MATHEMATICA MORAVICA
VOL. 5 (2001), 159–162

A COMMENT ON (n,m)–GROUPS FOR $n \geq 3m$

Janez Ušan

Abstract. In the present paper the following proposition is proved. Let $n \geq 3m$ and let (Q, A) be an (n,m)–groupoid. Then, (Q, A) is an (n,m)–group iff for some $i \in \{m+1, \ldots, n-2m+1\}$ the following conditions hold: (a) the $< i-1, i >$ -associative law holds in (Q, A); (b) the $< i, i+1 >$ -associative law holds in (Q, A); and (c) for every $a_1^n \in Q$ there is exactly one $x_1^m \in Q$ such that the following equality holds $A(a_1^{i-1}, x_1^m, a_i^{n-m}) = a_{n-m+1}^n$.

1. Introduction

1.1. Definitions. *Let $n \geq m+1$ $(n, m \in N)$ and (Q, A) be an (n,m)–groupoid $(A : Q^n \to Q^m)$. Then: (a) we say that (Q, A) is an (n,m)–***semigroup** *iff for every $i, j \in \{1, \ldots, n-m+1\}$, $i < j$, the following law holds*

$$A\Big(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-m}\Big) = A\Big(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-m}\Big)$$

[: $< i, j >$-associative law]; and (b) we say that (Q, A) is a **weak** (n,m)–**quasigroup** *iff for every $i \in \{1, \ldots, n-m+1\}$ and for every $a_1^n \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds*

$$A(a_1^{i-1}, x_1^m, a_i^{n-m}) = a_{n-m+1}^n;$$

*and (c) we say that (Q, A) is an (n,m)–***group** *iff (Q, A) is an (n,m)–semigroup and a weak (n,m)–quasigroup as well.*
(See, also [4].)

1.2. Remark. A notion of an (n,m)–group was introduced by G. Čupona in [3] as a generalization of the notion of a group (n–group – [1]). The paper [4] is mainly a survey on the known results for vector valued groupoids, semigroups and groups (to 1988).

AMS Subject Classification (2000). Primary: 20N15.
Key words and phrases: (n,m)–groupoids, (n,m)–semigroup, (n,m)–group.

2. Auxiliary propositions

2.1. Proposition. *Let* $n > m+2$, $i \in \{2,\dots,n-m\}$ *and let* (Q,A) *be an* $(n,m)-$*groupoid. Also let*
(*i*) *the* $< i-1, i >$*-associative law holds in* (Q,A);
(*ii*) *the* $< i, i+1 >$*-associative law holds in* (Q,A); *and*
(*iii*) *for every* $x_1^m, y_1^m, a_1^{n-m} \in Q$ *the following implication holds*

$$A(a_1^{i-1}, x_1^m, a_i^{n-m}) = A(x_1^m, y_1^m, a_1^{n-m}) \Rightarrow x_1^m = y_1^m.$$

Then (Q,A) *is an* (n,m)*-semigroup.*

The sketch of the part of the proof:
$i < n-m$:

$$A(a_1^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{n-m}) = A(a_1^i, A(a_{i+1}^{i+n}), a_{i+n+1}^{n-m}) \Rightarrow$$

$$A(c_1^i, A(a_1^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{n-m}), c_{i+1}^{n-m})) =$$

$$A(c_1^i, A(a_1^i, A(a_{i+1}^{i+n}), a_{i+n+1}^{n-m}), c_{i+1}^{n-m})) \Rightarrow$$

$$A(c_1^{i-1}, A(c_i, a_1^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{n-m-1}), a_{n-m}, c_{i+1}^{n-m}) =$$

$$A(c_1^{i-1}, A(c_i, a_1^i, A(a_{i+1}^{i+n}), a_{i+n+1}^{n-m-1}), a_{n-m}, c_{i+1}^{n-m}) \Rightarrow$$

$$A(c_i, a_1^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{n-m-1}) = A(c_i, a_1^i, A(a_{i+1}^{i+n}), a_{i+n+1}^{n-m-1})$$

[(*ii*), (*iii*)] *(See, also [5] and [7].)*

2.2. Remark. For $n = m+2$ the conditions (*i*) and (*ii*) are equivalent to the condition that (Q,A) is an $(n,m)-$semigroup. (For example: *a*) $m = 1$, $n = 3$; *b*) $m = 2$, $n = 2m$.)

2.3. Proposition [9]: *Let* $n \geq 2m$ *and let* (Q,A) *be an* $(n,m)-$*groupoid. Then,* (Q,A) *is an* $(n,m)-$*group iff the following statements hold:* (*a*) (Q,A) *is an* $< 1, n-m+1 >$ *and* $< 1, 2 >$*-associative* $n,m)-$*groupoid [or* $< 1, n-m+1 >$ *and* $< n-m, n-m+1 >$ *-associative* $(n,m)-$*groupoid]; and* (*b*) *for every* $a_1^n \in Q$ *there is* **at least one** $x_1^m \in Q^m$ *and* **at least one** $y_1^m \in Q^m$ *such that the following equalities hold*

$$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n \text{ and } A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n.$$

(See, also [4].)

Remark. For $m = 1$ Proposition 2.3 is proved in [6].

3. Result

3.1. Theorem. *Let* $n \geq 3m$ *and let* (Q,A) *be an* $(n,m)-$*groupoid. Then the following statements are equivalent*

(1) (Q,A) *is an* $(n,m)-$*group; and*

(2) There is at least one $i \in \{m+1,\dots,n-2m+1\}$ *such that the following conditions hold:* (*a*) *the* $< i-1, i >$ *-associative law holds in* (Q,A);

(b) the $< i, i+1 >$ – *associative law holds in* (Q, A)*; and (c) for every* $a_1^n \in Q$ *there is* **exactly one** $x_1^m \in Q^m$ *such that the following equality holds*

$$A(a_1^{i-1}, x_1^m, a_i^{n-m}) = a_{n-m+1}^n.$$

Proof. 1) $\Rightarrow$:

Let (1) holds. Then the implication (1) $\Rightarrow$ (2) holds tautologically.

2) $\Leftarrow$:

Let (2) holds. We prove respectively that the following propositions hold: 1° *(Q, A) is an (n, m)−semigroup;*

2° *For every* $a_1^n \in Q$ *there is at least one* $x_1^m \in Q^m$ *such that the following equality holds*

$$A(x_1^m, a_1^{n-m}) = a_{n-m+1}^n; \text{ and}$$

3° *for every there is at least one* $y_1^m \in Q^m$ *such that the following equality holds*

$$A(a_1^{n-m}, y_1^m) = a_{n-m+1}^n.$$

Proof of 1°:

a) For $m = 1$ and $n = 3(= m + 2)$ the conditions (*a*) and (*b*) are equivalent to the condition that (Q, A) is an (n, m)−semigroup (see, also 2.2); and

b) For $n \geq 3m$ and $(n, m) \neq (3, 1)$, by $(a), (b), (c)$ [i−cancelation] and Proposition 2.1, we conclude that (Q, A) is an (n, m)−semigroup.

Proof of 2°:

a) By 1° and (*c*) [i−cancelation], we conclude that for every $x_1^m, a_1^n, c_1^{n-m} \in Q$ th following sequence of equivalences holds:

$$A(x_1^m, a_1^{n-m}) = a_{n-m+1}^n \Leftrightarrow$$
$$A(c_1^{i-1}, A(x_1^m, a_1^{n-m}), c_i^{n-m}) = A(c_1^{i-1}, a_{n-m+1}^n, c_i^{n-m}) \Leftrightarrow$$
$$A(c_1^{i-1}, x_1^m, , A(a_1^{n-m}, c_i^{i+m-1}), c_{i+m}^{n-m}) = A(c_1^{i-1}, a_{n-m+1}^n, c_i^{n-m}),$$

i.e., the following equivalence holds

$$A(x_1^m, a_1^{n-m}) = a_{n-m+1}^n \Leftrightarrow$$
$$A(c_1^{i-1}, x_1^m, , A(a_1^{n-m}, c_i^{i+m-1}), c_{i+m}^{n-m}) = A(c_1^{i-1}, a_{n-m+1}^n, c_i^{n-m}).$$

b) By (*c*), we conclude that for every $a_1^n, c_1^{n-1} \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds:

$$A(c_1^{i-1}, x_1^m, , A(a_1^{n-m}, c_i^{i+m-1}), c_{i+m}^{n-m}) = A(c_1^{i-1}, a_{n-m+1}^n, c_i^{n-m}).$$

c) Finally, by a) and b), we conclude that the statement 2° holds.

Similarly, it is possible to prove the statement 3°.

By $1^\circ, 2^\circ, 3^\circ$ and Proposition 2.3, we conclude that the Theorem 3.1 holds.

Remarks. 1) For $m = 1$ Theorem 3.1 is proved in [8]. 2) A part of Theorem 1.4 in [2] is the following proposition. Let $n \geq 3$ and let (Q, A) be an $n-$**semigroup**. Then (Q, A) is an $n-$group iff for some $i \in \{2, \ldots, n-1\}$ and for every $a_1^n \in Q$ there is exactly one $x \in Q$ such that the following equality holds $A(a_1^{i-1}, x, a_i^{n-1}) = a_n$.

4. References

[1] Dörnte W.: *Untersuchengen über einen verallgemeinerten Gruppenbegriff*, Math. Z. 29 (1928), 1-19.

[2] Monk J. D., Sioson F. M.: *On the general theory of $m-$groups*, Fund. Math. 72(1971), 233-244.

[3] Čupona G.: *Vector valued semigroups*, Semigroup Forum, Vol. 26(1983), 65-74.

[4] Čupona G., Celakoski N., Markovski S., Dimovski D.: *Vector valued groupoids, semigroups and groups*, Vector valued semigroups and groups, Collection of papers edited by B. Popov, G. Čupona and N. Celakoski, Skopje 1988, 1–78.

[5] Ušan J.: *$n-$groups, $n \geq 2$, as variety of type $< n, n-1, n-2 >$*, Algebra and model Theory, Collection of papers edited by A. G. Pinus and K. N. Ponomaryov, Novosibirsk 1997, 182-208.

[6] Ušan J.: *On n-groups*, Maced. Acad. Sci. and Arts, Contributions, Sect. Math. Techn. Sci. XVIII 1-2 (1997), 17-20.

[7] Ušan J.: *Note on $(n, m)-$groups*, Math. Moravica Vol. 3(1999), 127-139.

[8] Ušan J.: *A note on $n-$groups for $n \geq 3$*, Novi Sad J. Math. 29-1(1999), 55-59.

[9] Ušan J.: *On $(n, m)-$groups*, Math. Moravica Vol. 4(2000), 119-122.

Institute of Mathematics,
University of Novi Sad
Trg D. Obradovića 4,
21000 Novi Sad, Yugoslavia