MATHEMATICA MORAVICA
VOL. 5 (2001), 111–118

FIXED POINTS AND APICES ON ARBITRARY SETS

Milan R. Tasković

Abstract. This paper presents new statements for fixed points and apices on arbitrary nonempty sets. Applications in fixed point theory and for general quasi-metric spaces are considered.

1. Introduction and some facts

Let X be a nonempty set. The problem of fixpoint for a given mapping $f|X$ is very easy to formulate: the question is if some $\xi \in X$ verifies $f(\xi) = \xi$. It is interesting that many problems are reducible to the existence of fixpoints of certain mappings, as on Figure 1. The question remains whether each statement could be equivalently expressed in the fixpoint language as well. The answer is affirmative, the answers, an example, were given in: Kurepa [2] and Tasković [6].

In this sense we consider a former concept of fixed apices for the mapping T of a nonempty set into itself. A map T of a nonempty set X to itself has a **fixed apex** $u \in X$ if there is $v \in X$ such that $T(u) = v$ and $T(v) = u$. The points $u, v \in X$ are called **fixed apices** of T if $T(u) = v$ and $T(v) = u$, as on Figure 2.

In general, for fixed integer $n \geq 2$ the points $u_1, \ldots, u_n \in X$ are called **fixed apices** of T if

$$u_1 = T(u_2), \ldots, u_{n-1} = T(u_n) \text{ and } u_n = T(u_1), \tag{1}$$

or if hold the following equalities

$$u_2 = T(u_1), \ldots, u_n = T(u_{n-1}) \text{ and } u_1 = T(u_n). \tag{2}$$

We begin with the following essential fact for the further statements in this paper.

AMS (MOS) Subject Classification 1991. Primary: 47H10, 54E70, 05A15. Secondary: 54H25, 54E35, 05A05, 06A10.

Key words and phrases: Fixed points, fixed apex, fixed apices, periodic points, fixed points on arbitrary sets, general quasi-metric spaces.

Lemma 1. *Let X be a nonempty set and T a mapping from X into X. Then the map T, for fixed integer $n \geq 2$, has fixed apices $u_1, \ldots, u_n \in X$ if and only if the map $T^n := T(T^{n-1})$ has a fixed point.*

A brief proof for $n = 2$ of this statement (in the case of partially ordered sets) may be found in Tasković [4].

Proof. If T has fixed apices $u_1, \ldots, u_n \in X$ then holds (1) or (2), i.e., we obtain the following equalities $u_1 = T(u_2) = \cdots = T^{n-1}(u_n) = T^n(u_1)$ or

$$u_n = T(u_{n-1}) = \cdots = T^{n-1}(u_1) = T^n(u_n),$$

which means that T^n has a fixed point. On the other hand, if the equation $x = T^n(x)$ has a solution $\xi = T^n(\xi)$ for some $\xi \in X$, then T has fixed apices ξ, $T^{n-1}(\xi)$, $T^{n-2}(\xi), \ldots, T(\xi) \in X$, because

$$\xi = T^n(\xi), T^{n-1}(\xi) = T(T^{n-2}(\xi)), \ldots, T(\xi) = T(\xi);$$

and with this, the proof is complete.

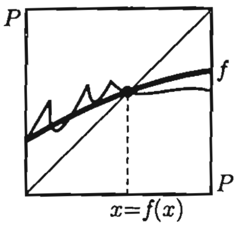

Figure 1

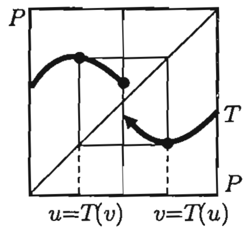

Figure 2

Let X be a nonempty set and T be a self-mapping of X. A point $x \in X$ is called a **periodic point** of T if there exists a positive integer k such that $T^k x = x$. The least positive integer satisfying this condition is called the **periodic index** of x.

In connection with this, we notice that the preceding Lemma 1 has the following equivalently formulation.

Lemma 1a. *Let X be a nonempty set and T a mapping from X into X. Then the map T, for fixed integer $n \geq 2$, has fixed apices $u_1, \ldots, u_n \in X$ if and only if the map T has the periodic point with periodic index $n \geq 2$.*

The proof of this fact is a totally analogous to the proof of the preceding Lemma 1.

2. Main statements on arbitrary sets

In recent years a great number of papers have presented the statements of fixed point on spaces and ordered sets. In this section we are now in a position to formulate the following main statements for fixed points and apices on arbitrary nonempty sets. This statements we consider on arbitrary nonempty sets without metrical and topological architecture.

Theorem 1. *Let X be a nonempty set, T be a self-mapping of X and let $P := (P, \preccurlyeq, \prec)$ be a partially ordered set, where for any $p \in P$ the case $p \prec p$ is impossible. If there exists a mapping $B : X \to P$ such that*

(B) $$B(Tx) \prec B(x) \quad \text{for every } x \in X \ (x \neq Tx)$$

or

(D) $$B(Tx) \succ B(x) \quad \text{for every } x \in X \ (x \neq Tx)$$

then T has a fixed point in X if and only if there exist integers m and n, $m > n \geq 0$, and a point $z \in X$ such that

(Co) $$T^m(z) = T^n(z).$$

Proof. Let $\xi \in X$ be a fixed point of T, i.e., $\xi = T\xi$. Then (Co) is true in case $m = 1$ and $n = 0$. Conversely, suppose that there exists a point $y \in X$ and two integers m and n, $m > n \geq 0$, such that (Co) is satisfied.

Then, (Co) is equivalent to the equality $T^{m-n}a = z$, where $a = T^n z$ and $k := m-n$ is the minimal integer satisfying $T^k a = a$ $(k \geq 1)$. Applying Lemma 1a to this case, we obtain that the map T has fixed apices $u_1, \ldots, u_k \in X$ for $k \geq 2$. Suppose that $u_1, \ldots, u_k \in X$ are not fixed points of T for $k \geq 2$. (If $k = 1$, then the statement directly holds). Thus, from (1) and (2), we obtain

(3) $$u_1 = T(u_2), \ldots, u_{k-1} = T(u_k) \text{ and } u_k = T(u_1)$$

or

(4) $$u_2 = T(u_1), \ldots, u_k = T(u_{k-1}) \text{ and } u_1 = T(u_k);$$

which means, by (B), it follows that the following inequalities hold

(5) $$B(u_1) = B(Tu_2) \prec B(u_2) \preccurlyeq \cdots \preccurlyeq B(u_k) = B(Tu_1) \prec B(u_1)$$

or

(6) $$\begin{aligned} B(u_1) = B(Tu_k) \prec B(u_k) = B(Tu_{k-1}) \prec B(u_{k-1}) \preccurlyeq \cdots \\ \cdots \preccurlyeq B(u_3) = B(Tu_2) \prec B(u_2) = B(Tu_1) \prec B(u_1), \end{aligned}$$

which is a contradiction. Therefore there is at least one point $u_i \in X$ $(i = 1, \ldots, k)$ such that $u_i = Tu_i$ for some $u_i \in X$.

In the case (D), from (3) and (4), we have

$$(7)\qquad \begin{aligned} B(u_k) = B(Tu_1) \succ B(u_1) = B(Tu_2) \succ B(u_2) \succcurlyeq \cdots \\ \cdots \succcurlyeq B(u_{k-1}) = B(Tu_k) \succ B(u_k) \end{aligned}$$

or

$$(8)\qquad \begin{aligned} B(u_1) = B(Tu_k) \succ B(u_k) = B(Tu_{k-1}) \succ B(u_{k-1}) \succcurlyeq \cdots \\ \cdots \succcurlyeq B(u_2) = B(Tu_1) \succ B(u_1), \end{aligned}$$

which is a contradiction. Therefore there is at least one point $u_i \in X$ ($i = 1, \dots, k$) such that $u_i = Tu_i$ for some $u_i \in X$, i.e., T has at least one fixed point in X. The proof is complete.

As an immediate consequence of the preceding statement we obtain the following result.

Theorem 2. *Let X be a nonempty set, T be a self-mapping of X and let $P := (P, \preccurlyeq, \prec)$ be a partially ordered set, where for any $p \in P$ the case $p \prec p$ is impossible. If there exists a mapping $A : X \times X \to P$ such that*

$$\text{(A)}\qquad A(Tx, Ty) \prec A(x, y) \quad \textit{for all } x, y \in X (x \neq y),$$

or

$$\text{(C)}\qquad A(Tx, Ty) \succ A(x, y) \quad \textit{for all } x, y \in X (x \neq y),$$

then T has a unique fixed point in X if and only if there exist integers m and n, $m > n \geq 0$, and a point $z \in X$ such that

$$\text{(Co)}\qquad T^m(z) = T^n(z).$$

Proof. For $y = Tx \neq x$, from (A) or (C) we have the following facts in the form

$$A(Tx, T^2x) \prec A(x, Tx) \text{ for every } x \in X \ (x \neq Tx)$$

or

$$A(Tx, T^2x) \succ A(x, Tx) \quad \text{for every } x \in X \ (x \neq Tx);$$

and hence, for $B(x) = A(x, Tx)$ with $x \neq Tx$, we obtain the condition (B) or (D) in Theorem 1. Since X satisfies all conditions of the preceding statement, applying Theorem 1 gives $T\xi = \xi$ for some $\xi \in X$ if and only if (Co) holds. Uniqueness follows immediately from the condition (A) or (C). The proof is complete.

In connection with the preceding facts, since de facto Lemma 1 is equivalent to Lemma 1a, thus we can Theorems 1 and 2 write in the following two extension forms.

Theorem 1a. *Let X be a nonempty set, T be a self-mapping of X and let $P := (P, \preccurlyeq, \prec)$ be a partially ordered set, where for any $p \in P$ the case $p \prec p$ is impossible. If there exists a mapping $B : X \to P$ such that (B) or (D), then T has a fixed point in X if and only if for T there exist fixed apices $u_1, \ldots, u_n \in X$ for fixed integer $n \geq 2$.*

Proof. Let $\xi \in X$ be a fixed point of T, i.e., $\xi = T\xi$; then the equation $x = T^n(x)$ has a solution $\xi = T^n(\xi)$ and thus, from Lemma 1, the map T has fixed apices $u_1, \ldots, u_n \in X$ for fixed integer $n \geq 2$.

Conversely, if there exist fixed apices $u_1, \ldots, u_n \in X$ for fixed integer $n \geq 2$, then we have (1) and (2). Suppose that $u_1, \ldots, u_n \in X$ are not fixed points of T for $n \geq 2$. Thus, by (B) it follows that (5) or (6) and by (D) it follows that (7) or (8), where the index k substitute with the index n. Since the cases (5) to (8) are impossible, therefore there is at least one fixed point of the points $u_i \in X$ $(i = 1, \ldots, n)$. The proof is complete.

Theorem 2a. *Let X be a nonempty set, T be a self-mapping of X and let $P := (P, \preccurlyeq, \prec)$ be a partially ordered set, where for any $p \in P$ the case $p \prec p$ is impossible. If there exists a mapping $A : X \times X \to P$ such that (A) or (C), then T has a unique fixed point in X if and only if for T there exist fixed apices $u_1, \ldots, u_n \in X$ for fixed integer $n \geq 2$.*

Proof. The necessary condition is obvious. The proof of the sufficiently condition is a totally analogous to the proof of the preceding Theorems 2 and 1a, and thus we omit it.

3. Further facts and consequences

In connection with the preceding statements, from our the Principle of Symmetry (see: Tasković, *Math. Japonica*, **35** (1990), p. 661), we obtain as an immediate extension of Theorem 1 the following statement.

Theorem 1b. *Let X be a nonempty set, T be a self-mapping of X and let $P := (P, \preccurlyeq, \prec)$ be a partially ordered set, where for any $p \in P$ the case $p \prec p$ is impossible. If there exists a mapping $B : X \to P$ and for each $x \in X$ there is a positive integer $r = r(x)$ such that*

$$B(T^r x) \prec B(x) \quad \text{for } x \neq T^r x \tag{9}$$

or

$$B(T^r x) \succ B(x) \quad \text{for } x \neq T^r x, \tag{10}$$

then T has a fixed point in X if and only if there exist integers m and n, $m > n \geq 0$, and a point $z \in X$ such that (Co).

On the other hand, from the Principle of Symmetry, similar to the preceding statement, as an immediate extension of Theorem 2, we obtain the following statement.

Theorem 2b. *Let X be a nonempty set, T be a self-mapping of X and let $P := (P, \preccurlyeq, \prec)$ be a partially ordered set, where for any $p \in P$ the case $p \prec p$ is impossible. If there exists a mapping $A : X \times X \to P$ and if for each $x \in X$ there is a positive integer $r = r(x)$ such that*

$$(11) \qquad A(T^r x, T^r y) \prec A(x, y) \quad \text{for every } y \in X,\ x \neq y,$$

or

$$(12) \qquad A(T^r x, T^r y) \succ A(x, y) \quad \text{for every } y \in X,\ x \neq y,$$

then T has a unique fixed point in X if and only if there exist integers m and n, $m > n \geq 0$, and a point $z \in X$ such that (Co).

Since Lemma 1 is equivalent to Lemma 1a, thus we obtain very similar to Theorem 1b the following statement which is an extension of Theorems 1 and 1a.

Theorem 1c. *Let X be a nonempty set, T be a self-mapping of X and let $P := (P, \preccurlyeq, \prec)$ be a partially ordered set, where for any $p \in P$ the case $p \prec p$ is impossible. If there exists a mapping $B : X \to P$ and for each $x \in X$ there is a positive integer $r = r(x)$ such that (9) or (10), then T has a fixed point in X if and only if for T there exist fixed apices $u_1, \ldots, u_n \in X$ for fixed integer $n \geq 2$.*

As in this statement, a totally analogous with the preceding facts is the following result, which is an extension of Theorems 2 and 2a.

Theorem 2c. *Let X be a nonempty set, T be a self-mapping of X and let $P := (P, \preccurlyeq, \prec)$ be a partially ordered set, where for any $p \in P$ the case $p \prec p$ is impossible. If there exists a mapping $A : X \times X \to P$ and if for each $x \in X$ there is a positive integer $r = r(x)$ such that (11) or (12), then T has a unique fixed point in X if and only if for T there exist fixed apices $u_1, \ldots, u_n \in X$ for fixed integer $n \geq 2$.*

For further applications, we give some fixed point theorems for mappings in general quasi-metric spaces. In this sense, let $(G, \preccurlyeq, \prec)$ be a partial order set satisfying the following conditions: θ is the minimal element in G;

for any $u, v \in G$ the element $\sup\{u, v\}$ exists and belongs to G; for any $u \in G$ the case $u \prec u$ is impossible; and for any $u, v, w \in G$, $u \prec w$ and $v \prec w$ implies $\sup\{u, v\} \prec w$, and $u \prec v \preccurlyeq w$ implies $u \prec w$.

Let X be a nonempty set. The pair (X, ρ) is called **general quasi-metric space** if $\rho : X \times X \to G := (G, \preccurlyeq, \prec)$ satisfies the following conditions: $\rho(x, y) = \theta$ if and only if $x = y$, and $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$.

Corollary 1. (Chang, Huang and Cho [1]). *Let (X, ρ) be a general quasi-metric space and T be a self-mapping of X. If assume that for any $x, y \in X$ $(x \neq y)$ the following fact holds*

$$\rho(Tx, Ty) \prec \sup\Big\{\rho(x, y), \rho(x, Tx), \rho(y, Ty), \rho(x, Ty), \rho(y, Tx)\Big\}, \tag{13}$$

then T has a unique fixed point in X if and only if there exists a periodic point $\xi \in X$ of T.

Proof. Let $a_n(x) = \sup\{\rho(T^i x, T^j x) : i, j \geq n\}$ for each $n \in \mathbb{N}$. This sequence is nonincreasing in G and $\theta \preccurlyeq a_n(x)$ for every $n \in \mathbb{N}$. Then, from (13) for $i, j \in \mathbb{N}$ we obtain the following facts

$$\rho(T^i x, T^j x) \prec \sup\Big\{\rho(T^{i-1}x, T^{j-1}x), \dots, \rho(T^{j-1}x, T^i x\Big\} \prec$$
$$\prec \sup\Big\{a_{n-1}(x), \dots, a_{n-1}(x)\Big\} = a_{n-1}(x),$$

i.e., $\rho(T^{i-1}(Tx), T^{j-1}(Tx)) \prec a_{n-1}(x)$ for every $x \in X$ and $x \neq Tx$. Thus, $a_{n-1}(Tx) \prec a_{n-1}(x)$ for every $x \in X$ and $x \neq Tx$. This means that the condition (B) in Theorem 1a holds for $B(x) = a_{n-1}(x)$; then X and G satisfy all the required hypotheses in Theorem 1a. Applying Theorem 1a and the preceding facts we obtain that T has a fixed point in X if and only if there exist fixed apices $u_1, \dots, u_n \in X$, i.e., from Lemmas 1 and 1a, if and only if there exists a periodic point in X. Uniqueness follows immediately from the condition (13). The proof is complete.

In connection with this, from the Principle of Symmetry, as an immediately analogy with this statement we obtain directly the following result.

Corollary 2. *Let (X, ρ) be a general quasi-metric space and T be a self-mapping of X. If assume that for every $x \in X$ there exists a positive integer $r = r(x)$ such that*

$$\rho(T^r x, T^r y) \prec \sup\Big\{\rho(x, y), \rho(x, T^r x), \rho(y, T^r y), \rho(x, T^r y), \rho(y, T^r x)\Big\}$$

for every $y \in X$, then T has a unique fixed point in X if and only if for T there exist fixed apices $u_1, \dots, u_n \in X$ for fixed integer $n \geq 2$.

Open problems. We notice that the preceding Theorems 1b, 2b, 1c and 2c are given (formulate) via our the Principle of Symmetry. Does new proofs of this statements (i.e., Theorems 1b, 2b, 1c and 2c) can be given elementary and directly without usage of the Principle of Symmetry?

4. References

[1] S.S. Chang, N. J. Huang and Y. J. Cho: *Fixed point theorems in general quasi-metric spaces and applications*, Math. Japonica, **41** (1995), 545-551.

[2] Đ. Kurepa: *Some cases in the fixed point theory*, Topology and its applications, Budva, 1972, 144-153.

[3] M. Kwapisz: *Some generalizations of an abstract contraction mapping principle*, Nonlinear Analysis, **3** (1979), 293-302.

[4] M. R. Tasković: *Characterizations of inductive posets with applications*, Proc. Amer. Math. Soc., **104** (1988), 650-659.

[5] M. R. Tasković: *Some new principles in fixed point theory*, Math. Japonica, **35** (1990), 645-666.

[6] M. R. Tasković: *The axiom of choice, fixed point theorems and inductive ordered sets*, Proc. Amer. Math. Soc., **116** (1992), 897-904.

Milan R. Tasković
Matematički fakultet
11000 Beograd, P.O. Box 550
Yugoslavia