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ONE CHARACTERIZATION OF NEAR-P-POLYAGROUP

Janez Ušan

Abstract. In the present paper the following proposition is proved. Let $k > 1, s \geq 1,\ n = k \cdot s + 1$ and let (Q, A) be an $n-$groupoid. Then, (Q, A) is an near-P-polyagroup (briefly: NP-polyagroup) of the type $(s, n-1)$ iff for some $i \in \{t \cdot s + 1 | t \in \{1, \ldots, k-1\}\}$ the following conditions hold: (a) the $< i - s, i > -$associative law holds in (Q, A); (b) the $< i, , i + s > -$associative law holds in (Q, A); and (c) for every $a_1^n \in Q$ there is exactly one $x \in Q$ such that the following equality holds $A(a_1^{i-1}, x, a_i^{n-1}) = a_n$.

1. Introduction

1.1 Definition [6]: Let $k > 1, s \geq 1,\ n = k \cdot s + 1$ and let (Q, A) be an $n-$groupoid. Then, we say that (Q, A) is an $Ps-$**associative** $n-$**groupoid** iff for every $i, j \in \{t \cdot s + 1 | t \in \{0, 1, \ldots, k\}\}, i < j$, the following law holds

$$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-1})$$

[$:< i, j > -$associative law*]*.

Remark: For $s = 1$ (Q, A) is a $(k+1)-$semigroup; $k > 1$.
A notion of an $s-$associative $n-$groupoid was introduced by F.M. Sokhatsky (for example [2]).

1.2. Definitions [6]: Let $k > 1, s \geq 1,\ n = k \cdot s + 1$ and let (Q, A) be an $n-$groupoid. Then: (a) we say that (Q, A) is a $P-$**polyagroup of the type** $(s, n-1)$ iff is an $Ps-$associative $n-$groupoid and a $n-$quasigroup; and (b) we say that (Q, A) is an **near-P-polyagroup (briefly: NP–polyagroup) of the type** $(s, n-1)$ iff for every $i \in \{t \cdot s + 1 | t \in \{0, 1, \ldots, k\}\}$ and for all $a_1^n \in Q$ there is exactly one $x_i \in Q$ such that the equalityty

$$A(a_1^{i-1}, x_i, a_i^{n-1}) = a_n.$$

holds.

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A notion of an **polyagroup** was introduced by F.M. Skohatsky (for example [3]).

Remark: Every P-polyagroup of the type $(s, n-1)$ is an NP–polyagroup of the type $(s, n-1)$. In [6] NP–polyagroups is described as algebras with laws of type $< n, n-1, n-2 >$. See, also [4].

2. Auxiliary propositions

2.1. Proposition: *Let $k > 1, s \geq 1,\ n = k \cdot s + 1,\ i \in \{t \cdot s + 1 | t \in \{0, 1, \ldots, k\}\}$ and let (Q, A) be an $n-$groupoid. Also let*

(i) the $< i - s, i > -$associative law holds in the (Q, A);

(ii) the $< i, i + s > -$associative law holds in the (Q, A); and

(iii) for every $x, y, a_1^{n-1} \in Q$ the following implication holds

$$A(a_1^{i-1}, x, a_i^{n-1}) = A(a_1^{i-1}, y, a_i^{n-1}) \Rightarrow x = y.$$

Then (Q, A) is an $Ps-$associative $n-$groupoid.

The sketch of the part of the proof:

$i = t \cdot s + 1,\ n = k \cdot s + 1$

$i < (k-1) \cdot s + 1:$

$$A(a_1^{t\cdot s}, A(b_1^{k\cdot s+1}), a_{t\cdot s+1}^{k\cdot s}) = A(a_1^{t\cdot s}, b_1^s, A(b_{s+1}^{k\cdot s+1}, a_{t\cdot s+1}^{(t+1)\cdot s}), a_{(t+1)\cdot s+1}^{k\cdot s}) \Rightarrow$$

$$A(c_1^{t\cdot s}, d_1^s, A(a_1^{t\cdot s}, A(b_1^{k\cdot s+1}), a_{t\cdot s+1}^{k\cdot s}), e_{(t+1)\cdot s+1}^{k\cdot s}) =$$

$$A(c_1^{t\cdot s}, d_1^s, A(a_1^{t\cdot s}, b_1^s, A(b_{s+1}^{k\cdot s+1}, a_{t\cdot s+1}^{(t+1)\cdot s}), a_{(t+1)\cdot s+1}^{k\cdot s}), e_{(t+1)\cdot s}^{k\cdot s}) \Rightarrow$$

$$A(c_1^{t\cdot s}, A(d_1^s, a_1^{t\cdot s}, A(b_1^{k\cdot s+1}), a_{t\cdot s+1}^{(k-1)\cdot s}), a_{(k-1)\cdot s+1}^{k\cdot s}, e_{(t+1)\cdot s}^{k\cdot s}) =$$

$$A(c_1^{t\cdot s}, A(d_1^s, a_1^{t\cdot s}, b_1^s, A(b_{s+1}^{k\cdot s+1}, a_{t\cdot s+1}^{(t+1)\cdot s}), a_{(t+1)\cdot s+1}^{(k-1)\cdot s}), a_{(k-1)\cdot s+1}^{k\cdot s}, e_{(t+1)\cdot s}^{k\cdot s}) \Rightarrow$$

$$A(d_1^s, a_1^{t\cdot s}, A(b_1^{k\cdot s+1}), a_{t\cdot s+1}^{(k-1)\cdot s}) =$$

$$A(d_1^s, a_1^{t\cdot s}, b_1^s, A(b_{s+1}^{k\cdot s+1}, a_{t\cdot s+1}^{(t+1)\cdot s}), a_{(t+1)\cdot s+1}^{(k-1)\cdot s})$$

[:(ii), (iii)].

Remark: *For $k = 2$ the conditions (i) and (ii) are equivalent to the condition that (Q, A) is an $Ps-$associative $n-$groupoid.*

2.2. Proposition [7]: *Let $k > 1, s \geq 1,\ n = k \cdot s + 1$ and let (Q, A) be an $n-$groupoid. Then, (Q, A) is an* **NP–polyagroup of the type** $(s, n-1)$ *iff the following statements hold:*

(a) (Q, A) is an $< 1, n > -$associative $n-$groupoid;

(b) (Q, A) is an $< 1, s+1 > -$associative $n-$groupoid **or** *$< (k-1) \cdot s + 1,\ k \cdot s + 1 > -$associative $n-$groupoid; and*

(c) *for every* $a_1^n \in Q$ *there is* **at least one** $x \in Q$ *and* **at least one** $y \in Q$ *such that the following equalities hold*

$$A(a_1^{n-1}, x) = a_n \text{ and } A(y, a_1^{n-1}) = a_n.$$

3. Result

3.1. Theorem: *Let* $k > 1, s \geq 1,\ n = k \cdot s + 1$ *and let* (Q, A) *be an* $n-$*groupoid. Then the following statements are equaivalent:*

(i) (Q, A) *is an NP–polyagroup of the type* $(s, n-1)$*;*

(ii) *There is at least one* $i \in \{t \cdot s + 1 | t \in \{1, \ldots, k-1\}\}$ *such that the following conditions hold:* (a) *the* $< i-s, i > -$*associative law holds in* (Q, A)*;* (b) *the* $< i, i+s > -$*associative law holds in* (Q, A)*; and* (c) *for every* $a_1^n \in Q$ *there is* **exactly one** $x \in Q$ *such that the following equality holds* $A(a_1^{i-1}, x, a_i^{n-1}) = a_n$.

Proof. 1) $\Rightarrow$:

Let (i) holds. Then the implication $(i) \Rightarrow (ii)$ holds tautologically.

2) $\Leftarrow$:

Let (ii) holds. We prove respectively that the following proposition hold:

1° (Q, A) is an $Ps-$associative $n-$groupoid;

2° For every $a_1^n \in Q$ there is at least one $x \in Q$ such that the following equality holds $A(x, a_1^{n-1}) = a_n$; and

3° For every $a_1^n \in Q$ there is at least one $y \in Q$ such that the following equality holds $A(a_1^{n-1}, y) = a_n$.

Proof of 1° :

a) For $k = 2$ the conditions (a) and (b) are equaivalent to the condition that (Q, A) is an $Ps-$associative $n-$groupoid.

b) For $k > 2$, by $(a), (b), (c)$ $[i-$cancelation$]$ and Proposition 2.1, we conclude that (Q, A) is an $Ps-$associative $n-$groupoid.

Proof of 2°

a) By 1° and (c) $[i-$cancelation$]$, we conclude that for every $x, a_1^{k\cdot s+1}, c_1^{k\cdot s} \in Q$ the following sequence of equivalences holds:

$$A(x, a_1^{k\cdot s}) = a_{k\cdot s+1} \Leftrightarrow$$

$$A(c_1^{t\cdot s} A(x, a_1^{k\cdot s}), c_{t\cdot s+1}^{k\cdot s}) = A(c_1^{t\cdot s}, a_{k\cdot s+1}, c_{t\cdot s+1}^{k\cdot s}) \Leftrightarrow$$

$$A(c_1^{t\cdot s}, x, a_1^{s-1}, A(a_s^{k\cdot s}, c_{t\cdot s+1}^{(t+1)\cdot s}), c_{(t+1)\cdot s+1}^{k\cdot s}) = A(c_1^{t\cdot s}, a_{k\cdot s+1}, c_{t\cdot s+1}^{k\cdot s}),$$

i.e. the following equivalence holds

$$A(x, a_1^{k\cdot s}) = a_{k\cdot s+1} \Leftrightarrow$$

$$A(c_1^{t\cdot s}, x, a_1^{s-1}, A(a_s^{k\cdot s}, c_{t\cdot s+1}^{(t+1)\cdot s}), c_{(t+1)\cdot s+1}^{k\cdot s}) = A(c_1^{t\cdot s}, a_{k\cdot s+1}, c_{t\cdot s+1}^{k\cdot s});$$

$i = t \cdot s + 1,\ t \in \{1, \ldots, k-1\}$.

b) By (c), we conclude that for every $a_1^{k\cdot s+1}, c_1^{k\cdot s} \in Q$ there is exactly one $x \in Q$ such that the following equality holds:

$A(c_1^{t\cdot s}, x, a_1^{s-1}, A(a_s^{k\cdot s}, c_{t\cdot s+1}^{(t+1)\cdot s}), c_{(t+1)\cdot s+1}^{k\cdot s}) = A(c_1^{t\cdot s}, a_{k\cdot s+1}, c_{t\cdot s+1}^{k\cdot s})$.

c) Finally, by a) and b), we conclude that the statement 2° holds.

Similarly, it is possible to prove the statement 3°.

By 1°, 2°, 3° and Proposition 2.2, we conclude that the Theorem 3.1 holds.

3.2. Remark: 1) For $s = 1$ Theorem 3.1 is proved in [5]. 2) A part of Theorem 1.4 in [1] is the following proposition. Let $n \geq 3$ and let (Q, A) be an $n-$**semigroup.** Then (Q, A) is an $n-$group iff for some $i \in \{2, \ldots, n-1\}$ and for every $a_1^n \in Q$ there is exactly one $x \in Q$ such that the following equality holds $A(a_1^{i-1}, x, a_i^{n-1}) = a_n$. See, also Th.1 in [9].

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Institute of Mathematics
University of Novi Sad
Trg D. Obradovića 4,
21000 Novi Sad, Yugoslavia