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SOME TYPES OF RELATIVE PARACOMPACTNESS

Vladimir Pavlović*

Abstract. This paper is a continuation of the study of relative topological properties. We use a characterization of paracompactness via a certain selection principle to introduce five types of relative paracompactness, provide examples showing that none of them coincide with each other and establish some results concerning finite unions of subspaces which are relatively paracompact in one or another of the defined senses.

1. Introduction

If X is a topological space and Y a subspace of X, then the properties of Y in general will depend on the way in which that subspace is "*placed*" in X. Conversely, it is not rarely the case that having a subspace of a certain type placed in a particular way can largely effect the properties of the whole space. This suggests that to each topological property $\mathcal{P}$ a kind of its "*relative*" version can be assigned, now viewed on the family of all subspaces of a space X, in attempt to grasp one aspect of the fact how a certain subspace can be placed in X. In that sense, we talk about "*a subspace Y being $\mathcal{P}$-placed in X*", or about "*Y relatively having the property $\mathcal{P}$ in X*". Thus, we say that Y is *relatively Haussdorff* in X (see [3]) if for each pair of distinct points $x, y \in Y$ there exist two disjoint open in X sets U, V such that $x \in U$ and $y \in V$; according to Ju. Smirnov (see [5]) a subspace Y of X is *normally placed in X* if for each open in X set U containing Y there is a F_σ-set $Z \subseteq X$ such that $U \subseteq Z \subseteq X$; S. Mrówka (see [5]) calls Y *regularly placed in X* if for each point $x \in X \setminus Y$ there is a F_σ-set $Z \subseteq X$ such that $Y \subseteq Z \subseteq X \setminus \{x\}$ etc.

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Relative properties have already been considered by many authors: relative compactness, countable compactness, relative dimensions, relative extremal disconnectedness, relative G_δ-diagonal, relative cardinal invariants in C_p theory and so on. Let us only mention the following result ([5], pages 305 and 306): A Tychonoff space X is Lindelöf (realcompact) if and only if X is normally (regularly) placed in βX. As opposed to the relative ones the classical topological properties will be referred to as *absolute* properties. Let us mention here that a systematic study of relative topological properties was started by A.V. Arhangel'skiĭ in [3] and continued later in a series of its papers (see for example [1], [2]).

In this paper we will be concerned with some relative versions of paracompactness, but in distinction from A.V. Arhangel'skiĭ who has already considered this relative topological property defining it starting from the usual definition of paracompactness, we take another characterization of paracompactness as a base for deriving five relative types of it (actually four because it turns out that one of the defined versions coincides with the absolute paracompactness of the subspace under consideration).

Let us first establish some terminology and notation. For any sets x, y the symbol $x \preceq y$ means $\forall z \in x\ \exists t \in y\ (z \subseteq t)$. $\mathcal{P}(X)$ denotes the set of all subsets of x. If g is a function we use the symbol $g_{\rightarrow}A$ ($g^{\leftarrow}A$) to denote the (inverse) image of A under g and write $ran(g)$ for the range of g. When we say that x is *point finite* (*point countable*) *on* y we mean that for each $b \in y$ the set $\{a \in x : b \in a\}$ is finite (countable). When a set X is looked at as a topological space the corresponding topology will be denoted by $\mathcal{T}_X$. If X is a space then $\mathcal{O}_X$ or just $\mathcal{O}$ (when there is no confusion to which X the notation refers) stands for the family of all open covers of X.

Definicija 1.1. ([4]) *Let a space X, a subspace Y of X and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\mathcal{P}(X))$ be given. The notation $S(\mathcal{A}, \mathcal{B}; X, Y)_{lf}$ ($S(\mathcal{A}, \mathcal{B}; X, Y)_{pf}$) stands for the following statement:*

For each sequence $\langle \mathcal{U}_n : n < \infty \rangle$ of elements of $\mathcal{A}$ there is a sequence $\langle \mathcal{V}_n : n < \infty \rangle$ of subsets of $\mathcal{P}(X)$ such that

- *for each n $\mathcal{V}_n \preceq \mathcal{U}_n$, and the family $\mathcal{V}_n$ is locally finite on Y with respect to the topology of X (point finite on Y),*
- $\bigcup\{\mathcal{V}_n : n < \infty\} \in \mathcal{B}$.

Also we introduce the abbreviation $S(\mathcal{A}, \mathcal{B})_{lf} \overset{\text{def}}{=} S(\mathcal{A}, \mathcal{B}; X, X)_{lf}$, and similarly for $S(\mathcal{A}, \mathcal{B})_{pf}$.

The notion we have just defined, for a convenient choice of the families $\mathcal{A}$ and $\mathcal{B}$, coincides exactly with the notion of paracompactness. More precisely, the following theorem holds (see [4]).

Theorem 1. *A regular space X is paracompact if and only if $S(\mathcal{O},\mathcal{O})_{lf}$ holds.*

Thus, as we are interested in relative paracompactness we will primarily be concerned with the S_{lf} principle. The main reason we consider the second one (S_{pf}) is that most of the theorems in the paper are valid for it too, with almost identical (or even easier) proofs.

Now we can give an alternative (to that of Arhangel'skiĭ) definition of relative paracompactness. We simply relativize the principle $S(\mathcal{O},\mathcal{O})_{lf}$.

Definicija 1.2. *Let a space X and its subspace Y be given. Put $\mathcal{O}_X(Y) = \{\mathcal{B} \subseteq \mathcal{T}_X : Y \subseteq \bigcup \mathcal{B}\}$ and denote with $i-(Y|X)_{lf}$ ($i-(Y|X)_{pf}$) the fact $S(\mathcal{A},\mathcal{B};X,Y)_{lf}$ ($S(\mathcal{A},\mathcal{B};X,Y)_{pf}$), where:*

for $i=1: \mathcal{A}=\mathcal{O}_X(Y),\ \mathcal{B}=\mathcal{O}_X(Y)$,
for $i=2: \mathcal{A}=\mathcal{O}_X(Y),\ \mathcal{B}=\mathcal{O}_Y$,
for $i=3: \mathcal{A}=\mathcal{O}_X,\ \mathcal{B}=\mathcal{O}_X$,
for $i=4: \mathcal{A}=\mathcal{O}_X,\ \mathcal{B}=\mathcal{O}_X(Y)$,
for $i=5: \mathcal{A}=\mathcal{O}_X,\ \mathcal{B}=\mathcal{O}_Y$.

If $i-(Y|X)_{lf}$ ($i-(Y|X)_{pf}$) holds we shall say that Y is a $i-lf$ ($i-pf$) subspace of X.

So, there are five (potentially different) relative variants of paracompactness to be considered.

A direct consequence of the preceding definition is the following proposition, in which some basic relations between this $S_{lf}-i$ ($i=\overline{1,5}$) relative properties are established.

Proposition 1. *For a space X and its subspace Y the following implications hold:*

$$\begin{array}{ccccc} 3-(Y|X)_{lf} & \Longrightarrow & 4-(Y|X)_{lf} & \Longrightarrow & 5-(Y|X)_{lf} \\ & & \Uparrow & & \Uparrow \\ & & 1-(Y|X)_{lf} & \Longrightarrow & 2-(Y|X)_{lf} \end{array}$$

If X is compact then $3-(Y|X)_{lf}$ for any subspace Y of X. Also, if Y is compact then $1-(Y|X)_{lf}$ for any X containing Y.

Now we list several easy facts about the $S_{lf}-i$ properties.

Proposition 2. *For a space X and $Y \subseteq X$ the following statements are true:*

1) *$4-(Y|X)_{pf} \Leftrightarrow 5-(Y|X)_{pf}$ i $1-(Y|X)_{pf} \Leftrightarrow 2-(Y|X)_{pf}$;*
2) *if $\overline{Y}=X$ then $4-(Y|X)_{lf} \Leftrightarrow 5-(Y|X)_{lf}$ i $1-(Y|X)_{lf} \Leftrightarrow 2-(Y|X)_{lf}$;*

3) *if* $Y = \overline{Y}$ *then* $1-(Y|X)_{ab} \Leftrightarrow 4-(Y|X)_{ab}$, $2-(Y|X)_{ab} \Leftrightarrow 5-(Y|X)_{ab}$, *where "ab" replaces any of the notations "lf" or "pf", as well as* $3-(Y|X)_{pf} \Leftrightarrow 4-(Y|X)_{pf}$;
4) *if* X *is a perfectly normal space and* $Y = \overline{Y}$, *then* $3-(Y|X)_{lf} \Leftrightarrow 4-(Y|X)_{lf}$;
5) $Y \in S(\mathcal{O}_Y, \mathcal{O}_Y)_{ab} \Leftrightarrow 2-(Y|X)_{ab}$, *where "ab" replaces any of the notations "lf" or "pf";*
6) $X \in S(\mathcal{O}, \mathcal{O})_{ab} \Rightarrow 3-(Y|X)_{ab}$, *where "ab" stands for either "lf" or "pf";*
7) *if* $Y = \overline{Y}$ *then* $X \in S(\mathcal{O}, \mathcal{O})_{ab} \Rightarrow 1-(Y|X)_{ab}$, *with "ab" as before;*
8) *if* $Z \subseteq Y$ *then* $i-(Y|X)_{ab} \Rightarrow i-(Z|X)_{ab}$, *for* $i \in \{3,4,5\}$ *and "ab" as before;*
9) *if* $Z \subseteq Y = \overline{Y}$ *then* $i-(Z|X)_{ab} \Rightarrow i-(Z|Y)_{ab}$, *for* $i \in \{3,4,5\}$ *and "ab" as above.*

Proof. The statements under 1) and 2) follow directly from the next few easy observations:
Let $\mathcal{L} \subseteq \mathcal{T}_Y$, $\mathcal{C} \subseteq \mathcal{T}_X$ and $\mathcal{L} \preceq \mathcal{C}$. For each $A \in \mathcal{L}$ take a $U_A \in \mathcal{C}$ and a $V_A \in \mathcal{T}_X$, such that $A = Y \cap V_A$ and $A \subseteq U_A$. Put $\mathcal{L}' = \{U_A \cap V_A : A \in \mathcal{L}\}$. Clearly, $\mathcal{L} \preceq \mathcal{L}' \preceq \mathcal{C}$ and $\mathcal{L}' \subseteq \mathcal{T}_X$. Then for each $y \in Y$ and each $A \in \mathcal{L}$ we must have $y \in A \Leftrightarrow y \in U_A \cap V_A$. Also, if $Y = \overline{X}$, then for each $y \in Y$, $G \in \mathcal{T}_X$ and $A \in \mathcal{L}$, where $y \in G$, we have that $A \cap G = \emptyset \Leftrightarrow (U_A \cap V_A) \cap G = \emptyset$. Consequently, if $\mathcal{L}$ is point-finite on Y so is $\mathcal{L}'$, and if addition $Y = \overline{Y}$, then if $\mathcal{L}$ is locally finite on Y so is $\mathcal{L}'$. Finally, $\bigcup \mathcal{L} \subseteq \bigcup \mathcal{L}'$.

Now let us prove the claim stated under 4). Let $4-(Y|X)_{lf}$, where $Y = \overline{Y}$ and let a sequence of open covers of X $\langle \mathcal{U}_n : n < \infty \rangle$ be given. Applying $4-(Y|X)_{lf}$ to that sequence we obtain another sequence $\langle \mathcal{V}_n : n < \infty \rangle$, where $\mathcal{V}_n \subseteq \mathcal{T}_X$, such that $\mathcal{V}_n \preceq \mathcal{U}_n$, each $\mathcal{V}_n$ a family locally finite on Y, and $Y \subseteq \bigcup \bigcup \{\mathcal{V}_n : n < \infty\}$. As X is perfectly normal and Y closed in X, there is a sequence of open sets $\langle U_n : n < \infty \rangle$ such that $Y = \bigcap_{n<\infty} U_n = \bigcap_{n<\infty} \overline{U_n}$. Put $\mathcal{U}'_n = \{U \cap (X \setminus \overline{U_n}) : U \in \mathcal{U}_n\}$. Then, for $\mathcal{M}_n = \mathcal{V}_n \cup \mathcal{U}'_n \subseteq \mathcal{T}_X$, we have that $\mathcal{M}_n \preceq \mathcal{U}_n$, each of the families $\mathcal{M}_n$ is locally finite on Y, and $\bigcup \{\mathcal{M}_n : n < \infty\}$ is an open cover of X.

The remaining statements of the proposition are trivially established. □

Apparently, the relation $2-(Y|X)_{lf}$, i. e. $2-(Y|X)_{pf}$ does not depend on the particular way in which Y is placed in X, it actually describes the absolute paracompactness of Y.

In the next two examples $\mathbf{R}$ will denote the set of reals. Also, (a,b) will stand both for the appropriate ordered pair and for the set $\{x \in \mathbf{R} : a <$

$x < b\}$, in which case, of course, $a, b \in \mathbf{R}$. It will be clear from the context which one is the case. $[a, b]$, where $a, b \in \mathbf{R}$, means as usual the closed segment $\{x \in \mathbf{R} : a \leq x \leq b\}$ of the real line.

Example 1. Let X be the *Niemytzky plane* [5]: the supporting set is $X = \{(x, y) \in \mathbf{R}^2 : y > 0\} \cup \mathbf{R}$, and it is topologized as follows: if $(a, b) \in X \setminus \mathbf{R}$ then the family of all the sets of the form $K(a, b; r) = \{(x, y) \in X : \sqrt{(x-a)^2 + (y-b)^2} < r\}$, where $r > 0$ is an arbitrary real number, constitutes an open neighborhood base at that point; if $a \in \mathbf{R}$ then an open neighborhood base at a is given by the family of all $B(a, b) = K(a, b; b) \cup \{a\}$, where $b > 0$ is an arbitrary real number.

It is not difficult to see that $\mathbf{R} \subseteq X$ is a discrete subspace of X. Therefore it is paracompact, so $2 - (\mathbf{R}|X)_{lf}$. Let us show, on the other hand, that $1 - (\mathbf{R}|X)_{lf}$does not hold.

Assume to the contrary that $\mathbf{R}$ is a $1 - lf$ subspace of X and consider for each n a family $\mathcal{A}_n = \mathcal{A} = \{B(x, 1) : x \in \mathbf{R}\}$ of open subsets of X covering the subspace $\mathbf{R}$. Then there is a sequence of families $\mathcal{L}_n$ of open subsets of X such that $\bigcup_{n<\infty} \mathcal{L}_n \supseteq \mathbf{R}$, and such that, for each n, $\mathcal{L}_n \preceq \mathcal{A}_n \equiv \mathcal{A}$ and $\mathcal{L}_n$ is locally finite on $\mathbf{R}$.

For each $x \in \mathbf{R}$ there is a n_x and a $U_x \in \mathcal{L}_{n_x}$ such that $x \in U_x$.

On the other hand, as $\mathcal{L}_{n_x} \preceq \mathcal{A}$, there is a $y \in \mathbf{R}$ such that $U_x \subseteq B(y, 1)$. From $\{y\} = \mathbf{R} \cap B(y, 1) \supseteq \mathbf{R} \cap U_x \supseteq \{x\}$ it follows $y = x$ and $\{x\} = \mathbf{R} \cap U_x$. Since U_x is open, there is a real number $\varepsilon_x > 0$ such that $x \in B(x, \varepsilon_x) \subseteq U_x$. So:

$$\{x\} = \mathbf{R} \cap B(x, \varepsilon_x) = \mathbf{R} \cap U_x; \quad x \in B(x, \varepsilon_x) \subseteq U_x; K(x, \varepsilon_x; \varepsilon_x) \subseteq B(x, \varepsilon_x) \tag{1}$$

$$x \neq y \Rightarrow U_x \neq U_y. \tag{2}$$

Put $S_n = \{x \in \mathbf{R} : n_x = n\}$. For each $n \in \mathbf{N}$, and each real number $a > 0$, let us call a closed segment I of the real line "(n, a)– **good**" if $\forall x \in S_n \cap I$ $(\varepsilon_x < a)$.

Claim. For each $m \in \mathbf{N}$, a real number $\theta > 0$ and a closed segment of the real line $[p, q]$, with $p < q$, there is a (m, θ)–**good** closed segment $I \subseteq [p, q]$ ... $(*)$.

Indeed:

Let us first note that $\forall x \in S_m$ $(U_x \in \mathcal{L}_m)$. Fix a $a \in (p, q)$. $\mathcal{L}_m$ is locally finite on the set $\mathbf{R}$ so there must be a real number $r > 0$ and a finite set $\{V_1, \ldots, V_k\} \subseteq \mathcal{L}_m$ such that $\forall V \in \mathcal{L}_m \setminus \{V_1, \ldots, V_k\}$ $(V \cap B(a, r) = \emptyset)$. By (2), this implies that there must be a finite set $\{x_1 \ldots, x_k\} \subseteq \mathbf{R}$ such that

$\forall x \in S_m (x \notin \{x_1, \ldots, x_k\} \Rightarrow U_x \cap B(a,r) = \emptyset)$. So, having in mind (1) as well as $K(a,r;r) \subseteq B(a,r)$:

$$\forall x \in S_m \setminus \{x_1, \ldots, x_k\} (K(x, \varepsilon_x; \varepsilon_x) \cap K(a,r;r) = \emptyset).$$

Consequently, for $x \in S_m$ with $x \neq x_i$ we must have:

$$\varepsilon_x + r \leq \sqrt{(x-a)^2 + (r - \varepsilon_x)^2} \leq |x-a| + |r - \varepsilon_x|. \tag{3}$$

Let $n > 0$ be any integer such that $\frac{1}{n} < \min(2r, \theta)$, $((a - \frac{1}{n}, a + \frac{1}{n}) \setminus \{a\}) \cap \{x_1, \ldots, x_k\} = \emptyset$ and $(a - \frac{1}{n}, a + \frac{1}{n}) \subseteq [p,q]$. Then, according to (4), for each $x \in S_m \cap ((a - \frac{1}{n}, a + \frac{1}{n}) \setminus \{a\})$ we have that:

$$\varepsilon_x + r \leq \frac{1}{n} + |r - \varepsilon_x|,$$

and

$$\theta > \frac{1}{n} \geq \varepsilon_x + r - |r - \varepsilon_x| = \begin{cases} \varepsilon_x + r - r + \varepsilon_x = 2\varepsilon_x \\ \varepsilon_x + r + r - \varepsilon_x = 2r, & \text{which is impossible} \\ & \text{because } \frac{1}{n} < 2r. \end{cases}$$

In other words, we conclude that each $x \in S_m \cap ((a - \frac{1}{n}, a + \frac{1}{n}) \setminus \{a\})$ must satisfy: $2\varepsilon_x < \theta$, implying that any closed segment $I \subseteq (a - \frac{1}{n}, a)$ is (m, θ)– **good**, which, together with $I \subseteq [p,q]$, proves $(*)$. •

Now let, according to $(*)$, $I_1 \subseteq [0,1]$ be a $(1,1)$– **good** segment (of length not greater than 1). If the segment I_n has been constructed so that it is $(i, \frac{1}{n})$– **good** for each $i \in \{1, \ldots, n\}$ and with length not greater than $\frac{1}{n}$, choose a segment $J_n \subseteq I_n$ of length not greater than $\frac{1}{n+1}$, as well as segments $J_n \supseteq I_1^{n+1} \supseteq I_2^{n+1} \supseteq \cdots \supseteq I_{n+1}^{n+1}$, such that for each $i \in \{1, \ldots, n+1\}$ I_i^{n+1} is $(i, \frac{1}{n+1})$– **good** and put $I_{n+1} = I_{n+1}^{n+1}$. Since $I_{n+1} \subseteq I_i^{n+1}$ and I_i^{n+1} is $(i, \frac{1}{n+1})$– **good**, the segment I_{n+1} must also be $(i, \frac{1}{n+1})$– **good** for each $i \in \{1, \ldots, n+1\}$. Finally, $I_{n+1} \subseteq I_n$ and the length of I_{n+1} is not greater than $\frac{1}{n+1}$.

So, we have constructed a decreasing sequence $\langle I_n : n < \infty \rangle$ of closed intervals with lengths converging to 0 such that for all n, m with $n \leq m$ I_m is $(n, \frac{1}{m})$– **good**. As there must be an $x \in \bigcap_{n<\infty} I_n$, and as every I_m is $(n_x, \frac{1}{m})$ **good** for $m \geq n_x$, it must be that $\varepsilon_x < \frac{1}{m}$ for all $m \geq n_x$, i. e. $\varepsilon_x = 0$, which is impossible.

According to Proposition 2. under (3, this is also an example of a pair Y, X with $5 - (Y|X)_{lf}$ but not $4 - (Y|X)_{lf}$. □

Example 2. Let X denote the same space as in the example above and fix an arbitrary compact K containing the set $\mathbf{R}$ as its discrete subspace. Call a pair $(U,V) \in \mathcal{T}_X \times \mathcal{T}_K$ a l-pair if $U \cap \mathbf{R} = V \cap \mathbf{R}$. Put $Y = X \cup K$, and define the following (Hausdorff) topology on the set Y: $\mathcal{T}_Y = \{U \cup V : (U,V) \text{ je } l\text{-par}\}$. Let us prove $4 - (\mathbf{R}|Y)_{lf}$.

Let $\mathcal{A} \in \mathcal{O}_Y$. Consider $\mathcal{A}_1 = \{V \in \mathcal{T}_K : \exists U \in \mathcal{T}_X \ ((U, V) \text{ is } l\text{-pair; } U \cup V \in \mathcal{A}\}$ and fix a function $f : \mathcal{A}_1 \to \mathcal{T}_X$ such that for each $V \in \mathcal{A}_1$ $(f(V), V)$ is a l-pair and $f(V) \cup V \in \mathcal{A}$. As $\mathcal{A}_1 \in \mathcal{O}_K$ and as K is compact there are $V_1, \ldots, V_n \in \mathcal{A}_1$ such that $K = \bigcup_{i=1}^n V_i$. Then $\mathbf{R} \subseteq \bigcup_{i=1}^n (f(V_i) \cup V_i)$, where, of course, for each $i = 1, \ldots, n$, $f(V_i) \cup V_i \in \mathcal{A}$, and the family $\{f(V_i) \cup V_i : i = 1 \ldots, n\}$ is finite (whence also locally finite at every point of the space Y). So obviously $4 - (\mathbf{R}|Y)_{lf}$.

On the other hand, it cannot be $1 - (\mathbf{R}|Y)_{lf}$: Since $\mathbf{R}$ is a discrete subspace of K we can, for each $x \in \mathbf{R}$, choose a $V_x \in \mathcal{T}_K$ such that $\{x\} = V_x \cap \mathbf{R}$. Putting, for each $n \in \mathbf{N}$, $\mathcal{L}_n = \{B(x, 1) \cup V_x : x \in \mathbf{R}\}$ (where $B(x, \varepsilon)$ stands for the same thing as before) and reasoning in the same manner as in the preceding example it is not difficult to see that there can be no sequence $\langle \mathcal{P}_n : n < \infty \rangle$ with $\mathcal{P}_n \subseteq \mathcal{T}_Y$ for each $n \in \mathbf{N}$, such that $\mathbf{R} \subseteq \bigcup \bigcup \{\mathcal{P}_n : n < \infty\}$ and for each n the family $\mathcal{P}_n$ is locally finite on $\mathbf{R}$ as well as $\mathcal{P}_n \preceq \mathcal{L}_n$. $\square$

Example 3. If X is compact and Y any non paracompact subspace of X then $i - (Y|X)_{lf}$ for $i \in \{3, 4, 5\}$, but not $2 - (Y|X)_{lf}$ (hence neither $1 - (Y|X)_{lf}$). $\square$

Example 4. If Y is a countable subspace of X then trivially $1 - (Y|X)_{lf}$ (so $4 - (Y|X)_{lf}$ too) holds. If in addition X is not Lindëlof and $\overline{Y} = X$, then it cannot be $3 - (Y|X)_{lf}$. This is because any family of open sets of a space which is locally finite on a dense countable subspace S of it (even point countable on S) must be countable. Thus the assertion follows if we choose any open cover $\mathcal{K}$ of X with no countable subcover and take all the $\mathcal{U}_n$-s in the definition of $3 - (Y|X)_{lf}$ (see Definition 1.) to be $\mathcal{K}$.

So, any non Lindëlof separable space (e.g. the Niemitcky plane) is an example showing that neither $1-(Y|X)_{lf}$ nor $4-(Y|X)_{lf}$ imply $3-(Y|X)_{lf}$. $\square$

In the next few theorems we will try to answer the question when a certain type of relative paracompactness of two (finitely many) subspaces implies the same property of their union.

A trivial, but useful observation is formulated as follows.

Lemma 1. *If $\mathcal{K}$ is an arbitrary locally finite (point-finite) on S family of subsets of the space X, where $S \subseteq X$, and $\mathcal{L}$ any partition of the set $\mathcal{K}$ (i. e. $\bigcup \mathcal{L} = \mathcal{K}$ i $\forall x, y \in \mathcal{L}$ $(x \neq y \ \Rightarrow \ x \cap y = \emptyset)$), then the family $\{\bigcup x : x \in \mathcal{L}\}$ is also locally finite (point-finite) on S.*

Proof. Elementary. $\square$

We shall often make use of this fact below.
A union of two closed $2 - lf$ subspaces is again a $2 - lf$ subspace.

Theorem 2. *If F_1, F_2 are both closed and paracompact subspaces of a space X then the subspace $F_1 \cup F_2$ is also paracompact.*

Proof. Let an arbitrary family $\mathcal{U} \subseteq \mathcal{T}_X$ be given such that $F_1 \cup F_2 \subseteq \bigcup \mathcal{U}$. As F_1 is paracompact there exists a $\mathcal{L} \subseteq \mathcal{T}_{F_1}$ such that $F_1 = \bigcup \mathcal{L}$ and $\mathcal{L} \preceq \mathcal{U}$, and such that the family $\mathcal{L}$ is locally finite on F_1 (whence on the whole space X, because F_1 is closed). Take any function $f : \mathcal{L} \to \mathcal{T}_X$ such that $\forall U \in \mathcal{L}\ (U = F_1 \cap f(U)\ \wedge\ \exists V \in \mathcal{U}\ (f(U) \subseteq V))$. Let $\mathcal{L}' = \{f(U) : U' \in \mathcal{L}\}$ i $\mathcal{U}' = \{(X \setminus F_1) \cap U : U \in \mathcal{U}\}$.

$\mathcal{L}' \cup \mathcal{U}'$ is a family of sets open in X which covers F_2 so, as F_2 is paracompact, there is a $\mathcal{V} \subseteq \mathcal{T}_{F_2}$, where $\mathcal{V} \preceq \mathcal{L}' \cup \mathcal{U}'$, $F_2 \subseteq \bigcup \mathcal{V}$ such that the family $\mathcal{V}$ is locally finite on F_2. Put $\mathcal{V}_1' = \{A \in \mathcal{V} : \exists U \in \mathcal{L}\ (A \subseteq f(U))\}$ and choose a $g : \mathcal{V}_1' \to \mathcal{L}$ such that $\forall A \in \mathcal{V}_1'\ (A \subseteq f(g(A)) = h(A))$, where $h = f \circ g$. For a $A \in \mathcal{V}_1'$ find a $O'(A) \in \mathcal{T}_X$ such that $A = F_2 \cap O'(A)$ and denote $O(A) = O'(A) \cap h(A) \in \mathcal{T}_X$. For a $U \in ran(g)$ let $\Theta(U) = \bigcup\{O(A) : A \in g^{\leftarrow}\{U\}\} \in \mathcal{T}_X$ and $\Theta_i(U) = F_i \cap \Theta(U), \quad i = 1, 2$. So, $\Theta_1(U) \cup \Theta_2(U) = \Theta(U) \cap (F_1 \cup F_2)$ is open in $F_1 \cup F_2$. Finally, put $\mathcal{V}_1 = \{\Theta_1(U) \cup \Theta_2(U) : U \in ran(g)\}$.

Clearly, from $\mathcal{L}' \preceq \mathcal{U}$ it follows $\mathcal{V}_1 \preceq \mathcal{U}$ (because $\Theta_1(U) \cup \Theta_2(U) \subseteq \Theta(U) = \bigcup\{O(A) : A \in g^{\rightarrow}\{U\}\} \subseteq \bigcup\{f(g(A)) : g(A) = U\} \subseteq f(U) \subseteq V$, for a $V \in \mathcal{U}$).

By construction we have that $\Theta_1(U) \subseteq U$ for each $U \in ran(g) \subseteq \mathcal{L}$, so as $\mathcal{L}$ is locally finite on X so is $\{\Theta_1(U) : U \in ran(g)\}$. Also, $\Theta_2(U) = \bigcup g^{\leftarrow}\{U\} = \bigcup\{A \in \mathcal{V}_1' : g(A) = U\}$, so by Lemma 1. (considering that $\mathcal{V}_1' \subseteq \mathcal{V}$ is locally finite on F_2, hence on X too) the family $\{\Theta_2(U) : U \in ran(g)\}$ is locally finite on X.

Therefore $\mathcal{V}_1 \subseteq \mathcal{T}_{F_1 \cup F_2}$ must be locally finite on X too (equivalently: on $F_1 \cup F_2$).

Put $\mathcal{V}_2 = \{A \in \mathcal{V} : \exists U \in \mathcal{U}\ (A \subseteq (X \setminus F_1) \cap U)\} \preceq \mathcal{U}$. One readily sees that, for each $A \in \mathcal{V}_2$ we must have that $A = (X \setminus F_1) \cap U \cap F_2 = \big((X \setminus F_1) \cap U\big) \cap (F_1 \cup F_2)$, which implies that $\mathcal{V}_2 \subseteq \mathcal{T}_{F_1 \cup F_2}$. From $\mathcal{V}_2 \subseteq \mathcal{V}$ it follows that $\mathcal{V}_2$ is locally finite on $F_1 \cup F_2$.

Finally, it can easily be seen that $F_2 \subseteq \bigcup(\mathcal{V}_1 \cup \mathcal{V}_2)$.

Now, we have defined a family $\mathcal{A} = \mathcal{V}_1 \cup \mathcal{V}_2 \subseteq \mathcal{T}_{F_1 \cup F_2}$ such that $\mathcal{A} \preceq \mathcal{U}$, $\mathcal{A}$ is locally finite on $F_1 \cup F_2$, and $F_2 \subseteq \bigcup \mathcal{A}$. Replacing the roles of the subspaces F_1 and F_2, in exactly the same way we can obtain a $\mathcal{B} \subseteq \mathcal{T}_{F_1 \cup F_2}$ with the property that $\mathcal{B} \preceq \mathcal{U}$, $\mathcal{B}$ is locally finite on $F_1 \cup F_2$ and $F_1 \subseteq \bigcup \mathcal{B}$.

$\mathcal{A} \cup \mathcal{B}$ is the required family which is locally finite on $F_1 \cup F_2$, refines $\mathcal{U}$, with $F_1 \cup F_2 = \bigcup(\mathcal{A} \cup \mathcal{B})$, and which witnesses for the paracompactness of the subspace $F_1 \cup F_2$. $\square$

The analoguous property for "$3 - lf$" subspaces is established as shown below.

Theorem 3. *If $F_1, F_2 \subseteq X$ are such that $3-(F_i|X)_{lf}, \quad i=1,2$, then $3-(F_1\cup F_2|X)_{lf}$.*

Proof. Let a sequence $\langle \mathcal{U}_{k,i} : k,i<\infty\rangle$ be given such that $\mathcal{U}_{k,i}\subseteq \mathcal{T}_X$ and such that any of the families $\mathcal{U}_{k,i}$ covers X.

Apply the fact that $3-(F_1|X)_{lf}$ to each of the sequences $\langle \mathcal{U}_{k,i} : i<\infty\rangle$ in order to obtain for each $k\in \mathbf{N}$ a sequence $\langle \mathcal{L}_{k,i} : i<\infty\rangle$ such that $\mathcal{L}_{k,i}\subseteq \mathcal{T}_X$, families $\mathcal{L}_{k,i}$ are all locally finite on F_1, each of the sets $\bigcup_{i<\infty}\mathcal{L}_{k,i}$ covers X and such that $\mathcal{L}_{k,i}\preceq \mathcal{U}_{k,i}$.

Now apply $3-(F_2|X)_{lf}$ to the sequence $\langle \bigcup_{i<\infty}\mathcal{L}_{k,i} : k<\infty\rangle$ so as to obtain a sequence $\langle \mathcal{J}_k : k<\infty\rangle$, where $\mathcal{J}_k\subseteq\mathcal{T}_X$, each $\mathcal{J}_k$ is locally finite on F_2, the family $\bigcup_k \mathcal{J}_k$ covers X and $\mathcal{J}_k\preceq\bigcup_{i<\infty}\mathcal{L}_{k,i}$. For each k,i put $\mathcal{J}'_{k,i}=\{A\in\mathcal{J}_k : \exists U\in\mathcal{L}_{k,i}\ (A\subseteq U)\}$ and take any $g_{k,i} : \mathcal{J}'_{k,i}\to\mathcal{L}_{k,i}$ such that $\forall A\in\mathcal{J}'_{k,i}\ (A\subseteq g_{k,i}(A))$. Denote $\mathcal{J}_{k,i}=\{\bigcup g^{\leftarrow}_{k,i}\{U\} : U\in\mathcal{L}_{k,i}\}$. Obviously: $\bigcup\mathcal{J}'_{k,i}=\bigcup\mathcal{J}_{k,i}$, $\mathcal{J}_{k,i}\preceq\mathcal{L}_{k,i}\preceq\mathcal{U}_{k,i}$. As the family $\mathcal{J}_{k,i}$ is of the form $\{V_U : U\in\mathcal{L}_{k,i}\}$, where $V_U\subseteq U$, and as $\mathcal{L}_{k,i}$ is locally finite on F_1, we conclude that $\mathcal{J}_{k,i}$ must also be locally finite on F_1. The family $\mathcal{J}'_{k,i}$ is locally finite on F_2 (because such is $\mathcal{J}_k$), so (in view of Lemma 1. as well as the way in which we constructed the $\mathcal{J}_{k,i}$-s) the family $\mathcal{J}_{k,i}$ is also locally finite on F_2. And so, we have defined the families $\mathcal{J}_{k,i}\prec\mathcal{U}_{k,i}$ of open sets each of which is locally finite on $F_1\cup F_2$.
Since $\mathcal{J}_k\preceq\bigcup_{i<\infty}\mathcal{L}_{k,i}$ it must be $\mathcal{J}_k=\bigcup_{i<\infty}\mathcal{J}'_{k,i}$, and therefore:

$$\bigcup\mathcal{J}_k=\bigcup\bigcup_{i<\infty}\mathcal{J}'_{k,i}=\bigcup_{i<\infty}\bigcup\mathcal{J}'_{k,i}=\bigcup_{i<\infty}\bigcup\mathcal{J}_{k,i}=\bigcup\bigcup_{i<\infty}\mathcal{J}_{k,i}.$$

So

$$X=\bigcup_{k<\infty}\bigcup\mathcal{J}_k=\bigcup_{k<\infty}\bigcup\bigcup_{i<\infty}\mathcal{J}_{k,i}=\bigcup\bigcup_{k<\infty}\bigcup_{i<\infty}\mathcal{J}_{k,i}=\bigcup\bigcup_{k,i<\infty}\mathcal{J}_{k,i}$$

i. e. $\bigcup_{k,i<\infty}\mathcal{J}_{k,i}$ is a cover of X.

And so, the sequence $\langle\mathcal{J}_{k,i} : k,i<\infty\rangle$ proves $3-(F_1\cup F_2|X)_{lf}$. $\square$

Practically repeating the proof of the previous theorem we obtain the next one.

Theorem 4. *If $F_1, F_2 \subseteq X$ are such that $3-(F_i|X)_{pf}, \quad i=1,2$, then $3-(F_1\cup F_2|X)_{pf}$.*

And finally Theorems 4.,3. and Proposition 2. (under 3) and 4)) give us the next two observations.

Theorem 5. *If F_1, F_2 are both closed subspaces of a space X with $4-(F_i|X)_{pf}, \quad i=1,2$, then $4-(F_1\cup F_2|X)_{pf}$.*

Theorem 6. *If F_1, F_2 are both closed subspaces of a perfectly normal space X with $4-(F_i|X)_{lf}, \quad i=1,2$ then also $4-(F_1\cup F_2|X)_{lf}$.*

2. References

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Faculty of Sciences and Mathematics
University of Niš
Višegradska 33, 18000 Niš
Serbia & Montenegro (Yugoslavia)