

ON SCHAUDER'S 54th PROBLEM IN SCOTTISH BOOK REVISITED

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Abstract. The most famous of many problems in nonlinear analysis is Schauder's problem (*Scottish book*, problem 54) of the following form, that if C is a nonempty convex compact subset of a linear topological space does every continuous mapping $f : C \rightarrow C$ has a fixed point? The answer we give in this paper is yes.

In this paper we prove that if C is a nonempty convex compact subset of a linear topological space, then every continuous mapping $f : C \rightarrow C$ has a fixed point.

On the other hand, in this sense, we extend and connected former results of Brouwer, Schauder, Tychonoff, Markoff, Kakutani, Darbo, Sadovskij, Browder, Krasnoselskij, Ky Fan, Reinermann, Hukuhara, Mazur, Hahn, Ryll-Nardzewski, Day, Riedrich, Jahn, Eisenack-Fenske, Idzik, Kirk, Göhde, Caristi, Granas, Dugundji, Klee and some others.

1. Introduction

Brouwer's theorem of fixed point is one of the oldest and best known results in mathematics.

Schauder's theorem of fixed point is a generalization of Brouwer's theorem to infinite dimensional normed linear spaces. Schauder's theorem states that every continuous mapping of a compact convex subset of a normed linear space into itself has a fixed point.

Schauder's problem (*Scottish book*, problem 54) is the following form: *Does every continuous mapping $f : C \rightarrow C$ of a nonempty convex compact subset C in arbitrary linear topological space have a fixed point?*

AMS (1980) Mathematics Subject Classification (1985 Revision). Primary: 47H10. Secondary: 54H25, 52A07, 55M20.

Key words and phrases: Fixed point theorems, Brouwer's theorem, Schauder's theorem, Tychonof's theorem, Markoff's theorem, Kakutani's theorem, Sadovskij's theorem, Schauder's Conjecture (*Scottish book*, problem 54).

*Research supported by Science Fund of Serbia under Grant 1457.

For locally convex spaces the answer is affirmative from Tychonoff. Namely, in 1935, Tychonoff had shown that if C is a nonempty convex compact subset of a locally convex space, then every continuous map $f : C \rightarrow C$ has a fixed point.

In this paper we give the complete solution of the preceding well known Schauder's problem fixed point. Also, this solution is answering a question of S. Ulam. In connection with this, in this paper, we extend well known Markoff-Kakutani theorem to arbitrary linear topological spaces as an immediate consequence of the preceding solution of Schauder's problem.

On the other hand, in this sense, we extend and connected former results of Brouwer, Schauder, Tychonoff, Markoff, Kakutani, Darbo, Sadovskij, Krasnoselskij, Browder, Ky Fan, Reinermann, Hahn, Ryll-Nardzewski, Granas, Dugundji, Hukuhara, Mazur, Riedrich, Jahn, Eisenack-Fenske, Day and some others.

2. Main result

In connection with the preceding, let X topological space, let $T : X \rightarrow X$ and let $A : X \times X \rightarrow \mathbb{R}_+^0 := [0, +\infty)$ be a function. A topological space X satisfies the condition of **CS-konvergence** iff $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in X and if $A(x_n, x_{n+1}) \rightarrow 0 (n \rightarrow \infty)$ implies that $\{x_n\}_{n \in \mathbb{N}}$ has a convergent subsequence.

An annotation. With this definition we precision and correction our the former definiton of CS-convergence in: Tasković [3, p.123].

On the other hand, a function T satisfies the condition of **general A-variation** iff there exists a continuous function $G : X \rightarrow \mathbb{R}_+^0$ and if for any $x \in X$, with $x \neq Tx$, there exists $y \in X$ BS $\{x\}$ such that

$$(AG) \quad A(x, y) \leq |G(x) - G(y)|$$

for some function $A : X \times X \rightarrow \mathbb{R}_+^0$ with property $A(a, c) \leq A(a, b) + A(b, c)$ for all $a, b, c \in X$.

We are now in a position to formulate the following general statement, which is an extension to former results of Brouwer, Schauder, Tychonoff and some others.

Theorem 1. (General A-variation Principle). *Let T be a general A-variation mapping of topological space X into itself, where X satisfies the condition of CS-convergence. If $y \mapsto A(x, y)$ is continuous and if $A(x, y) = 0$ iff $x = y$, then T has a fixed point $\xi \in X$.*

A brief proof of this statement based on Zorn's lemma may be found in Tasković [3], [4] and [5].

Proof of Theorem 1. As is well known, the use of Zorn's lemma may be replaced by an induction argument (involving the Axiom of Choice) along the following lines. In this sense defines

$$R = \{Q \subset X : A(x, y) \leq |G(x) - G(y)| \text{ for all } x, y \in Q\}.$$

It is easy to verify that (R, \preceq) is a partially ordered set (asymmetric and transitive relation), where $Q_1 \preceq Q_2$ iff $Q_1 \subset Q_2$. Namely, in view of Zorn's lemma, there exists a maximal set $M \subset R$ such that

$$(1.1) \quad A(x, y) \leq |G(x) - G(y)| \text{ for all } x, y \in M.$$

Denote by α the greatest lower bound of the set $\{G(x) : x \in M\}$, i.e., $\alpha := \inf\{G(x) : x \in M\}$. Thus there exists a sequence of points $\{a_n\}_{n \in \mathbb{N}}$ from M such that $\{G(a_n)\}_{n \in \mathbb{N}}$ is decreasing and $G(a_n) \rightarrow \alpha$ ($n \rightarrow \infty$). It follows from (1) and from

$$A(a_n, a_{n+1}) \leq |G(a_n) - G(a_{n+1})|$$

that $A(a_n, a_{n+1}) \rightarrow 0$ ($n \rightarrow \infty$). This implies (from CS-convergence) that its sequence $\{a_n\}_{n \in \mathbb{N}}$ contains a convergent subsequence $\{a_{n(k)}\}_{k \in \mathbb{N}}$ with limit $\xi \in X$.

For any $x \in M$, if $G(x) \neq \alpha$, then for sufficiently large k we have the following inequalities

$$A(\xi, x) \leq A(\xi, a_{n(k)}) + A(a_{n(k)}, x) \leq A(\xi, a_{n(k)}) + |G(x) - G(a_{n(k)})|.$$

If $G(b) = \alpha$ for some $b \in M$, then we obtain in a similar way $A(\xi, b) = 0$. For any $x \in M$, if $G(x) \neq \alpha$, then we have

$$A(x, a_{n(k)}) \leq |G(x) - G(a_{n(k)})|$$

and thus by the continuity of G and A , we obtain that $A(x, \xi) \leq |G(x) - G(\xi)|$. This means that $\xi \in M$ and that there is no point $y \in X$ such that $\xi \neq y$ and $A(\xi, y) \leq |G(\xi) - G(y)|$, because such y would belong to M . Then it must be so that $A(\xi, T\xi) = 0$, i. e. , $\xi = T\xi$. This completes the proof.

3. A geometric lemma and its applications

Further, we notice, the set C in linear space is **convex** if for $x, y \in C$ and $\lambda \in [0, 1]$ implies $\lambda x + (1 - \lambda)y \in C$. The metric space (X, ρ) is called **convex** (or **metric convex**) if for any two different points $x, y \in X$ there is a point $z \in X$ ($z \neq x, y$) such that

$$(1.2) \quad \rho[x, y] = \rho[x, z] + \rho[z, y].$$

In connection with this, if $C \subset X$ convex set of a normed linear space X , then C also and metric convex set with $\rho[x, y] = \|x - y\|$, because for any two different points $x, y \in C$ there is a point $z := (x + y)/2 \in C$ ($z \neq x, y$) such that (2).

Lemma 1. *Let (X, ρ) be a metric space. If C is a metric convex set in X and if map $T : C \rightarrow C$ with the property that there is a point $a \in C$ which is not fixed point, then there exists a continuous function $G : C \rightarrow \mathbb{R}_+^0$ such that T is a general ρ -variation mapping.*

Proof. Let $a \in C$ be a fixed element such that $a \neq Ta$ and let $x \in C$ be an arbitrary point with $x \neq a$. Since C is a convex (metric convex) set in X , it follows from definition that for $a \in C$ and for all $x \in C \setminus BS\{a\}$ there exists a point $y \neq a, x$ in C such that $\rho[a, x] = \rho[a, y] + \rho[y, x]$. Hence, we have, also and the following inequality

$$(1.3) \quad 3^{-1}\rho[x, y] \leq \rho[x, y] = \rho[a, x] - \rho[a, y] \quad \text{for all } x \in C \setminus BS\{a\}.$$

On the other hand, analogous to the preceding construction, we also have the following inequality

$$(1.4) \quad 3^{-1}\rho[x, y] \leq \rho[x, y] = \rho[Ta, x] - \rho[Ta, y] \quad \text{for all } x \in C \setminus BS\{Ta\}.$$

Also, immediately to join and take away the expression $\rho[Ta, a]$ on the right side of inequality (3) we obtain the following equivalent inequality with (3), that is

$$(3') \quad 3^{-1}\rho[x, y] \leq \rho[a, x] + \rho[Ta, a] - (\rho[a, y] + \rho[Ta, a])$$

for all $x \in C \setminus BS\{a\}$.

From inequalities (3') and (4) define function $G : C \rightarrow \mathbb{R}_+^0$ such that

$$(1.5) \quad G(x) = \begin{cases} 3\rho[Ta, x] & \text{for } x = a, \\ 3(\rho[a, x] + \rho[Ta, a]) & \text{for } x \in C \setminus BS\{a\}. \end{cases}$$

Then, clearly, from (3'), (4) and (5) we have for any $x \in C$ there exists $y \neq x$ in C such that $\rho[x, y] \leq |G(x) - G(y)|$. Thus, for any $x \in C$ with $x \neq Tx$ there exists $y \in X \setminus BS\{x\}$ such that (AG), where $A(x, y) := \rho[x, y]$. Hence, it follows that T is a general ρ -variation mapping. The proof is complete.

Some remarks. In the proper second manner, for the existing continuous functions $G : C \rightarrow \mathbb{R}_+^0$ (in Lemma 1) instead (5) we can, from the proof of Lemma 1, define function $G : C \rightarrow \mathbb{R}_+^0$ such that

$$(1.6) \quad G(x) = \begin{cases} 3\rho[a, x] & \text{for } x = Ta, \\ 3(\rho[Ta, x] + \rho[a, Ta]) & \text{for } x \in C \setminus \{Ta\}; \end{cases}$$

then, from (4) and (6), also for any $x \in C$ with $x \neq Tx$ there exists $y \in X \setminus BS\{x\}$ such that (AG), where $A(x, y) := \rho[x, y]$. In this case, also, it follows that T is a general ρ -variation mapping.

We are now in a position to formulate our the following famous applications.

Corollary 1. (Brouwer, 1912). *Suppose that C is a nonempty convex, compact subset of \mathbb{R}^n , and that $T : C \rightarrow C$ is a continuous mapping. Then T has a fixed point in C .*

Proof. From the preceding Lemma 1, we have that $T : C \rightarrow C$ is a general A -variation mapping, where A is a metric on \mathbb{R}^n . The set C is compact in X , and thus C satisfies the condition of CS-convergence.

From the preceding remarks, it is easy to see that T satisfy all the required hypotheses in Theorem 1. Hence, it follows from Theorem 1 that T has a fixed point in C .

Let X, Y be topological spaces. A continuous map $F : X \rightarrow Y$ is called **compact** if $F(X)$ is contained in a compact subset of Y . If X and Y are Banach's spaces and $T : D(T) \subset X \rightarrow Y$, then T is called **compact** if T is continuous and T maps bounded sets into relatively compact sets. Compact operators play a central role in nonlinear functional analysis. Schauder's theorem is a generalization of Brouwer's theorem to infinite dimensional normed linear spaces, with the preceding fact.

We can now formulate Brouwer's theorem in a manner valid for all normed linear spaces.

Corollary 2. (Schauder, 1930). *Let C be a nonempty, closed, bounded, convex subset of the Banach space X , and suppose $T : C \rightarrow C$ is a compact operator. Then T has a fixed point in C .*

Also, we have and an alternate version of the preceding Schauder fixed point theorem.

Corollary 3. (Schauder, 1930). *Let C be a nonempty, compact, convex subset of a Banach space X , and suppose $T : C \rightarrow C$ is a continuous operator. Then T has a fixed point.*

This corollary is the direct translation of the Brouwer fixed point theorem to Banach spaces.

Proof of Corollary 3. Since C is a convex subset of Banach space, from Lemma 1, we have that $T : C \rightarrow C$ is a general A -variation, where $A(x, y) = \|x - y\|$. The set C is closed in X , and thus C is a complete space. It is easy to see that T satisfy all the required hypotheses in Theorem 1. Hence, it follows from the Theorem 1 that T has a fixed point in C

Corollary 4. (Banach Contraction Principle, 1922). *Let (X, ρ) be a complete metric space and $T : X \rightarrow X$ contractive. Then T has a unique fixed point ξ , and $T^n x \rightarrow \xi (n \rightarrow \infty)$ for each $x \in X$.*

Proof. From the condition of contraction, it is easy to see that T is general ρ -(bounded) variation. Precisely, every contraction mapping is bounded variation and continuous. Hence, it follows from the Theorem 1 that T has a fixed point.

At the end, we notice, also in this paper, we extend and results of Darbo, Browder, Sadovskij, Tychonoff, Krasnoselskij, Ky Fan, Dugundji, Granas, Kirk, Caristi, Kakutani and some others. In connection with this, proofs are the analogous to the proofs of the preceding statements of Brouwer, Schauder and Banach.

4. Answer to Schauder's problem is affirmative

From the preceding statement and some further facts we are now in the position to formulate the following fact which is an extension of the former results of Brouwer, Schauder, Tychonoff, Mazur, Hukuhara, Ky Fan, Browder, Sadovskij, Darbo, Krasnoselskij, Reinermann, Dugundji, Granas, Klee, Idzik, Riedrich, Eisenack-Fenske, Jahn and some others.

Theorem 2. (Answer is yes for Schauder's problem). *Let C be a non-empty convex compact subset of a linear topological space X and suppose $T : C \rightarrow C$ is a continuous mapping. Then T has a fixed point in C .*

To prove this statement, the following facts are essential. In this sense we have the following facts:

Let X be a nonempty set and $q : X \times X \rightarrow \mathbb{R}_+^0$. The function q is called a *quasimetric* (or *pseudometric*) on X iff $q(x, y) = q(y, x)$, $q(x, z) \leq q(x, y) + q(y, z)$ and if $x = y$ implies $q(x, y) = 0$ for all $x, y, z \in X$. The pair (X, q) is called a *quasimetric* (or a *pseudometric*) space.

Lemma 2. *Let X be a nonempty set, let $\phi : X \rightarrow \mathbb{R}_+^0$ an arbitrary function and let us define $q : X \times X \rightarrow \mathbb{R}_+^0$ by the equalities*

$$q(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \max\{\phi(x), \phi(y)\} & \text{if } x \neq y, \end{cases}$$

then q is a quasimetric function on X . Also, if $\phi(x) = \phi(y) = 0$ implies $x = y$, then q is a metric function.

The proof of this statement is very elementary. Thus we omit it.

To prove Theorem 2 and the following fact is essential.

Lemma 3. (Application of Lemmas 1 and 2). *Let X be a linear space. If C is a convex set in X and if T is a map of C into itself, then there exists a continuous function $G : C \rightarrow \mathbb{R}_+^0$ such that T is a general A -variation mapping for some function $A : C \times C \rightarrow \mathbb{R}_+^0$.*

Proof. Consider the convex set C of linear space X as a quasi-metric space (from Lemma 2) with the quasi-metric q , where $q : C \times C \rightarrow \mathbb{R}_+^0$ is defined by

$$(1.7) \quad q(x, y) = \begin{cases} 0 & \text{for } x = y, \\ \max\{K(x), K(y)\} & \text{for } x \neq y, \end{cases}$$

for a strictly convex function $K : C \rightarrow \mathbb{R}_+^0$. Then it is easy to see that q is a quasi-metric, i.e., that for all $x, y, z \in C$ we have : $q(x, y) = q(y, x)$, $q(x, z) \leq q(x, y) + q(y, z)$, $q(x, y) \geq 0$ and that $x = y$ implies $q(x, y) = 0$.

On the other hand, if $q(x, y) = 0$ and $x \neq y$, i.e., if $K(x) = K(y) = 0$, then since K is a strictly convex function, we obtain

$$0 = \frac{K(x) + K(y)}{2} > K\left(\frac{x + y}{2}\right) \geq 0,$$

which is a contradiction. Consequently $x = y = \frac{x+y}{2}$, i.e., $x = y$. Thus $q(x, y) = 0$ implies $x = y$, i.e., q is a metric on C .

Applying Lemma 1 to this case, we obtain then that there exists a continuous function $G : C \rightarrow \mathbb{R}_+^0$ such that T is a general q -variation mapping, where $A = \rho \equiv q$. The proof is complete.

Proof of Theorem 2. From Lemma 3 there exists a continuous function $G : C \rightarrow \mathbb{R}_+^0$ such that T is a general A -variation, mapping where $A(x, y) := q(x, y)$ and q defined in (7).

Since T is a continuous mapping, the function of the following form $x \mapsto A(x, Tx) = q(x, Tx)$ is a continuous function. Also and the function $y \mapsto A(x, y) = q(x, y)$ is continuous. The set C is a compact in space X and thus C satisfies the condition of CS-convergence.

It is easy to see that T satisfies all the required hypotheses in Theorem 1. Hence, it follows from the Theorem 1 that T has a fixed point $\xi \in C$. The proof is complete.

5. Some further applications

As an immediate corollary of the preceding solved problem (Theorem 2), we obtain one of the basic results in nonlinear functional analysis which is an extension of the Markoff-Kakutani theorem.

Theorem 3. *Let C be a nonempty convex compact set in a linear topological space X and let \mathcal{F} be a commuting family of continuous affine maps of C into itself. Then \mathcal{F} has a common fixed point $\xi \in C$.*

A brief proof of this result based on Theorem 2 may be found in Tasković [3], [4] and [5].

On the other hand, as an immediate consequence of Theorem 1, we obtain the following geometrical fact on fixed points.

Theorem 4. *Let T be a self-map on a topological space X and $A : X \times X \rightarrow \mathbb{R}_+^0$ be a function with properties : $A(a, b) = 0$ iff $a = b$ and $A(a, c) \leq A(a, b) + A(b, c)$ for all $a, b, c \in X$. Suppose that there exists a continuous function $G : X \rightarrow \mathbb{R}_+^0$ such that*

$$A(x, Tx) \leq |G(x) - G(Tx)|$$

for every $x \in X$. If X satisfies the condition of CS-convergence and if $b \mapsto A(a, b)$ is continuous, then T has a fixed point $\xi \in X$.

A brief proof of this statement, based on Theorem 1, may be first found in Tasković [3].

6. References

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Received July 15, 2001.