $k = 1, 2, \ldots, 6$

$n \setminus k$	1	2	3	4	5	6
1	**1**	**0**	0	0	0	0
2	1	−24	**276**	**−2024**	10626	−42504
3	1	−24	252	−1448	**4278**	**−552**
4	1	−24	252	−1472	4854	−6600
5	1	−24	252	−1472	4830	−6024
6	1	−24	252	−1472	4830	−6048

Lemma 4. *For $n \in \mathbb{N}$ we have*

$$\tau(n) = a_{\left[\frac{n+1}{2}\right]}(n) - 24 \sum_{k=1}^{[n/2]} \tau(k).$$

Proof. Firstly, applying the relation (2) and Definition 2. we have

$$\tau(k) = a_n(k), \qquad (k \le n). \tag{12}$$

Secondly, denote the polynomial by $R_n(x) = x\prod_{k=1}^{n-1}(1-x^k)^{24}$. Then

$$\begin{aligned} R_n(x) &= (1-x^{n-1})^{24}R_{n-1}(x) \\ &= \sum_{j=0}^{12n-1}\sum_{i=1}^{n-1}\sum_{k=0}^{j}(-1)^k\binom{24}{k}a_{n-1}((j-k)(n-1)+i)x^{j(n-1)+i} \\ &+ a_{n-1}(12(n-1)(n-2)+1)x^{12n(n-1)+1} \end{aligned}$$

Hence, for $(i = 1, 2, \ldots, n-1;\ j = 0, 1, \ldots, 12n-1)$ we have:

$$a_n(j(n-1)+i) = \sum_{k=0}^{j}(-1)^k\binom{24}{k}a_{n-1}((j-k)(n-1)+i), \tag{13}$$

and

$$a_n(12n(n-1)+1) = a_{n-1}(12(n-1)(n-2)+1) = 1. \tag{14}$$

Finally, applying Definition 2. and the relations (12), (13) and (14), for $n \in \mathbb{N}$, $t \in \mathbb{N}_0$, and $A = \left[\frac{t-1}{n-1}\right]$, we have

$$a_n(t) = \begin{cases} \sum_{k=0}^{A}(-1)^k\binom{24}{k}a_{n-1}(t-k(n-1)), & 1 \le t < 12n(n-1)+1 \\ 1, & t = 12n(n-1)+1, \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

By the recursion (15) we have for $t = n$

$$a_n(n) = a_{\left[\frac{n+1}{2}\right]}(n) - \binom{24}{1}\sum_{k=1}^{[n/2]}a_{n-k}(k).$$

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