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DYNAMICS ON $(P_{cp}(X), H_d)$ GENERATED BY A FINITE FAMILY OF MULTI-VALUED OPERATORS ON (X, d)

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Abstract. The main purpose of this paper is to give some partial answers to the following problem: If F_i, $i \in \{1, ..., m\}$ is a finite family of weakly Picard multi-valued operators, is the operator $T_F : P(X) \to P(X)$, $T_F(Y) := \bigcup_{i=1}^m F_i(Y)$ a weakly Picard operator too?

1. Introduction

Let (X, d) be a complete metric space and $P(X)$ the space of all nonempty subsets of X. Denote by $P_p(X)$ the space of all nonempty subsets of X having the property (or properties) p (where p could be cp = compact, cl = closed, b = bounded, etc.). If H_d is the Hausdorff-Pompeiu functional, it is well known that $(P_{b,cl}(X), H_d)$ is a complete metric space. If $F : X \to P(X)$ is a multi-valued operator then $x^* \in X$ is a fixed point for T if and only if $x^* \in F(x^*)$.

By definition (see [14]), the operator $F : X \to P(X)$ is a **multi-valued weakly Picard operator** (briefly m.w.P.o.) if and only if for each $x \in X$ and each $y \in F(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

i) $x_0 = x$, $x_1 = y$

ii) $x_{n+1} \in F(x_n)$, for all $n \in \mathbb{N}$

iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of F.

Let $F_1, \ldots, F_m : X \to P_{cp}(X)$ be a finite family of u.s.c. multi-valued operators. We define the following fractal operator(see [1], [6], [12]) generated by $F = (F_1, \ldots, F_m)$:

$$T_F : P_{cp}(X) \to P_{cp}(X), \quad T_F(Y) = \bigcup_{i=1}^{m} F_i(Y). \tag{1}$$

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Problem 1. (see [13]) If F_i, $i \in \{1, \ldots, m\}$ are m.w.P. operators, is T_F a w.P. operator?

The following problems are in connection with Problem 1.

Problem 1.a. If F_i, $i \in \{1, \ldots, m\}$ are generalized contractions, is T_F a generalized contraction?

Problem 1.b. If $F_i : X \to P_{cl}(X)$ are a-contractions, is $T_F : P(X) \to P(X)$ an a-contraction?

Problem 1.c. Let (X, d) be a complete generalized metric space (in Luxemburg-Jung'sense, i.e., $d(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$). If $F_i : X \to P_{cl}(X)$, $i \in \{1, \ldots, m\}$ are a-contractions, is $T_F : P(X) \to P(X)$ an a-contraction?

The purpose of this paper is to give some partial answers to these problems.

2. Main results

Let (X, d) be a metric space and $\mathcal{P}(X)$ be the space of all subsets of X. Let us consider now some functionals on $\mathcal{P}(X) \times \mathcal{P}(X)$ (see for example [2], [14]).

(i) the gap functional D defined by:

$$D : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+$$

$$D(A, B) = \begin{cases} \inf\{d(a, b) \mid a \in A,\ b \in B\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{otherwise} \end{cases}$$

(ii) the excess generalized functional ρ defined by:

$$\rho : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\},$$

$$\rho(A, B) = \begin{cases} \sup\{D(a, B) \mid a \in A\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{if } A = \emptyset \\ +\infty, & \text{if } B = \emptyset \neq A \end{cases}$$

(iii) the Hausdorff-Pompeiu generalized functional H defined by:

$$H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\},$$

$$H(A, B) = \begin{cases} \max\{\rho(A, B), \rho(B, A)\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{if } A = \emptyset = B \\ +\infty, & \text{if } A = \emptyset \neq B \text{ or } A \neq \emptyset = B. \end{cases}$$

The following lemma is an easy consequence of (iii).

Lemma 1. *Let (X, d) be a metric space and $A_k, B_k \in P(X)$ for $k \in \{1, 2, ..., m\}$. Then*

$$H\left(\bigcup_{k=1}^{m} A_k, \bigcup_{k=1}^{m} B_k\right) \leq \max\{H(A_k, B_k)|\ k \in \{1, \ldots, m\}\}.$$

Definition 2. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be:

a) a comparison function iff φ is monotone increasing and $(\varphi^n(t))_{n\in\mathbb{N}}$ converges to 0 for all $t \geq 0$.

b) a strict comparison function iff φ is a continuous comparison function and $t - \varphi(t)$ converges to $+\infty$, as $t \to +\infty$.

Let us recall now some contractivity type conditions for multi-valued operators (see [3], [4], [8], [10], [13],[14]).

Definition 3. Let (X, d) be a metric space.

i) The multi-valued operator $F : X \to P(X)$ is an a-**contraction** iff $a \in [0, 1[$ and

$$H(T(x), T(y)) \leq ad(x, y), \text{ for each } x, y \in X.$$

ii) The multi-valued operator $F : X \to P(X)$ is a φ-**contraction** iff φ is a comparison function and

$$H(T(x), T(y)) \leq \varphi(d(x, y)), \text{ for each } x, y \in X.$$

iii) The multi-valued operator $F : X \to P(X)$ is a **Meir-Keeler type operator** iff:

for each $\eta > 0$ there is $\delta > 0$ such that $\eta \leq d(x, y) < \eta + \delta$ implies $H(F(x), F(y)) < \eta$.

iv) The multi-valued operator $F : X \to P(X)$ is an ε-**locally Meir-Keeler type operator** (where $\varepsilon > 0$) iff:

for each $\eta \in]0, \varepsilon[$ there is $\delta > 0$ such that $\eta \leq d(x, y) < \eta + \delta$ implies $H(F(x), F(y)) < \eta$.

v) The multi-valued operator $F : X \to P(X)$ is **contractive** iff:

$$H(F(x), F(y)) < d(x, y), \text{ for each } x, y \in X, x \neq y.$$

If Y is a nonempty set and $f : Y \to Y$ is an operator, then a fixed point of f is an element $x^* \in Y$ such that $x^* = f(x^*)$. The set of all fixed points for the operator f will be denoted by $Fixf$.

Remark 4. *In[9] the notion of a-contraction is given for a multi-valued operator $F : X \to P_{b,cl}(X)$. For example, $F : \mathbb{R}_+ \to \mathbb{R}_+$, $F(x) = 2^{-1}x +$*

$\mathbb{R}_+$ is a multi-valued contraction in the sense of Definition 3., but is not a multivalued contraction in Nadler'sense.

Our first result is:

Theorem 5. *Let (X, d) be a metric space and $F_1, \ldots, F_m : X \to P(X)$ be a-contractions. Then the operator $T_F : (P(X), H) \to (P(X), H)$ defined by (1) is an a-contraction.*

Proof. Suppose that for each $x, y \in X$

$$H(F_i(x), F_i(y)) \leq ad(x, y), \text{ for } i \in \{1, \ldots, m\},$$

(where $a \in [0, 1[$). Then for $Y_1, Y_2 \in P(X)$ we deduce successively (see also Lemma 1)

$$H(T_F(Y_1), T_F(Y_2)) = H\left(\bigcup_{i=1}^{m} F_i(Y_1), \bigcup_{i=1}^{m} F_i(Y_2)\right) \leq$$

$$\leq \max\{H(F_1(Y_1), F_1(Y_2)), \ldots, H(F_m(Y_1), F_m(Y_2))\} \leq aH(Y_1, Y_2). \square$$

Remark 6. *See [9] for the case $F : X \to P_{b,cl}(X)$. (see also [4])*

An existence and data dependence theorem for T_F is the following:

Theorem 7. *Let (X, d) be a complete metric space and $F_i, G_i : X \to P_{cp}(X)$, $i \in \{1, \ldots, m\}$ be a-contractions. Then:*

a) $FixT_F = \{A^*\}$ and $FixT_G = \{B^*\}$.

b) If $H(F_i(x), G_i(x)) \leq \eta$ for each $x \in X$ and $i \in \{1, \ldots, m\}$ then

$$H(A^*, B^*) \leq \frac{\eta}{1-a}.$$

Proof. From Theorem 5 we have that $T_F : (P_{cp}(X), H) \to (P_{cp}(X), H)$ is an (single-valued) a-contraction. From Banach fixed point principles $FixT_F = \{A^*\}$. An identical approach can be made for T_G and hence $FixT_G = \{B^*\}$.

For the second part, we have successively:

$$H(A^*, B^*) = H(T_F(A^*), T_G(B^*)) \leq H(T_F(A^*), T_F(B^*)) +$$

$$+ H(T_F(B^*), T_G(B^*)) \leq aH(A^*, B^*) + H(T_F(B^*), T_G(B^*)).$$

On the other hand, if $u \in T_F(B^*)$ there exist $k \in \{1, \ldots, m\}$ and $a \in B^*$ such that $u \in F_k(a)$.

For this we can choose $v \in G_k(a)$ such that

$$d(u, v) \leq H(F_k(a), G_k(a)) \leq \eta.$$

Hence for each $u \in T_F(B^*)$ there is an element $v \in T_G(B^*)$ such that $d(u,v) \leq \eta$. Interchanging the roles of $T_F(B^*)$ and $T_G(B^*)$ we obtain that for each $u \in T_G(B^*)$ there is $v \in T_F(B^*)$ such that $d(u,v) \leq \eta$.

It follows that $H(T_F(B^*), T_G(B^*)) \leq \eta$ and hence $H(A^*, B^*) \leq \frac{\eta}{1-a}$.

□

Remark 8. *By definition, the set A^* is called the attractor of the system $F = (F_1, F_2, ..., F_m)$. Hence, Theorem 7 is a data dependence result of an attractor.(see also [6], [11])*

By a similar way we have:

Theorem 9. *Let (X,d) be a complete metric space and $F_i, G_i : X \to P_{cp}(X)$, $i \in \{1, \dots, m\}$ be φ-contractions, where φ is a strict comparison function. Let $t_\eta := sup\{t | t - \varphi(t) \leq \eta\}$ Then:*

a) $FixT_F = \{A^*\}$ and $FixT_G = \{B^*\}$

b) If $H(F_i(x), G_i(x)) \leq \eta$ for each $x \in X$ and $i \in \{1, \dots, m\}$ then

$$H(A^*, B^*) \leq t_\eta.$$

Another fixed point principle for the operator T_F is:

Theorem 10. *Let (X,d) be a complete metric space and $F_1, \dots, F_m : X \to P_{cl}(X)$ be a-contractions such that for each nonempty closed set Y of X, the set $F_i(Y) \in P_{cl}(X)$, for every $i \in \{1, \dots, m\}$.*

Then:

a) If there exists $A \in P_{cl}(X)$ such that $H(A, T_F(A)) < \infty$, then the operator $T_F : P_{cl}(X) \to P_{cl}(X)$, has fixed point.

b) If $H(A, T_F(A)) < \infty$, for all $A \in P_{cl}(X)$, then the operator $T_F : P_{cl}(X) \to P_{cl}(X)$ is a m.w.P.o.

Proof. If it well known that $(P_{cl}(X), H)$ is a complete generalized metric space. The conclusions follow from Theorem 5 and Luxemburg-Jung fixed point principle (see [7]). □

Some other results of this type are the following:

Theorem 11. *Let (X,d) be a complete metric space and $F_i : X \to P_{cp}(X)$, $i \in \{1, \dots, m\}$ be a finite family of multi-valued Meir-Keeler type operators. Then the operator $T_F : P_{cp}(X) \to P_{cp}(X)$ defined by (1) is a (singlevalued) Meir-Keeler type operator an has a unique fixed point.*

Proof. Let us suppose that for each $\eta > 0$ there exists $\delta > 0$ such that $\eta \le d(x,y) < \eta + \delta$ implies

$$H(F_i(x), F_i(y)) < \eta \text{ for } i \in \{1,\ldots,m\}. \tag{2}$$

From (2), it follows that F_i is contractive and hence F_i is upper semicontinuous, for $i \in \{1,\ldots,m\}$. As consequence $T_F : P_{cp}(X) \to P_{cp}(X)$.

Let us consider $\eta > 0$ and $Y_1, Y_2 \in P_{cp}(X)$ such that $\eta \le H(Y_1, Y_2) < \eta + \delta$. We will prove that $H(T_F(Y_1), T_F(Y_2)) < \eta$.

For this purpose, let $u \in T_F(Y_1)$ be arbitrary. Then there exist $k \in \{1,\ldots,m\}$ and $y_1 \in Y_1$ such that $u \in F_k(Y_1)$. For this $y_1 \in Y_1$ there is $y_2 \in Y_2$ such that $d(y_1, y_2) \le H(Y_1, Y_2) < \eta + \delta$.

If $d(y_1, y_2) \ge \eta$, then from (2) we get that $H(F_k(y_1), F_k(y_2)) < \eta$. It follows that there is $v \in F_k(y_2)$ such that $d(u,v) < \eta$ and hence $D(u, T_F(Y_2)) \le d(u,v) < \eta$.

On the other hand if $0 < d(y_1, y_2) < \eta$ the from (2) we deduce that

$$H(F_k(y_1), F_k(y_2)) < d(y_1, y_2) < \eta$$

and as before $D(u, T_F(Y_2)) < \eta$.

Because $T_F(Y_1)$ is a compact set, we have that $\rho(T_F(Y_1), T_F(Y_2)) < \eta$. Interchanging the roles of $T_F(Y_1)$ and $T_F(Y_2)$ we obtain $\rho(T_F(Y_2), T_F(Y_1)) < \eta$ and the conclusion $H(T_F(Y_1), T_F(Y_2)) < \eta$ follows.

So $T_F : P_{cp}(X) \to P_{cp}(X)$ is a Meir-Keeler type operator and by Theorem in Meir-Keeler [8] has a unique fixed point, i.e. $A^* \in P_{cp}(X)$ such that $T(A^*) = A^*$. □

Corollary 12. *Let (X,d) be a complete metric space and $F_i : X \to P_{cp}(X)$, $i \in \{1,\ldots,m\}$ be a finite family of multi-valued Meir-Keeler type operators. Then the operator $T_F : P_{cp}(X) \to P_{cp}(X)$ is a w.P.o.*

Proof. The conclusion follows from Theorem 11 and the Theorem in Meir-Keeler [8]. □

Recall that a metric space (X,d) is said to be ε-**chainable** if for every $x, y \in X$ there exists a finite set of elements $x = x_0, x_1, \ldots, x_n = y$ in X such that $d(x_{k-1}, x_k) < \varepsilon$ for $k \in \{1,\ldots,n\}$.

Theorem 13. *Let (X,d) be a complete ε-chainable metric space (where $\varepsilon > 0$) and $F_i : X \to P_{cp}(X)$, $i \in \{1,\ldots,m\}$ be a finite family of multi-valued ε-locally Meir-Keeler type operators.*

Then the operator $T_F : P_{cp}(X) \to P_{cp}(X)$ given by (1) is an (singlevalued) ε-locally Meir-Keeler type operator, having a fixed point.

Proof. The proof runs exactly as in Theorem 11, but instead of using Theorem in Meir-Keeler [8], the conclusion follows from Proposition 1 in M.K. Xu [15]. □

Using an ε-locally Boyd-Wong type condition (see [3] and [15]) one can also prove:

Theorem 14. *Let (X, d) be a complete ε-chainable metric and let $F_i : X \to P_{cp}(X)$, $i \in \{1, \ldots, m\}$ be multi-valued operators such that*

$$H(F_i(x), F_i(y)) \leq k(d(x, y))d(x, y), \text{ for all } x, y \in X \tag{3}$$

with $0 < d(x, y) < \varepsilon$, where $k : (0, \infty) \to (0, 1)$ is a real function with the property:

$$\text{(P)} \quad \begin{cases} \text{For each } 0 < t < \varepsilon \text{ there exist } e(t) > 0 \text{ and } s(t) < 1 \\ \text{such that } k(r) \leq s(t) \text{ provided } t \leq r < t + e(t). \end{cases}$$

Then, the operator $T_F : P_{cp}(X) \to P_{cp}(X)$ given by (1) satisfy the condition:

$$H(T_F(Y_1), T_F(Y_2)) \leq k(H(Y_1, Y_2))H(Y_1, Y_2),$$

for all $Y_1, Y_2 \in P_{cp}(X)$ with $0 < H(Y_1, Y_2) < \varepsilon$ and has a fixed point.

Proof. Let $Y_1, Y_2 \in P_{cp}(X)$ such that $0 < H(Y_1, Y_2) < \varepsilon$. Then

$$H(T_F(Y_1), T_F(Y_2)) \leq \max\{H(F_k(Y_1), F_k(Y_2)) \mid k \in \{1, \ldots, m\}\} \leq$$
$$\leq k(H(Y_1, Y_2))H(Y_1, Y_2).$$

The conclusion follows now from Theorem 2 in H.K. Xu [15]. □

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