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# TAUBERIAN THEOREMS FOR CONVERGENCE AND SUBSEQUENTIAL CONVERGENCE WITH MODERATELY OSCILLATORY BEHAVIOR

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**Abstract.** In the classical Tauberian theory, the main objective is to obtain convergence of a swquence $\{u_n\}$ by imposing conditions about the oscillatory behavior of $\{u_n\}$ in addition to the existence of certain continuous limits. However, there are some conditions of considerable interest from which it is not possible to obtain convergence of $\{u_n\}$. This situation motivates a different kind of Tauberian theory where we do not look for convergence recovery of $\{u_n\}$, rather we are concerned with the subsequential behavior of the sequence $\{u_n\}$. The first section includes definitions, notations and an overview of classical results. Succinct proofs of the Hardy-Littlewood theorem and the generalizad Littlewood theorem are given using the corollary to Karamata's Hauptsatz. In the second section subsequential Tauberian theory is introduced and some related Tauberian theorems are proved. Finally, in the last section we study convergence and subsequential convergence of regularly generated sequences.

## 1. Introduction

### 1.1 Definitions and notations

The classical Tauberian theory studies convergence of sequences $\{u_n\}$ for which

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}u_nx^n \tag{1}$$

exists. If we denote the class of all sequences $\{u_n\}$ for which (1) exists by $\mathcal{U}$, then the main objective of the theory is identifying subclasses $\mathcal{U}_c$ of $\mathcal{U}$ such

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that $\{u_n\} \in \mathcal{U}_c$ implies

$$\lim_n u_n = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} u_n x^n. \tag{2}$$

The subclasses $\mathcal{U}_c$ are defined by requiring additional information about $\{u_n\} \in \mathcal{U}$. The Kronecker identities

$$\begin{aligned} u_n - \sigma_n^{(1)}(u) &= V_n^{(0)}(\Delta u) \\ \sigma_n^{(m-1)}(u) - \sigma_n^{(m)}(u) &= V_n^{(m-1)}(\Delta u), \end{aligned} \tag{3}$$

for $m \geq 1$, where $\sigma_n^{(1)}(u) = \frac{1}{n+1}\sum_{k=0}^{n} u_k$, $\sigma_n^{(0)}(u) = u_n$, $V_n^{(0)}(\Delta u) = \frac{1}{n+1}\sum_{k=0}^{n} k\Delta u_k$, $u = \{u_n\}$ and $\Delta u_n = u_n - u_{n-1}$, $u_{-1} = 0$, are extensively used in the classical and neoclassical theory. From (3) and

$$\sigma_n^{(1)}(u) = \sum_{k=1}^{n} \frac{V_k^{(0)}(\Delta u)}{k} + u_0 \tag{4}$$

we have the following representation of $\{u_n\}$

$$\{u_n\} = \{V_n^{(0)}(\Delta u) + \sum_{k=1}^{n} \frac{V_k^{(0)}(\Delta u)}{k} + u_0\} \tag{5}$$

in terms of the sequence $\{V_n^{(0)}(\Delta u)\}$. The closer look at the above identity gives an idea how to set conditions on the generator sequence $\{V_n^{(0)}(\Delta u)\}$ of $\{u_n\}$ and consequently how to define subclasses of $\mathcal{U}$ for which (2) holds. For example, Tauber [1] proved that, if for a sequence $\{u_n\} \in \mathcal{U}$ we assume that

$$V_n^{(0)}(\Delta u) = o(1), \quad n \to \infty, \tag{6}$$

then (2) holds. That is, the condition (6) defines a subclass of $\mathcal{U}$ for which (2) holds. Since $n\Delta u_n = o(1)$, $n \to \infty$ implies (6), we see that determining of subclasses $\mathcal{U}_c$ of $\mathcal{U}$ in the early theory is obtained by restricting the order of magnitude of the sequence $\{n\Delta u_n\}$. Littlewood [2] succeeded in replacing $n\Delta u_n = o(1)$, $n \to \infty$ by $n\Delta u_n = O(1)$, $n \to \infty$, and together with (1) he obtained (2). However, the condition

$$V_n^{(0)}(\Delta u) = O(1), \quad n \to \infty \tag{7}$$

alone is not sufficient to construct a subclass $\mathcal{U}_c$. Even

$$V_n^{(0)}(|\Delta u|) = \frac{1}{n+1}\sum_{k=0}^{n} k\,|\Delta u_k| = O(1), \quad n \to \infty \tag{8}$$

is not sufficient as shown in [3, 4, 5, 12]. But it is shown in [3, 4, 5] that (8) and the existence of

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n \tag{9}$$

imply

$$\lim_n \sigma_n^{(1)}(u) = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n, \tag{10}$$

and consequently $u_n = O(1)$, $n \to \infty$.

In addition to (7), if we assume that for every positive $\varepsilon$ there exists a positive integer $n_0(\varepsilon)$ and $\delta(\varepsilon) > 0$ such that for all integers $n \geq n_0(\varepsilon)$ and every $k \in \{1, 2, 3, \ldots, [n_0(\varepsilon).\delta(\varepsilon)]\}$

$$\left|V_{n+k}^{(0)}(\Delta u) - V_n^{(0)}(\Delta u)\right| < \varepsilon \tag{11}$$

then together with the existence of (1) we have (2). The above concept found by Landau [6] is known as slow-oscillation of $\{V_n^{(0)}(\Delta u)\}$. From the representation (5), we observe that $\{V_n^{(0)}(\Delta u)\}$ is bounded and slowly-oscillating if and only if $\{u_n\}$ is slowly-oscillating. See [11] for the proof. Later Schmidt [7] defined the slow-oscillation of $\{u_n\}$ as follows:

A sequence $\{u_n\}$ is slowly-oscillating if

$$\lim_{\substack{N>M \\ \frac{N}{M}\to 1}}(u_N - u_M) = 0. \tag{12}$$

Since (12) is a restricted Cauchy property of $\{u_n\}$, it is clear that every convergent sequence is slowly-oscillating. But the converse is not true. For example, the sequence $\{\sum_{k=1}^{n}\frac{1}{k}\}$ is slowly-oscillating, but it does not converge. From the definition of slowly-oscillating sequences, it is also clear that if we have a slowly-oscillating sequence $\{u_n\}$, then $\Delta u_n = o(1)$, $n \to \infty$.

Throughout this work a different definition of slow-oscillation is better tailored for our purposes. In [8], the following definition of slow-oscillation is proposed:

**Definition 1.** *A sequence $\{u_n\}$ is slowly-oscillating if*

$$\lim_{\lambda\to 1^+}\overline{\lim_n}\max_{n+1\leq k\leq[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta u_j\right| = 0.$$

From this definition and the representation (5), it follows that the sequence $\{e^{u_n}\}$ where $\{u_n\}$ is slowly-oscillating has a special property given by Karamata in [9] as follows:

**Definition 2.** *A positive sequence $\{R(n)\}$ is regularly varying if for $\lambda > 1$*

$$\lim_n \frac{R[\lambda n]}{R(n)} = \lambda^\rho, \quad \rho \geq 0.$$

Consider a slowly-oscillating sequence $\{u_n\}$, i.e., $u_n = B_n + \sum_{k=1}^{n} \frac{B_k}{k}$ where $\{B_n\}$ is bounded and slowly-oscillating. Let $R(n) = e^{u_n}$. Then

$$\lim_n \frac{R[\lambda n]}{R(n)} = e^{\lim_n (B_{[\lambda n]} - B_n + \sum_{k=n+1}^{[\lambda n]} \frac{B_k}{k})} =$$

$$e^{\lim_n (B_{[\lambda n]} - B_n + C.\lg \frac{[\lambda n]}{n})} = e^{0 + C \lg \lambda} = \lambda^C.$$

That is, slowly-oscillating sequences are logarithms of certain regularly varying sequences.

With Definition 1. it is also easy to show that the Hardy-Littlewood [10] condition

$$V_n^{(0)}(|\Delta u|, p) = \frac{1}{n+1} \sum_{k=0}^{n} k^p |\Delta u_k|^p = O(1), \quad n \to \infty, \quad p > 1 \tag{13}$$

implies that $\{u_n\}$ is slowly-oscillating. The proof is given in [11]. Recall that if $\{u_n\}$ is slowly-oscillating and the limit (1) exists, then (2) holds. That is, the Hardy-Littlewood condition for $p > 1$ defines another subclass $\mathcal{U}_c$ of $\mathcal{U}$. However, for $p = 1$ (13) is no longer a Tauberian condition, i.e., a condition on $\{u_n\}$ that together with (1) implies (2). In other words, the existence of the limit (1) and

$$V_n^{(0)}(|\Delta u|) = \frac{1}{n+1} \sum_{k=0}^{n} k |\Delta u_k| = O(1), \quad n \to \infty$$

do not imply (2). Rényi [12] showed that if $\lim_n V_n^{(0)}(|\Delta u|)$ exists, then $\lim_n u_n = \lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} u_n x^n$. In this case, the sequence $\{V_n^{(0)}(|\Delta u|, p)\}$ for $p > 1$ does not have to be bounded as shown in the following example by Rényi:

Let

$$\Delta u_k = \begin{cases} \dfrac{(-1)^n}{n} & \text{if} \quad k = 2^n, \quad n = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

The sequence $\{u_n\}$ converges by Leibnitz's test, and so $\lim_{x \to 1^-} \sum_{k=0}^{\infty} \Delta u_k x^k$ exists.

Now we show that $\lim_n V_n^{(0)}(|\Delta u|)$ exists and in particular for this example that the limit is zero. It is enough to consider $V_n^{(0)}(|\Delta u|)$ for indices $n = 2^s$:

$$V_{2^s}^{(0)}(|\Delta u|) = \frac{1}{2^s}\sum_{j=1}^{s}\frac{2^j}{j} \le \frac{1}{2^s}\sum_{j=1}^{[s/2]} 2^j + \frac{1}{2^s[s/2]}\sum_{j=[s/2]+1}^{s} 2^j$$

$$= O(2^{s/2}.\frac{1}{2^s} + \frac{2^s}{2^s.s}) = o(1), \quad s \to \infty.$$

But for $p > 1$

$$\frac{1}{2^s}\sum_{j=1}^{n}\frac{2^{jp}}{j^p} \ge \frac{1}{2^s.s^p}\sum_{j=1}^{s} 2^{jp} \ge \frac{2^{sp}}{2^s.s^p} \to \infty.$$

That is, the sequence $\{u_n\}$ is convergent even though $\{V_n^{(0)}(|\Delta u|\,, p)\}$ is not bounded for $p > 1$.

As we mentioned on page 4, the Hardy-Littlewood condition for $p = 1$ (8) is not a Tauberian condition for the convergence recovery of $\{u_n\}$. It is shown in [3, 4, 5] that (8) is a Tauberian condition for the recovery of convergence of $\{\sigma_n^{(1)}(u)\}$ from the existence of $\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n$. Also Renyi [12] showed that sequences satisfying the condition (8) are not slowly-oscillating but it is possible that they are an extension of the slowly-oscillating sequences. Indeed, consider $\max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta u_j\right|$, compute and estimate as follows:

$$\max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta u_j\right| \le \max_{n+1\le k\le[\lambda n]}\sum_{j=n+1}^{k}|\Delta u_j|$$

$$\le \sum_{j=n+1}^{[\lambda n]}|\Delta u_j| = \sum_{j=n+1}^{[\lambda n]}\frac{j\,|\Delta u_j|}{j} \le \frac{1}{n+1}\sum_{j=n+1}^{[\lambda n]} j\,|\Delta u_j| \le \frac{1}{n+1}\sum_{j=1}^{[\lambda n]} j\,|\Delta u_j|$$

$$\le \frac{1}{n+1}\frac{[\lambda n]}{[\lambda n]}\sum_{j=1}^{[\lambda n]} j\,|\Delta u_j| = \frac{[\lambda n]}{n+1}\frac{1}{[\lambda n]}\sum_{j=1}^{[\lambda n]} j\,|\Delta u_j|.$$

Taking limsup in $n$ of both sides we obtain,

$$\overline{\lim_n}\max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta u_j\right| \le \overline{\lim_n}\frac{[\lambda n]}{n+1}\frac{1}{[\lambda n]}\sum_{j=1}^{[\lambda n]} j\,|\Delta u_j|$$

$$\le \overline{\lim_n}\frac{[\lambda n]}{n+1}\overline{\lim_n}\frac{1}{[\lambda n]}\sum_{j=1}^{[\lambda n]} j\,|\Delta u_j| \le \lim_n\frac{[\lambda n]}{n+1}\overline{\lim_n}\frac{1}{[\lambda n]}\sum_{j=1}^{[\lambda n]} j\,|\Delta u_j|$$

$$\le \lambda.C < \infty,$$

where $C$ is an absolute constant that comes from (8).

The important special case of the Hardy-Littlewood condition (8) motivates the following definition [8].

**Definition 3.** *A sequence $\{u_n\}$ is moderately oscillatory if*

$$\overline{\lim_n} \max_{n+1\le k\le[\lambda n]} \left| \sum_{j=n+1}^{k} \Delta u_j \right| < \infty, \quad \lambda > 1.$$

In the case of moderately oscillatory behavior of $\{u_n\}$, we cannot conclude in general (2), however we can obtain some information about subsequential behavior of $\{u_n\}$. This will be discussed in details in the following sections.

The definition of slow-oscillation in [8] is designed to suit a version of de la Vallee Poussin mean of $\{u_n\}$, [8],

$$\tau_{n,[\lambda n]} = \tau_n(\lambda, u_n) = \frac{1}{[\lambda n]-n} \sum_{k=n+1}^{[\lambda n]} u_k, \quad \lambda > 1$$

and its relations to $\{u_n\}$ and $\{\sigma_n^{(1)}(u)\}$.

For $\lambda > 1$ we have

$$u_n = \tau_{n,[\lambda n]} - \frac{1}{[\lambda n]-n} \sum_{k=n+1}^{[\lambda n]} (u_k - u_n) = \tau_{n,[\lambda n]} - \frac{1}{[\lambda n]-n} \sum_{k=n+1}^{[\lambda n]} \sum_{j=n+1}^{k} \Delta u_j$$

$$= \tau_{n,[\lambda n]} - \sigma_n^{(1)}(u) + \sigma_n^{(1)}(u) - \frac{1}{[\lambda n]-n} \sum_{k=n+1}^{[\lambda n]} \sum_{j=n+1}^{k} \Delta u_j.$$

From

$$([\lambda n]+1)\sigma_{[\lambda n]}^{(1)}(u) - (n+1)\sigma_n^{(1)}(u) = \sum_{k=n+1}^{[\lambda n]} u_k,$$

we have

$$\frac{1}{[\lambda n]-n} \sum_{k=n+1}^{[\lambda n]} u_k = \tau_{n,[\lambda n]} = \frac{([\lambda n]+1)\sigma_{[\lambda n]}^{(1)}(u) - (n+1)\sigma_n^{(1)}(u)}{[\lambda n]-n}$$

and

$$\tau_{n,[\lambda n]} - \sigma_n^{(1)}(u) = \frac{[\lambda n]+1}{[\lambda n]-n}(\sigma_{[\lambda n]}^{(1)}(u) - \sigma_n^{(1)}(u)).$$

This is an important relation between $\{\tau_{n,[\lambda n]}(u)\}$ and $\{\sigma_n^{(1)}(u)\}$. From

$$u_n = \tau_{n,[\lambda n]} - \sigma_n^{(1)}(u) + \sigma_n^{(1)}(u) - \frac{1}{[\lambda n]-n} \sum_{k=n+1}^{[\lambda n]} \sum_{j=n+1}^{k} \Delta u_j,$$

we obtain the first important identity

$$(14)\quad u_n=\sigma_n^{(1)}(u)+\frac{[\lambda n]+1}{[\lambda n]-n}(\sigma_{[\lambda n]}^{(1)}(u)-\sigma_n^{(1)}(u))-\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}\Delta u_j.$$

To obtain the second important identity, for $1<\lambda<2$ consider our version of de la Vallee Poussin mean

$$\tau_{n-[(\lambda-1)n],n}=\tau_{n-[(\lambda-1)n],n}(u)=\frac{1}{[(\lambda-1)n]+1}\sum_{k=n-[(\lambda-1)n]}^{n}u_k.$$

Notice that $n-[(\lambda-1)n]<n$ because $1<\lambda<2$, and compute

$$u_n=\tau_{n-[(\lambda-1)n],n}+\frac{1}{[(\lambda-1)n]+1}\sum_{k=n-[(\lambda-1)n]}^{n}(u_n-u_k)$$

$$=\tau_{n-[(\lambda-1)n],n}+\frac{1}{[(\lambda-1)n]+1}\sum_{k=n-[(\lambda-1)n]}^{n}\sum_{j=k+1}^{n}\Delta u_j.$$

Observe that

$$\frac{(n+1)\sigma_n^{(1)}(u)-(n-[(\lambda-1)n])\sigma_{n-[(\lambda-1)n]-1}^{(1)}(u)}{[(\lambda-1)n]+1}=$$

$$\frac{1}{[(\lambda-1)n]+1}\sum_{k=n-[(\lambda-1)n]}^{n}u_k$$

and

$$\tau_{n-[(\lambda-1)n],n}-\sigma_{n-[(\lambda-1)n]-1}^{(1)}(u)$$

$$=\frac{(n+1)\sigma_n^{(1)}(u)-(n-[(\lambda-1)n]+[(\lambda-1)n]+1)\sigma_{n-[(\lambda-1)n]-1}^{(1)}(u)}{[(\lambda-1)n]+1}$$

$$=\frac{(n+1)}{[(\lambda-1)n]+1}(\sigma_n^{(1)}(u)-\sigma_{n-[(\lambda-1)n]-1}^{(1)}(u)).$$

From

$$u_n=\tau_{n-[(\lambda-1)n],n}-\sigma_{n-[(\lambda-1)n]-1}^{(1)}(u)$$

$$+\sigma_{n-[(\lambda-1)n]-1}^{(1)}(u)+\frac{1}{[(\lambda-1)n]+1}\sum_{k=n-[(\lambda-1)n]}^{n}\sum_{j=k+1}^{n}\Delta u_j,$$

it follows that

$$(15)\quad \begin{aligned} u_n&=\sigma_{n-[(\lambda-1)n]-1}^{(1)}(u)-\frac{n+1}{[(\lambda-1)n]+1}(\sigma_{n-[(\lambda-1)n]-1}^{(1)}(u)-\sigma_n^{(1)}(u))\\ &+\frac{1}{[(\lambda-1)n]+1}\sum_{k=n-[(\lambda-1)n]}^{n}\sum_{j=k+1}^{n}\Delta u_j,\end{aligned}$$

the second important identity. This identity is similar to the identity obtained in [11] and it is also obtained by the author in the Graduate Research Seminar [21].

## 1.2 Overview of classical theory

In the classical Tauberian theory, the main objective is determining subclasses $\mathcal{U}_c$ of $\mathcal{U}$ by setting certain conditions on $\{\Delta u_n\}$ such that (2) holds. For example, the condition

$$n\Delta u_n = o(1), \quad n \to \infty \tag{16}$$

where $\{u_n\} \in \mathcal{U}$ implies $\lim_n u_n = \lim_{x\to 1^-} (1-x) \sum_{n=0}^{\infty} u_n x^n$. This is a corollary to the original Tauber theorem [1]. The following proof of this result can be found in [4].

For every $\varepsilon > 0$, choose a positive integer $N$ such that for every $n > N$

$$|n\Delta u_n| < \frac{\varepsilon}{3}, \tag{17}$$

$$\frac{1}{n}\sum_{k=1}^{n} |k\Delta u_k| < \frac{\varepsilon}{3} \tag{18}$$

and

$$\left|\sum_{k=0}^{\infty} \Delta u_k (1-\frac{1}{n})^k - s\right| < \frac{\varepsilon}{3}, \tag{19}$$

where $\lim_{x\to 1^-} \sum_{k=0}^{\infty} \Delta u_k x^k = s$. Consider the difference

$$\sum_{k=0}^{n} \Delta u_k - s = \sum_{k=0}^{\infty} \Delta u_k x^k - s + \sum_{k=1}^{n} \Delta u_k (1-x^k) - \sum_{k=n+1}^{\infty} \Delta u_k x^k$$

for every $n > N$ and $x < 1$. Observe that $1-x^k = (1-x)(1+x+\ldots+x^{k-1}) \leq k(1-x)$ and $|\Delta u_k| = \frac{|k\Delta u_k|}{k} < \frac{\varepsilon}{3k}$. Next, for all $x$ in $(0,1)$, we have

$$\left|\sum_{k=0}^{n} \Delta u_k - s\right| \leq \left|\sum_{k=0}^{\infty} \Delta u_k x^k - s\right| + (1-x)\sum_{k=1}^{n} |k\Delta u_k| + \frac{\varepsilon}{3n(1-x)}.$$

Since (17), (18), and (19) hold, we may choose $x = 1 - \dfrac{1}{n}$ to obtain that $\left|\sum_{k=0}^{n} \Delta u_k - s\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$ for every $n > N$. This completes the proof that (16) and the existence of the limit (1) imply (2). Different proofs can be found in [8, 11, 13, 14].

A generalization of (16) was given by Tauber [1]. He proved that if for any sequence $\{u_n\} \in \mathcal{U}$ we assume

$$V_n^{(0)}(\Delta u) = o(1), \quad n \to \infty, \tag{20}$$

then (2) holds. The proof is given in [14] as follows:

Denote $v_n = \sum_{k=1}^{n} k\Delta u_k = (n+1)V_n^{(0)}(\Delta u)$. We have

$$\begin{aligned}
&(1-x)\sum_{k=1}^{\infty}\frac{v_k}{k+1}x^k - \frac{v_n}{n+1} + \sum_{k=1}^{n}\frac{v_k}{k(k+1)}(x^k-1) + \sum_{k=n+1}^{\infty}\frac{v_k}{k(k+1)}x^k \\
&= \frac{\Delta u_1}{2}x + \sum_{k=2}^{\infty}(\frac{v_k}{k+1} - \frac{v_{k-1}}{k})x^k - \frac{v_n}{n+1} - \sum_{k=1}^{n}\frac{v_k}{k(k+1)} + \sum_{k=1}^{\infty}\frac{v_k}{k(k+1)}x^k \\
&= \frac{\Delta u_1}{2}x + \sum_{k=2}^{\infty}\frac{(k+1)(v_k - v_{k-1}) - v_k}{k(k+1)}x^k - \\
&\quad \frac{v_n}{n+1} - \sum_{k=1}^{n}\frac{v_k}{k(k+1)} + \sum_{k=1}^{\infty}\frac{v_k}{k(k+1)}x^k \\
&= \frac{\Delta u_1}{2}x + \sum_{k=2}^{\infty}\Delta u_k x^k + \frac{\Delta u_1}{2}x - \frac{v_n}{n+1} - \sum_{k=1}^{n}\frac{v_k}{k(k+1)} \\
&= \sum_{k=1}^{\infty}\Delta u_k x^k - \frac{v_n}{n+1} - \sum_{k=1}^{n}v_k(\frac{1}{k} - \frac{1}{k+1}) \\
&= \sum_{k=1}^{\infty}\Delta u_k x^k - \frac{v_n}{n+1} - \Delta u_1 - \sum_{k=1}^{n-1}\frac{v_k - v_{k+1}}{k+1} + \frac{v_n}{n+1} \\
&= \sum_{k=1}^{\infty}\Delta u_k x^k - \Delta u_1 - \sum_{k=1}^{n-1}\frac{\Delta u_{k+1}(k+1)}{k+1} = \sum_{k=0}^{\infty}\Delta u_k x^k - \sum_{k=0}^{n}\Delta u_k.
\end{aligned}$$

Hence we obtain the following identity:

$$\begin{aligned}
\sum_{k=0}^{\infty}\Delta u_k x^k - \sum_{k=0}^{n}\Delta u_k &= (1-x)\sum_{k=1}^{\infty}\frac{v_k}{k+1}x^k - \frac{v_n}{n+1} + \\
&\sum_{k=1}^{n}\frac{v_k}{k(k+1)}(x^k-1) + \sum_{k=n+1}^{\infty}\frac{v_k}{k(k+1)}x^k
\end{aligned} \tag{21}$$

Without loss of generality, assume that $\lim_{x\to 1^-}\sum_{k=0}^{\infty}\Delta u_k x^k = 0$. Since $v_n = o(n)$, $n \to \infty$ and $\left|x^k - 1\right| \leq (1-x)k$, it follows from (21) that for each $\varepsilon > 0$, for some positive integer $N = N(\varepsilon)$, and $n \geq n_0$,

$$\left|\sum_{k=0}^{\infty} \Delta u_k x^k - \sum_{k=0}^{n} \Delta u_k\right| \leq C(1-x)\sum_{k=1}^{N} x^k + \varepsilon(1-x)\sum_{k=1}^{\infty} x^k +$$

$$\varepsilon + C(1-x)N + \varepsilon n(1-x) + \frac{\varepsilon}{n+1}\sum_{k=1}^{\infty} x^k$$

$$\leq 2CN(1-x) + 2\varepsilon + \varepsilon n(1-x) + \frac{\varepsilon}{(n+1)(1-x)}$$

where $C$ is an absolute constant from (20). As before, we may choose $1-x = \frac{1}{n}$, then the sum of the second, third and fourth terms in the last inequality reduce to $4\varepsilon$, and the first term becomes $\frac{2NC}{n}$. Hence

$$\left|\sum_{k=0}^{\infty} \Delta u_k (1-\frac{1}{n})^k - \sum_{k=0}^{n} \Delta u_k\right| \leq 2\frac{NC}{n} + 4\varepsilon$$

Recalling that $\lim_n \sum_{k=0}^{\infty} \Delta u_k (1-\frac{1}{n})^k = 0$, we have $\overline{\lim_n} \left|\sum_{k=0}^{n} \Delta u_k x^k\right| \leq 4\varepsilon$, and since $\varepsilon$ can be made arbitrarily small, we finally obtain $\lim_n \sum_{k=0}^{n} \Delta u_k = 0$.

After the original Tauber Theorem many significant generalizations of sufficient conditions for the convergence recovery of $\{u_n\}$ have been obtained. One of them is Littlewood's [2] result. He considerably weakened the condition (16) to obtain the following theorem:

**Theorem 1.** (Littlewood [2]) *Let* $\{u_n\} \in \mathcal{U}$ *and*

$$n\Delta u_n = O(1), \quad n \to \infty. \tag{22}$$

*Then* $\lim_n u_n = \lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} u_n x^n$.

Hardy and Littlewood [10] conjectured that a generalization of (22),

$$V_n^{(0)}(|\Delta u|, p) = O(1), \quad n \to \infty, \quad p > 1, \tag{23}$$

might be a Tauberian condition. Szsz [15] proved that Hardy and Littlewood's conjecture was correct. However Szsz's proof is rather complicated. Using Landau's definition of slow-oscillation [6], the proof becomes simple by showing that the condition (23) implies the slow- oscillation of $\{u_n\}$. Glancing through the references, one wonders about the time gap between Landau's publication [6] of his very general Tauberian condition slow-oscillation and the Hardy-Littlewood conjecture [10].

Assuming that $\{u_n\}$ is slowly-oscillating and $\{u_n\} \in \mathcal{U}$, Schmidt [7] tried to prove a general Tauberian theorem. Vijayaraghavan [16] gave a corrected proof of so-called generalized Littlewood Tauberian theorem.

**Theorem 2.** (Generalized Littlewood Tauberian Theorem) *Let* $\{u_n\} \in \mathcal{U}$. *If*

$$\lim_{\substack{N>M \\ \frac{N}{M}\to 1}} (u_N - u_M) = 0$$

*then (2) holds.*

Notice that all previous conditions, namely (16), (22), and (23) are special cases of the slow-oscillation.

Landau [6] discovered that the oscillatory behavior of $\{u_n\}$ can be controlled by one-sided boundedness of the sequence $\{n\Delta u_n\}$. That is, $\{n\Delta u_n\}$ is one sidedly bounded if for some $C \geq 0$ and all positive integers $n$, $n\Delta u_n \geq -C$. Clearly, one sided boundedness is applicable to real sequences.

**Theorem 3.** (Landau [6]) *For a real* $\{u_n\}$, *let* $\lim_n \sigma_n^{(1)}(u)$ *exist and*

$$n\Delta u_n \geq -C \tag{24}$$

*for some* $C \geq 0$ *and all* $n$. *Then* $\lim_n u_n = \lim_n \sigma_n^{(1)}(u)$.

One sided boundness is one of many legacies of the classical Tauberian theory. Hardy and Littlewood [10] proved that the existence of the limit (1) and the condition (24) imply (2). That is, they generalized Theorem 3. The proof of Hardy-Littlewood's [10] theorem is complicated. A simpler proof will be given later in this section.

Karamata's [17] significant corollary to his Hauptsatz led to various generalizations of the classical Tauberian theorems and provided a deeper insight into the classical Tauberian theory.

**Theorem 4.** (Karamata [17]) *For a real* $\{u_n\}$, *let the limit (1) exist. If for some* $C \geq 0$ *and all nonnegative integers* $n$

$$u_n \geq -C$$

*then*

$$\lim_n \sigma_n^{(1)}(u) = \lim_{x\to 1^-} (1-x) \sum_{n=0}^{\infty} u_n x^n. \tag{25}$$

In 1952, Wielandt [18] proved a version of this theorem. For the original proof see [11].

The proof of the generalized Littlewood Tauberian Theorem in [16] is very long and complicated. However, using a version of corollary to Karamata's Hauptsatz, the proof becomes quite transparent. To this end, we shall use the following theorem.

**Theorem 5.** (Corollary to Karamata's Hauptsatz) *For a real sequence* $\{B_n\}$, *let*

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} B_n x^n = 0 \tag{26}$$

*If for some* $C\geq 0$ *and all nonnegative integers* $n$ $B_n \geq -C$, *then* $\lim_n \sigma_n^{(1)}(B) = 0$.

## The proof of the generalized Littlewood Tauberian Theorem

**Proof 1.** (The proof for real case). Since $\{u_n\}$ is slowly-oscillating, the generator sequence $\{V_n^{(0)}(\Delta u)\}$ is slowly-oscillating and bounded, and therefore there exists $C \geq 0$ such that $V_n^{(0)}(\Delta u) \geq -C$ for all nonnegative integers $n$. From the existence of the limit (1), it follows that

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}u_n x^n$$

and

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}V_n^{(0)}(\Delta u)x^n = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}(u_n-\sigma_n^{(1)}(u))x^n = 0.$$

Then by the corollary to Karamata's Hauptsatz, we have

$$\sigma_n^{(1)}(V^{(0)}(\Delta u)) = \frac{1}{n+1}\sum_{k=0}^{n}V_k^{(0)}(\Delta u) = V_n^{(1)}(\Delta u) = o(1), \quad n\to\infty.$$

Since $\{V_n^{(0)}(\Delta u)\}$ is slowly-oscillating and $(C,1)$-summable, from the identity (14), it follows that $V_n^{(0)}(\Delta u) = o(1)$, $n\to\infty$. Using the identity

$$\sigma_n^{(1)}(u) - \sigma_n^{(2)}(u) = n(\sigma_n^{(2)}(u) - \sigma_{n-1}^{(2)}(u)) = V_n^{(1)}(\Delta u)$$

we have that $\lim_n \sigma_n^{(2)}(u) = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n$ and consequently $\lim_n \sigma_n^{(1)}(u)$ exists. Finally from the Kronecker identity $u_n - \sigma_n^{(1)}(u) = V_n^{(0)}(\Delta u)$, it follows that

$$\lim_n u_n = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}u_n x^n.$$

**Proof 2.** (The proof for complex case) Since $\{u_n\}$ is slowly-oscillating, $\{V_n^{(0)}(\Delta u)\}$ is bounded and slowly-oscillating, i.e., $u_n - \sigma_n^{(1)}(u) = n\Delta\sigma_n^{(1)}(u) = O(1)$, $n\to\infty$. The existence of the limit (1) implies that

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}u_n x^n.$$

Then by the Littlewood Theorem, we obtain

$$\lim_n \sigma_n^{(1)}(u) = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n.$$

Since $\{u_n\}$ is slowly-oscillating and $(C,1)$-summable, again from the identity (6) it follows that $\lim_n u_n = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} u_n x^n$.

Similarly, using the corollary to Karamata's Hauptsatz, the proof of the Hardy-Littlewood Theorem is much more simpler than the original proof.

**Theorem 6.** (Hardy-Littlewood [10]) *For a real sequence $\{u_n\}$, let the limit (1) exist. If for some $C \geq 0$ and all nonnegative integers $n$, $n\Delta u_n \geq -C$, then $\lim_n u_n = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} u_n x^n$.*

**Proof.** Since $\{n\Delta u_n\}$ is one sidedly bounded, we have that $V_n^{(0)}(\Delta u) \geq -C$ for some $C \geq 0$ and all nonnegative integers n. From the existence of the limit (1), it follows that

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} V_n^{(0)}(\Delta u)x^n = 0.$$

By the corollary to Karamata's Hauptsatz, we obtain that

$$V_n^{(1)}(\Delta u) = o(1), \quad n\to\infty.$$

Using the identity $\sigma_n^{(1)}(u) - \sigma_n^{(2)}(u) = n\Delta\sigma_n^{(2)}(u) = V_n^{(1)}(\Delta u)$, we have

$$n\Delta\sigma_n^{(2)}(u) = o(1), \qquad n\to\infty.$$

By the corollary to the original Tauber theorem, we obtain

$$\lim_n \sigma_n^{(2)}(u) = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} u_n x^n$$

and consequently

$$\lim_n \sigma_n^{(1)}(u) = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} u_n x^n$$

Next we need following identities that were introduced in the previous section:

for $\lambda > 1$

$$\text{(i)}\quad u_n = \sigma_n^{(1)}(u) + \frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]}^{(1)}(u) - \sigma_n^{(1)}(u)\right) - \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}\Delta u_j$$

for $1 < \lambda < 2$

$$u_n = \sigma^{(1)}_{n-[(\lambda-1)n]-1}(u) - \frac{n+1}{[(\lambda-1)n]+1}(\sigma^{(1)}_{n-[(\lambda-1)n]-1}(u) - \sigma^{(1)}_n(u))$$

(ii)
$$+\frac{1}{[(\lambda-1)n]+1}\sum_{k=n-[(\lambda-1)n]}^{n}\sum_{j=k+1}^{n}\Delta u_j$$

From the identity (i) we get

$$\overline{\lim_n}\, u_n \le \overline{\lim_n}\, \sigma^{(1)}_n(u) + \frac{\lambda}{\lambda-1}\overline{\lim_n}(\sigma^{(1)}_{[\lambda n]}(u) - \sigma^{(1)}_n(u))$$
$$+\overline{\lim_n}\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}(-\Delta u_j)$$

In the above inequality, the second term on the right hand side vanishes because $\lim_n \sigma^{(1)}_n(u)$ exists. Since $-\Delta u_j \le \frac{C}{j}$, we get

$$\overline{\lim_n}\, u_n \le \overline{\lim_n}\, \sigma^{(1)}_n(u) + \overline{\lim_n}\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}\frac{C}{j}$$
$$\le \overline{\lim_n}\, \sigma^{(1)}_n(u) + C\,\overline{\lim_n}\max_{n+1\le k\le[\lambda n]}\sum_{j=n+1}^{k}\frac{1}{j}$$
$$\le \overline{\lim_n}\, \sigma^{(1)}_n(u) + C.\overline{\lim_n}\sum_{j=n+1}^{[\lambda n]}\frac{1}{j} \le \overline{\lim_n}\, \sigma^{(1)}_n(u) + C.\lg\lambda$$

Thus

$$\lim_{\lambda\to 1^+}\overline{\lim_n}\, u_n \le \lim_{\lambda\to 1^+}\overline{\lim_n}\, \sigma^{(1)}_n(u).$$

Neither of the limits depend on $\lambda$, therefore $\overline{\lim_n}\, u_n \le \lim_n \sigma^{(1)}_n(u)$. From the identity (ii) we have

$$\underline{\lim_n}\, u_n \ge \underline{\lim_n}\, \sigma^{(1)}_{n-[(\lambda-1)n]-1}(u) - \frac{1}{\lambda-1}\underline{\lim_n}(\sigma^{(1)}_{n-[(\lambda-1)n]-1}(u) - \sigma^{(1)}_n(u))$$
$$+\underline{\lim_n}\frac{1}{[(\lambda-1)n]+1}\sum_{k=n-[(\lambda-1)n]}^{n}\sum_{j=k+1}^{n}\Delta u_j.$$

Again as in the case before, the second term on the right side of the above inequality vanishes, thus

$$\begin{aligned}\underline{\lim_n} u_n &\geq \underline{\lim_n} \sigma_n^{(1)}(u) + \underline{\lim_n} \frac{1}{[(\lambda-1)n]+1} \sum_{k=n-[(\lambda-1)n]}^{n} \sum_{j=k+1}^{n} \left(-\frac{C}{j}\right) \\ &\geq \lim_n \sigma_n^{(1)}(u) - C \underline{\lim_n} \frac{1}{[(\lambda-1)n]+1} \sum_{k=n-[(\lambda-1)n]}^{n} \left(\frac{1}{k}\right) \\ &\geq \underline{\lim_n} \sigma_n^{(1)}(u) - C. \lg \lambda\end{aligned}$$

Finally,

$$\lim_{\lambda \to 1^+} \underline{\lim_n} u_n \geq \lim_{\lambda \to 1^+} \underline{\lim_n} \sigma_n^{(1)}(u),$$

i.e., $\underline{\lim_n} u_n \geq \lim_n \sigma_n^{(1)}(u)$. Combining both estimations we obtain

$$\lim_n \sigma_n^{(1)}(u) \leq \underline{\lim_n} u_n \leq \overline{\lim_n} u_n \leq \lim_n, \sigma_n^{(1)}(u)$$

therefore

$$\lim_n u_n = \lim_n \sigma_n^{(1)}(u) = \lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} u_n x^n.$$

## 2. Subsequential Tauberian theorems

### 2.1 Introduction

In the first chapter we pointed out that in order to obtain convergence of a sequence $\{u_n\}$ we need to impose certain conditions on $\{u_n\}$ in addition to the existence of the limit $\lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} u_n x^n$. Finding such conditions is the basic problem of the classical Tauberian theory. However, there are some conditions of considerable interest from which it is not possible to infer convergence of $\{u_n\}$ from the existence of the limit (1). This situation motivates a different kind of Tauberian theory. For the rudimentary examples of this kind of Tauberian theorems, see [3, 4, 5, 8]. In the theory we propose, convergence of $\{u_n\}$ out of the existence of the limit (1) does not necessarily follow even if we assume some of the classical Tauberian conditions. Hence we are concerned with the following questions:

(i) What kind of divergence of $\{u_n\}$ may we expect ?

(ii) How is the structure of $\{u_n\}$ related to the manner $\{u_n\}$ diverges?

The main objective of this chapter and to some extent of chapter 3 is to study Tauberian problems described in (i) and (ii).

Recall from page 2 that the condition $V_n^{(0)}(\Delta u) = O(1)$, $n \to \infty$ is not sufficient to identify sequences in the class $\mathcal{U}$ for which (2) holds. Nevertheless, some information about the subsequential behavior of $\{u_n\}$ can be obtained assuming an additional mild condition $\Delta V_n^{(0)}(\Delta u) = V_n^{(0)}(\Delta u) - V_{n-1}^{(0)}(\Delta u) = o(1)$, $n \to \infty$. Indeed, for a real sequence $\{u_n\}$ if $V_n^{(0)}(\Delta u) = O(1)$, $n \to \infty$, then from the Kronecker identity $u_n - \sigma_n^{(1)}(u) = V_n^{(0)}(\Delta u)$, it follows that $u_n - \sigma_n^{(1)}(u) = O(1)$, $n \to \infty$ , i.e., for some $C \geq 0$ and all nonnegative integers $n$,

$$u_n - \sigma_n^{(1)}(u) = n(\sigma_n^{(1)}(u) - \sigma_{n-1}^{(1)}(u)) \geq -C.$$

Also, the existence of $\lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} u_n x^n$ implies that

$$\lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} \sigma_n^{(1)}(u) x^n$$

exists and

$$\lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} \sigma_n^{(1)}(u) x^n = \lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} u_n x^n.$$

Applying corollary to Karamata's Hauptsatz, we obtain that

$$\frac{1}{n+1} \sum_{k=0}^{n} k(\sigma_k^{(1)}(u) - \sigma_{k-1}^{(1)}(u)) = o(1), \quad n \to \infty.$$

This is the original Tauber condition on $\{\sigma_n^{(1)}(u)\}$. Hence

$$\lim_n \sigma_n^{(1)}(u) = \lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} u_n x^n.$$

Consequently, $u_n = O(1)$, $n \to \infty$. The assumption $\Delta V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$ implies $\Delta u_n = o(1)$, $n \to \infty$. Now we shall show that $u_n = O(1)$, $n \to \infty$ and $\Delta u_n = o(1)$, $n \to \infty$ imply that every point in the interval $(\underline{\lim}_n u_n, \overline{\lim}_n u_n)$ is an accumulation point of $\{u_n\}$.

Here we outline the proof of the above observation [21].

Let $\underline{\lim}_n u_n = l$, and $\overline{\lim}_n u_n = L$. If $l = L$, there is nothing to prove. Assume that $(l, L)$ is not a singleton, and that $x \in (l, L)$ is not an accumulation point of $\{u_n\}$. Then (i) there exist distinct numbers $b$ and $c$ such that

$$l < b < x < c < L,$$

(ii) there exists a positive integer $n_1$ such that for all $n \geq n_1$, in $[b, c]$ there are no points of $\{u_n\}$.

From the assumption $u_n - u_{n-1} = o(1)$, $n \to \infty$, it follows that there is a positive integer $n_2$ such that for all $n \geq n_2$

$$|u_n - u_{n-1}| < c - b.$$

Since $l$ and $L$ are two distinct accumulation points, there is a positive integer $m > \max(n_1, n_2)$ such that

$$u_m < b.$$

Hence for some $n > m$ $u_n < b$, because there is no point of $\{u_n\}$ in $[b, c]$. Then

$$u_{n+1} \leq u_n + |u_{n+1} - u_n| < b + c - b = c.$$

Thus $u_{n+1} < c$ but $u_{n+1} \notin [b, c]$. So $u_{n+1} < b$. By the finite induction on n, we have that for all $n > m$ $u_n < b$. Hence

$$\overline{\lim_n} u_n = L \leq b < c < L,$$

which is a contradiction. Consequently every point in $(\underline{\lim}_n u_n, \overline{\lim}_n u_n)$ is an accumulation point of $\{u_n\}$. That is, for every $z \in (\underline{\lim}_n u_n, \overline{\lim}_n u_n)$ there exists a subsequence $\{u_{n(z)}\}$ of $\{u_n\}$ such that

$$\lim_{n(z)} u_{n(z)} = z.$$

It is clear that every non-convergent bounded sequence contains a convergent subsequence. If $\Delta V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$ is not assumed but $u_n = O(1)$, $n \to \infty$, then we still have at least one convergent subsequence of $\{u_n\}$. A natural question is that under what conditions $\{u_n\}$ would be bounded. For instance, as we proved in this section that for a sequence $\{u_n\} \in \mathcal{U}$, $V_n^{(0)}(\Delta u) = O(1)$, $n \to \infty$ implies that $u_n = O(1)$, $n \to \infty$. Recalling the definition of moderately oscillatory behavior of $\{u_n\}$, we can have $u_n = O(1)$, $n \to \infty$ provided that the sequence $\{u_n\}$ in $\mathcal{U}$ is moderately-oscillatory. Observe that, if $\{u_n\} \in \mathcal{U}$ and it is moderately oscillatory, then $\{\sigma_n^{(1)}(u)\}$ is convergent and since $V_n^{(0)}(\Delta u) = O(1)$, $n \to \infty$ consequently $u_n = O(1)$, $n \to \infty$. Indeed, we do not need to assume that $\{u_n\}$ is moderately oscillatory to have boundedness of $\{u_n\}$. If the generator sequence $\{V_n^{(0)}(\Delta u)\}$ is moderately oscillatory, then we still have bounded $\{u_n\}$.

The following three-way generalization of Çanak's Theorem [5] is needed for our main result.

**Theorem 7.** *For a real sequence $\{u_n\}$, let the limit*

(27)
$$\lim_{x \to 1^-} (1 - x) \sum_{n=0}^{\infty} \sigma_n^{(1)}(u) x^n$$

*exist. If $\{V_n^{(0)}(\Delta u)\}$ is moderately oscillatory and $\Delta V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$, then there exists an interval $I$ such that for every $z \in I$ there is a subsequence $\{u_{n(z)}\}$ of $\{u_n\}$ converging to $z$, i.e., $\lim_{n(z)} u_{n(z)} = z$.*

**Proof.** Since $\{V_n^{(0)}(\Delta u)\}$ is moderately oscillatory, we have

$$V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u) = O(1), \quad n \to \infty \tag{28}$$

and

$$V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u) = O(1), \quad n \to \infty \tag{29}$$

i.e., for some $C \geq 0$ and all nonnegative integers $n$

$$V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u) = n(V_n^{(2)}(\Delta u) - V_{n-1}^{(2)}(\Delta u) \geq -C. \tag{30}$$

From the existence of the limit (27), it follows that

$$\lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} V_n^{(1)}(\Delta u) x^n = \lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} (\sigma_n^{(1)}(u) - \sigma_n^{(2)}(u)) x^n = 0 \tag{31}$$

and

$$\lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} V_n^{(2)}(\Delta u) x^n = \lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} (\sigma_n^{(2)}(u) - \sigma_n^{(3)}(u)) x^n = 0.$$

By the corollary to Karamata's Hauptsatz, it follows from (30) and (31) that

$$\frac{1}{n+1} \sum_{k=0}^{n} k(V_k^{(2)}(\Delta u) - V_{k-1}^{(2)}(\Delta u)) = o(1), \quad n \to \infty,$$

This is the original Tauber condition on $\{V_n^{(2)}(\Delta u)\}$. Hence $V_n^{(2)}(\Delta u) = o(1)$, $n \to \infty$. Since $\{V_n^{(1)}(\Delta u)\}$ is slowly-oscillating, from the identity

$$V_n^{(1)}(\Delta u) = V_n^{(2)}(\Delta u) + \frac{[\lambda n] + 1}{[\lambda n] - n} (V_{[\lambda n]}^{(2)}(\Delta u) - V_n^{(2)}(\Delta u))$$
$$- \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \sum_{j=n+1}^{k} (V_j^{(1)}(\Delta u) - V_{j-1}^{(1)}(\Delta u))$$

it follows that $V_n^{(1)}(\Delta u) = o(1)$, $n \to \infty$. Consequently $V_n^{(0)}(\Delta u) = O(1)$, $n \to \infty$. Notice that

$$\sigma_n^{(1)}(u) - \sigma_n^{(2)}(u) = n(\sigma_n^{(2)}(u) - \sigma_{n-1}^{(2)}(u)) = V_n^{(1)}(\Delta u) = o(1), \quad n \to \infty. \tag{32}$$

Since the existence of the limit (27) implies that $\lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} \sigma_n^{(2)}(u) x^n$ exists, by the corollary to the original Tauber theorem, we have $\lim_n \sigma_n^{(2)}(u) =$

$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} u_n x^n$. Hence, it follows from (32) that $\lim_n \sigma_n^{(1)}(u) = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} u_n x^n$. Finally from $u_n - \sigma_n^{(1)}(\Delta u) = O(1)$, $n\to\infty$, we get $u_n = O(1)$, $n\to\infty$. The assumption $\Delta V_n^{(0)}(\Delta u) = o(1)$, $n\to\infty$ is equivalent to $\Delta u_n = o(1)$, $n\to\infty$. As we proved earlier, $u_n = O(1)$, $n\to\infty$ and $\Delta u_n = o(1)$, $n\to\infty$ imply that every point in the interval $(\underline{\lim}_n u_n, \overline{\lim}_n u_n)$ is an accumulation point of $\{u_n\}$. Therefore, for every $z\in(\underline{\lim}_n u_n, \overline{\lim}_n u_n)$ there exists a subsequence $\{u_{n(z)}\}$ of $\{u_n\}$ such that

$$\lim_{n(z)} u_{n(z)} = z.$$

The above theorem could be proved for restricted complex $\{u_n\}$ if the definition of monotonicity in Petrovic angle given in [22] is used.

One might ask is it possible to have unbounded sequence and still obtain some information about certain subsequences of $\{u_n\}$. In other words, is there any way to get subsequential information about unbounded sequences? To answer these questions, assume that $\{u_n\}$ is slowly-oscillating, not belonging to $\mathcal{U}$ and unbounded. Since $\{u_n\}$ is slowly-oscillating, $\{\sigma_n^{(1)}(u)\}$ is slowly-oscillating too, and the sequence $\{u_n - \sigma_n^{(1)}(u)\}$ is bounded. From (3) it follows that $\{V_n^{(0)}(\Delta u)\}$ is slowly-oscillating and therefore $\Delta V_n^{(0)}(\Delta u) = o(1)$, $n\to\infty$. Hence $\Delta(u_n - \sigma_n^{(1)}(u)) = o(1)$, $n\to\infty$. Now by the fact that has been proved earlier, there exists an interval $I$ such that for every $z\in I$ there exists a subsequence $\{u_{n(z)} - \sigma_{n(z)}^{(1)}(u)\}$ of $\{u_n - \sigma_n^{(1)}(u)\}$ such that

$$\lim_{n(z)}(u_{n(z)} - \sigma_{n(z)}^{(1)}(u)) = z.$$

In other words, by knowing just the slow-oscillation of a sequence, we obtain subsequential information about the behavior of the difference between the sequence and its average.

In the next theorem we introduce another condition which implies subsequential information about the sequence $\{V_n^{(0)}(\Delta u)\}$.

**Theorem 8.** *For a real $\{V_n^{(0)}(\Delta u)\}$ let $\lim_n V_n^{(1)}(\Delta u)$ exist. If*

$$\Delta V_n^{(0)}(\Delta u) = o(1), \qquad n\to\infty$$

*and*

$$\overline{\lim_n}\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta V_j^{(0)}(\Delta u)\right| < \infty, \quad \lambda > 1 \tag{33}$$

*then there exists an interval $I$ such that for every $z \in I$ there is a subsequence $\{V_{n(z)}^{(0)}(\Delta u)\}$ of $\{V_n^{(0)}(\Delta u)\}$ such that $\lim_{n(z)} V_{n(z)}^{(0)}(\Delta u) = z$.*

**Proof.** First we need the following identities:

for $\lambda > 1$

(i)
$$V_n^{(0)}(\Delta u) = V_n^{(1)}(\Delta u) + \frac{[\lambda n]+1}{[\lambda n]-n}(V_{[\lambda n]}^{(1)}(\Delta u) - V_n^{(1)}(\Delta u)) - \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}\Delta V_j^{(0)}(\Delta u)$$

for $1 < \lambda < 2$

(ii)
$$V_n^{(0)}(\Delta u) = V_{n-[(\lambda-1)n]-1}^{(1)}(\Delta u) - \frac{n+1}{[(\lambda-1)n]+1}(V_{n-[(\lambda-1)n]-1}^{(1)}(\Delta u) - V_n^{(1)}(\Delta u)) + \frac{1}{[(\lambda-1)n]+1}\sum_{k=n-[(\lambda-1)n]}^{n}\sum_{j=k+1}^{n}\Delta V_j^{(0)}(\Delta u).$$

From identity (i), we have

$$V_n^{(0)}(\Delta u) \leq V_n^{(1)}(\Delta u) + \frac{[\lambda n]+1}{[\lambda n]-n}\left|V_{[\lambda n]}^{(1)}(\Delta u) - V_n^{(1)}(\Delta u)\right| + \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta V_j^{(0)}(\Delta u)\right|.$$

Taking limsup in $n$ of both sides of the above inequality, we get

$$\overline{\lim_n} V_n^{(0)}(\Delta u) \leq \overline{\lim_n} V_n^{(1)}(\Delta u) + \frac{\lambda}{\lambda-1}\overline{\lim_n}\left|V_{[\lambda n]}^{(1)}(\Delta u) - V_n^{(1)}(\Delta u)\right| + \overline{\lim_n}\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta V_j^{(0)}(\Delta u)\right|.$$

Since $\lim_n V_n^{(1)}(\Delta u)$ exists, the second term in the above inequality vanishes. Thus,

$$\overline{\lim_n} V_n^{(0)}(\Delta u) \leq \lim_n V_n^{(1)}(\Delta u) + \overline{\lim_n}\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta V_j^{(0)}(\Delta u)\right|.$$

Hence

$$\overline{\lim_n} V_n^{(0)}(\Delta u) \leq \lim_n V_n^{(1)}(\Delta u) + K,$$

where $K = \overline{\lim_n} \frac{1}{[\lambda n]-n} \sum_{k=n+1}^{[\lambda n]} \left| \sum_{j=n+1}^{k} \Delta V_j^{(0)}(\Delta u) \right| < \infty.$

From identity (ii), we have

$$V_n^{(0)}(\Delta u) \geq V_{n-[(\lambda-1)n]-1}^{(1)}(\Delta u) - \frac{n+1}{[(\lambda-1)n]+1}(V_{n-[(\lambda-1)n]-1}^{(1)}(\Delta u) - V_n^{(1)}(\Delta u))$$
$$-\frac{1}{[(\lambda-1)n]+1} \sum_{k=n-[(\lambda-1)n]}^{n} \left| \sum_{j=k+1}^{n} \Delta V_j^{(0)}(\Delta u) \right|.$$

Taking liminf in $n$ of both sides of the above inequality, we obtain

$$\underline{\lim_n} V_n^{(0)}(\Delta u) \geq \underline{\lim_n} V_{n-[(\lambda-1)n]-1}^{(1)}(\Delta u) -$$
$$\frac{1}{\lambda-1} \underline{\lim_n}(V_{n-[(\lambda-1)n]-1}^{(1)}(\Delta u) - V_n^{(1)}(\Delta u))$$
$$+\underline{\lim_n}(-\frac{1}{[(\lambda-1)n]+1} \sum_{k=n-[(\lambda-1)n]}^{n} \left| \sum_{j=k+1}^{n} \Delta V_j^{(0)}(\Delta u) \right|).$$

Since $\lim_n V_n^{(1)}(\Delta u)$ exists, the second term in the above inequality vanishes. Thus,

$$\underline{\lim_n} V_n^{(0)}(\Delta u) \geq \lim_n V_n^{(1)}(\Delta u) +$$
$$\underline{\lim_n}(-\frac{1}{[(\lambda-1)n]+1} \sum_{k=n-[(\lambda-1)n]}^{n} \left| \sum_{j=n+1}^{k} \Delta V_j^{(0)}(\Delta u) \right|)$$
$$\geq \lim_n V_n^{(1)}(\Delta u) - \overline{\lim_n}(\frac{1}{[(\lambda-1)n]+1} \sum_{k=n-[(\lambda-1)n}^{n} \left| \sum_{j=n+1}^{k} \Delta V_j^{(0)}(\Delta u) \right|).$$

Hence

$$\underline{\lim_n} V_n^{(0)}(\Delta u) \geq \lim_n V_n^{(1)}(\Delta u) - M,$$

where

$$M = \overline{\lim_n} \frac{1}{[(\lambda-1)n]+1} \sum_{k=n-[(\lambda-1)n]}^{n} \left| \sum_{j=k+1}^{n} \Delta V_j^{(0)}(\Delta u) \right| < \infty.$$

Consequently,

$$\lim_n V_n^{(1)}(u) - M \leq \underline{\lim_n} V_n^{(0)}(\Delta u) \leq \overline{\lim_n} V_n^{(0)}(\Delta u) \leq \lim_n V_n^{(1)}(\Delta u) + K.$$

That is, $V_n^{(0)}(\Delta u) = O(1)$, $n \to \infty$. Since $\Delta V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$ and $V_n^{(0)}(\Delta u) = O(1)$, $n \to \infty$, it follows from the previous results that there exists an interval $I$ such that for every $z \in I$ there is a subsequence $\{V_{n(z)}^{(0)}(\Delta u)\}$ of $\{V_n^{(0)}(\Delta u)\}$ such that $\lim_{n(z)} V_{n(z)}^{(0)}(\Delta u) = z$.

As a generalization of this theorem, if we assume that limit (27) exists, we obtain subsequential information about $\{u_n\}$.

**Theorem 9.** *For a real sequence $\{u_n\}$, let the limit (27) and $\lim_n V_n^{(1)}(\Delta u)$ exist. If $\Delta V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$ and*

$$\overline{\lim_n} \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \left| \sum_{j=n+1}^{k} \Delta V_j^{(0)}(\Delta u) \right| < \infty, \quad \lambda > 1 \tag{34}$$

*then there exists an interval $I$ such that for every $z \in I$ there is a subsequence $\{u_{n(z)}\}$ of $\{u_n\}$ such that $\lim_{n(z)} u_{n(z)} = z$.*

**Proof.** The proof follows the lines of the proof of Theorem 8. However, we need the existence of $\lim_n \sigma_n^{(1)}(u)$. Since $\lim_n V_n^{(1)}(\Delta u)$ exists, from the identity $\sigma_n^{(1)}(u) - \sigma_n^{(2)}(u) = V_n^{(1)}(\Delta u)$, it follows that $n\Delta\sigma_n^{(2)}(u) = O(1)$, $n \to \infty$. The existence of the limit (27) implies that $\lim_{x\to 1^-} (1-x) \sum_{n=0}^{\infty} \sigma_n^{(2)}(u)x^n$ exists. By the Littlewood Theorem, $\lim_n \sigma_n^{(2)}(u) = \lim_{x\to 1^-} (1-x) \sum_{n=0}^{\infty} \sigma_n^{(1)}(u)x^n$. Consequently, $\lim_n \sigma_n^{(1)}(u)$ exists.

## 2.2 Subsequential Tauberian theorems for sequences with moderately oscillatory moduli

In the previous section we proposed a new approach to the Tauberian theory by pointing out that if the conditions for the oscillatory behavior of $\{u_n\}$ are considerably weakened, then the existence of the limit (1) does not necessarily imply (2). That is, from some weaker conditions we may not conclude convergence of $\{u_n\} \in \mathcal{U}$ but we may obtain a deeper insight into the structure of the sequence. Also we can obtain some other asymptotic subsequential information. For example, we showed that if $\{u_n\} \in \mathcal{U}$, $u_n = O(1)$, $n \to \infty$ and $\Delta u_n = o(1)$, $n \to \infty$ then for every $z \in (\underline{\lim}_n u_n, \overline{\lim}_n u_n)$ there exists a subsequence $\{u_{n(z)}\}$ of $\{u_n\}$ such that $\lim_{n(z)} u_{n(z)} = z$, Later, we observed that the conditions

(i) $V_n^{(0)}(\Delta u) = O(1)$, $n \to \infty$,

(ii) $\{u_n\}$ is moderately-oscillatory,

(iii) $\{V_n^{(0)}(\Delta u)\}$ is moderately-oscillatory

together with $\Delta u_n = o(1)$, $n \to \infty$ are also examples of conditions that imply some kind of subsequential convergence. Namely, from (i), (ii) or (iii), and from the existence of the limit (1), it is not difficult to obtain that for every $z \in (\underline{\lim}_n u_n, \overline{\lim}_n u_n)$ there exists a subsequence $\{u_{n(z)}\}$ of $\{u_n\}$ such that $\lim_{n(z)} u_{n(z)} = z$. The above examples and other results suggest a new kind of convergence given in the following definition [21].

**Definition 4.** *A real sequence $\{u_n\}$ converges subsequentially if there exists a finite interval $I(u)$ such that all accumulation points of $\{u_n\}$ are in $I(u)$ and every point of $I(u)$ is an accumulation point of $\{u_n\}$.*

It is clear from Definition that subsequential convergence implies boundedness of the sequence. However, the converse is not true. Namely, there are bounded sequences which do not converge subsequentially. For instance, $\{(-1)^n\}$ is a bounded sequence but it is not subsequentially convergent. Since boundedness is a necessary condition for subsequential convergence, we might think of it as some kind of summability. This situation is analogous to the classical one. The existence of the limit (1) which is a necessary condition for the existence of the limit (2) is called Abel's summability.

As we discussed earlier, conditions for convergence or subsequential convergence of $\{u_n\}$ were based on the order of magnitudes of $\{V_n^{(0)}(\Delta u)\}$ or the classical modulo denoted by $\omega_n^{(0)}(u) = n\Delta u_n$. Now we propose a new modulo of the oscillatory behavior of order one [19]. Namely,

$$\omega_n^{(1)}(u) = \omega_n^{(0)}(u) - \sigma_n^{(1)}(\omega^{(0)}(u)) =$$
$$= n\Delta u_n - V_n^{(0)}(\Delta u) = n\Delta V_n^{(0)}(\Delta u) = \omega_n^{(0)}(V^{(0)}(\Delta u))$$

which controls the oscillatory behavior of $\{u_n\}$ via $\{\omega_n^{(0)}(V^{(0)}(\Delta u))\}$, the classical modulo of $\{V_n^{(0)}(\Delta u)\}$. This leads to new results and more succinct proofs of the classical theorems. For example, if $\{\omega_n^{(1)}(u)\}$ is moderately oscillatory and the limit $\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n$ exists, then we have convergence of $\{u_n\}$. See [11] for the proof. Since

$$\sigma_n^{(1)}(\omega^{(1)}(u)) = V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u) = n\Delta V_n^{(1)}(\Delta u) = \omega_n^{(0)}(V^{(1)}(\Delta u)),$$

assuming that $\{\sigma_n^{(1)}(\omega^{(1)}(u))\}$ is moderately oscillatory, we cannot conclude convergence of $\{u_n\}$ from the existence of the limit (1). But we can obtain subsequential information about $\{u_n\}$.

In the previous examples of Tauberian theorems, the conditions of Tauberian nature were placed either on $\{u_n\}$ or on $\{\sigma_n^{(1)}(\omega^{(0)}(u))\}$. In the following theorem we are going to place corresponding Tauberian conditions on $\{\sigma_n^{(1)}(\omega^{(1)}(u))\}$.

**Theorem 10.** *For a real sequence $\{u_n\}$, let the limit (27) exist, and let $\Delta V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$. If*

(35) $$\sigma_n^{(1)}(\omega^{(1)}(u)) = O(1), \quad n \to \infty$$

*then $\{u_n\}$ is subsequentially convergent.*

**Proof.** From (35) we have

(36) $$V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u) = n\Delta V_n^{(1)}(\Delta u) = O(1), \quad n \to \infty.$$

The existence of the limit (27) implies that

(37)
$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} V_n^{(1)}(\Delta u)x^n = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}(\sigma_n^{(1)}(u) - \sigma_n^{(2)}(u))x^n = 0.$$

Applying the Littlewood Theorem, we obtain from (36) and (37) that

$$V_n^{(1)}(\Delta u) = o(1), \qquad n \to \infty.$$

Hence we have

(38) $$\sigma_n^{(1)}(u) - \sigma_n^{(2)}(u) = V_n^{(1)}(\Delta u) = o(1), \quad n \to \infty$$

i.e.,

(39) $$n\Delta\sigma_n^{(2)}(u) = o(1), \quad n \to \infty.$$

The existence of the limit (27) implies the existence of the limit

(40) $$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(2)}(u)x^n.$$

Applying the corollary to the original Tauber theorem, we obtain from (39) and (40) that

$$\lim_n \sigma_n^{(2)}(u) = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(2)}(u)x^n.$$

The identity (38) yields that $\lim_n \sigma_n^{(1)}(u) = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n$. Since $V_n^{(1)}(\Delta u) = o(1)$, $n \to \infty$, from (36) it follows that $V_n^{(0)}(\Delta u) = O(1)$, $n \to \infty$. Finally, from the Kronecker identity $u_n - \sigma_n^{(1)}(u) = V_n^{(0)}(\Delta u)$, we obtain $u_n = O(1)$, $n \to \infty$. Since $\Delta u_n = o(1)$, $n \to \infty$ and $u_n = O(1)$, $n \to \infty$ from the previous results it follows that $\{u_n\}$ is subsequentially convergent.

Next, we propose a generalization of the above theorem by weakening the boundedness of $\{\sigma_n^{(1)}(\omega^{(1)}(u))\}$ to moderately oscillatory behavior.

**Theorem 11.** *For a real sequence $\{u_n\}$ such that $\Delta V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$, let the limit*

$$\lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} \sigma_n^{(1)}(u) x^n \tag{41}$$

*exist. If $\{\sigma_n^{(1)}(\omega^{(1)}(u))\}$ is moderately oscillatory, then $\{u_n\}$ is subsequentially convergent.*

**Proof.** Since $\{\sigma_n^{(1)}(\omega^{(1)}(u))\}$ is moderately oscillatory, i.e., $\{V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u)\}$ is moderately oscillatory, $\{V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u)\}$ is slowly-oscillating,

$$V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u) - ((V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u)) = O(1), \quad n \to \infty, \tag{42}$$

and

$$V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u) - ((V_n^{(2)}(\Delta u) - V_n^{(3)}(\Delta u)) = O(1), \quad n \to \infty. \tag{43}$$

That is,

$$n\Delta(V_n^{(2)}(\Delta u) - V_n^{(3)}(\Delta u)) \geq -C \tag{44}$$

for some $C \geq 0$ and all nonnegative integers $n$. On the other hand, the existence of the limit (41) implies that

$$\lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} V_n^{(1)}(\Delta u) x^n = \lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} (\sigma_n^{(1)}(u) - \sigma_n^{(2)}(u)) x^n = 0 \tag{45}$$

and consequently,

$$\lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} V_n^{(2)}(\Delta u) x^n = \lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} V_n^{(3)}(\Delta u) x^n = 0. \tag{46}$$

Applying the corollary to Karamata's Hauptsatz, from (44), (45) and (46) it follows that

$$\frac{1}{n+1} \sum_{k=0}^{n} k(V_k^{(2)}(\Delta u) - V_k^{(3)}(\Delta u) - (V_{k-1}^{(2)}(\Delta u) - V_{k-1}^{(3)}(\Delta u))) = o(1), \quad n \to \infty.$$

But this is the original Tauber condition. Hence, we obtain $V_n^{(2)}(\Delta u) - V_n^{(3)}(\Delta u) = o(1)$, $n \to \infty$. From (43), it follows that $V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u) = O(1)$, $n \to \infty$. Consequently, from (42) we obtain $V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u) = O(1)$, $n \to \infty$. The rest of the proof follows from the proof of Theorem 10. We would like to remark that Theorem 11. could have been proved had we used the generalized Littlewood Theorem.

Notice that if we assume that $\{V_n^{(0)}(\Delta u)-V_n^{(1)}(\Delta u)\}$ is slowly-oscillating, then we obtain convergence of $\{u_n\}$. For the proof see [11]. Recall that

$$\omega_n^{(0)}(u) = n\Delta u_n,$$
$$\omega_n^{(1)}(u) = \omega_n^{(0)}(u) - \sigma_n^{(1)}(\omega^{(0)}(u)) = n\Delta u_n - V_n^{(0)}(\Delta u).$$

A close analysis of the previous theorems shows that there is another modulo of oscillatory behavior whose part we have used in the above theorems. Namely,

$$\omega_n^{(2)}(u) = \omega_n^{(1)}(u) - \sigma_n^{(1)}(\omega^{(1)}(u)) =$$
$$n\Delta u_n - V_n^{(0)}(\Delta u) - (V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u)).$$

This suggests that we have to go on with different orders of oscillatory behavior. In general, we have

$$\omega_n^{(m)}(u) = \omega_n^{(m-1)}(u) - \sigma_n^{(1)}(\omega^{(m-1)}(u)) \text{ for } m \geq 1 \text{ [19]}.$$

In Theorems 10. and 11. we set conditions on

$$\sigma_n^{(1)}(\omega^{(1)}(u)) = n\Delta V_n^{(1)}(\Delta u).$$

In the next theorem we will use $\{n\Delta V_n^{(1)}(\Delta V^{(0)}(\Delta u))\}$ as a modulo of oscillatory behavior of $\{u_n\}$.

**Theorem 12.** *For real $\{u_n\}$ such that $\Delta V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$, let the limit (1) exist. If $\{V_n^{(0)}(\Delta V^{(0)}(\Delta u)) - V_n^{(1)}(\Delta V^{(0)}(\Delta u))\}$ is moderately oscillatory, then $\{u_n\}$ is subsequentially convergent.*

**Proof.** Since $\{V_n^{(0)}(\Delta V^{(0)}(\Delta u)) - V_n^{(1)}(\Delta V^{(0)}(\Delta u))\}$ is moderately oscillatory, $\{V_n^{(1)}(\Delta V^{(0)}(\Delta u)) - V_n^{(2)}(\Delta V^{(0)}(\Delta u))\}$ is slowly-oscillating, and

$$\begin{aligned} &V_n^{(0)}(\Delta V^{(0)}(\Delta u)) - V_n^{(1)}(\Delta V^{(0)}(\Delta u)) \\ &-[V_n^{(1)}(\Delta V^{(0)}(\Delta u)) - V_n^{(2)}(\Delta V^{(0)}(\Delta u))] = O(1), n \to \infty \end{aligned} \tag{47}$$

i.e.,

$$\begin{aligned} &V_n^{(0)}(\Delta V^{(0)}(\Delta u)) - V_n^{(1)}(\Delta V^{(0)}(\Delta u)) \\ &-[V_n^{(1)}(\Delta V^{(0)}(\Delta u)) - V_n^{(2)}(\Delta V^{(0)}(\Delta u))] \geq -C \end{aligned} \tag{48}$$

for all $C \geq 0$ and all nonnegative integers $n$.

From the existence of the limit (1), we have

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} V_n^{(0)}(\Delta u)x^n = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}(u_n - \sigma_n^{(1)}(u))x^n = 0,$$

and

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} V_n^{(1)}(\Delta u)x^n = 0.$$

Hence

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}V_n^{(0)}(\Delta V^{(0)}(\Delta u))x^n =$$

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}(V_n^{(0)}(\Delta u)-V_n^{(1)}(\Delta u))x^n = 0, \tag{49}$$

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}(V_n^{(0)}(\Delta V^{(0)}(\Delta u))-V_n^{(1)}(\Delta V^{(0)}(\Delta u))x^n = 0 \tag{50}$$

and

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}(V_n^{(1)}(\Delta V^{(0)}(\Delta u))-V_n^{(2)}(\Delta V^{(0)}(\Delta u)))x^n = 0, \tag{51}$$

Applying the corollary to Karamata's Hauptsatz, we obtain from (48), (50) and (51) that

$$\frac{1}{n+1}\sum_{k=0}^{n}k\Delta[V_k^{(1)}(\Delta V^{(0)}(\Delta u))-V_k^{(2)}(\Delta V^{(0)}(\Delta u))] = o(1), \quad n\to\infty.$$

But this is the original Tauber condition. Hence we have

$$V_n^{(1)}(\Delta V^{(0)}(\Delta u))-V_n^{(2)}(\Delta V^{(0)}(\Delta u)) = o(1), \quad n\to\infty.$$

From (47) we obtain

$$V_n^{(0)}(\Delta V^{(0)}(\Delta u))-V_n^{(1)}(\Delta V^{(0)}(\Delta u)) = O(1), \quad n\to\infty \tag{52}$$

i.e.,

$$V_n^{(0)}(\Delta V^{(0)}(\Delta u))-V_n^{(1)}(\Delta V^{(0)}(\Delta u)) \geq -C \tag{53}$$

for all $C\geq 0$ and all nonnegative integers $n$.

Applying again the corollary to Karamata's Hauptsatz, we obtain from (50) and (53) that

$$\frac{1}{n+1}\sum_{k=0}^{\infty}k(V_k^{(1)}(\Delta V^{(0)}(\Delta u))-V_{k-1}^{(1)}(\Delta V^{(0)}(\Delta u))) = o(1), \quad n\to\infty.$$

But this is the original Tauber condition. Hence we have $V_n^{(1)}(\Delta V^{(0)}(\Delta u)) = o(1)$, $n\to\infty$. From (52) we obtain

$$V_n^{(0)}(\Delta V^{(0)}(\Delta u)) = V_n^{(0)}(\Delta u)-V_n^{(1)}(\Delta u) = O(1), \quad n\to\infty.$$

The rest of the proof follows from the proof of Theorem 10.

Replacing the moderately oscillatory condition for

$$\{V_n^{(0)}(\Delta V^{(0)}(\Delta u))-V_n^{(1)}(\Delta V^{(0)}(\Delta u))\}$$

by a stronger condition, $V_n^{(0)}(\Delta V^{(0)}(\Delta u))-V_n^{(1)}(\Delta V^{(0)}(\Delta u)) = O(1), n\to\infty$, we obtain a corollary to Theorem 12.

**Corollary 1.** *For a real sequence $\{u_n\}$ such that $\Delta V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$, let the limit (1) exist. If*

$$V_n^{(0)}(\Delta V^{(0)}(\Delta u)) - V_n^{(1)}(\Delta V^{(0)}(\Delta u)) = O(1), \quad n \to \infty,$$

*then $\{u_n\}$ is subsequentially convergent.*

In the next theorem we will use $\{n\Delta V_n^{(1)}(\Delta\sigma^{(1)}(\omega^{(1)}(u)))\}$ to control the oscillatory behavior of $\{u_n\}$.

**Theorem 13.** *For a real $\{u_n\}$ such that $\Delta V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$, let the limit (1) exist. If $\{V_n^{(0)}(\Delta\sigma^{(1)}(\omega^{(1)}(u))) - V_n^{(1)}(\Delta\sigma^{(1)}(\omega^{(1)}(u)))\}$ is moderately oscillatory, then $\{u_n\}$ is subsequentially convergent.*

**Proof.** Since $\{V_n^{(0)}(\Delta\sigma^{(1)}(\omega^{(1)}(u))) - V_n^{(1)}(\Delta\sigma^{(1)}(\omega^{(1)}(u)))\}$ is moderately oscillatory, $\{V_n^{(1)}(\Delta\sigma^{(1)}(\omega^{(1)}(u))) - V_n^{(2)}(\Delta\sigma^{(1)}(\omega^{(1)}(u)))\}$ is slowly-oscillating and

$$\begin{aligned} V_n^{(0)}(\Delta\sigma^{(1)}(\omega^{(1)}(u))) - V_n^{(1)}(\Delta\sigma^{(1)}(\omega^{(1)}(u))) \\ -[V_n^{(1)}(\Delta\sigma^{(1)}(\omega^{(1)}(u))) - V_n^{(2)}(\Delta\sigma^{(1)}(\omega^{(1)}(u)))] = O(1), n \to \infty \end{aligned} \tag{54}$$

i.e.,

$$n\Delta[V_n^{(1)}(\Delta\sigma^{(1)}(\omega^{(1)}(u))) - V_n^{(2)}(\Delta\sigma^{(1)}(\omega^{(1)}(u)))] = O(1), \quad n \to \infty. \tag{55}$$

From the existence of the limit (1), we have

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} V_n^{(0)}(\Delta u)x^n =$$

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} (u_n - \sigma_n^{(1)}(u))x^n = 0$$

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} \sigma_n^{(1)}(\omega^{(1)}(u))x^n = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}(V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u))x^n = 0$$

$$\begin{aligned} &\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} V_n^{(0)}(\Delta\sigma^{(1)}(\omega^{(1)}(u)))x^n \\ &= \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}[V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u) - (V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u))]x^n = 0 \end{aligned} \tag{56}$$

and

$$\begin{aligned} &\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} V_n^{(1)}(\Delta\sigma^{(1)}(\omega^{(1)}(u)))x^n = \\ &\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} V_n^{(2)}(\Delta\sigma^{(1)}(\omega^{(1)}(u)))x^n = 0 \end{aligned} \tag{57}$$

By the Littlewood Theorem, it follows from (55) and (57) that

$$V_n^{(1)}(\Delta\sigma^{(1)}(\omega^{(1)}(u))) - V_n^{(2)}(\Delta\sigma^{(1)}(\omega^{(1)}(u))) = o(1), \quad n \to \infty.$$

From (54), we obtain

$$V_n^{(0)}(\Delta\sigma^{(1)}(\omega^{(1)}(u))) - V_n^{(1)}(\Delta\sigma^{(1)}(\omega^{(1)}(u))) = O(1), \quad n \to \infty, \tag{58}$$

i.e.,

$$n\Delta V_n^{(1)}(\Delta\sigma^{(1)}(\omega^{(1)}(u))) = O(1), \quad n \to \infty. \tag{59}$$

Again by the Littlewood Theorem, (57) and (59) imply that

$$V_n^{(1)}(\Delta\sigma^{(1)}(\omega^{(1)}(u))) = o(1), \quad n \to \infty.$$

Consequently, it follows from (58) that

$$V_n^{(0)}(\Delta\sigma^{(1)}(\omega^{(1)}(u))) = \sigma_n^{(1)}(\omega^{(1)}(u)) - \sigma_n^{(2)}(\omega^{(1)}(u)) = O(1), \quad n \to \infty \tag{60}$$

i.e.,

$$n\Delta\sigma_n^{(2)}(\omega^{(1)}(u)) = O(1), \quad n \to \infty. \tag{61}$$

Applying the Littlewood Theorem once more, we obtain $\sigma_n^{(2)}(\omega^{(1)}(u)) = o(1)$, $n \to \infty$. It follows from (60) that

$$\sigma_n^{(1)}(\omega^{(1)}(u)) = V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u) = n\Delta V_n^{(1)}(\Delta u) = O(1), \quad n \to \infty. \tag{62}$$

By the same argument, $V_n^{(1)}(\Delta u) = o(1)$, $n \to \infty$. Finally, from (62) we obtain $V_n^{(0)}(\Delta u) = O(1)$, $n \to \infty$. As we showed earlier, $V_n^{(0)}(\Delta u) = O(1)$, $n \to \infty$ and $\Delta V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$, together with the existence of the limit (1) imply subsequential convergence of $\{u_n\}$.

Replacing the moderately oscillatory condition for $\{V_n^{(0)}(\Delta\sigma^{(1)}(\omega^{(1)}(u))) - V_n^{(1)}(\Delta\sigma^{(1)}(\omega^{(1)}(u)))\}$ by a stronger condition,

$$V_n^{(0)}(\Delta\sigma^{(1)}(\omega^{(1)}(u))) - V_n^{(1)}(\Delta\sigma^{(1)}(\omega^{(1)}(u))) = O(1), \quad n \to \infty,$$

we obtain a corollary to Theorem 13.

**Corollary 2.** *For a real sequence $\{u_n\}$ such that $\Delta V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$, let the limit (1) exist. If*

$$V_n^{(0)}(\Delta\sigma^{(1)}(\omega^{(1)}(u))) - V_n^{(1)}(\Delta\sigma^{(1)}(\omega^{(1)}(u))) = O(1), \quad n \to \infty$$

*then $\{u_n\}$ is subsequentially convergent.*

We will close this section by the following observation and restating some of the results we have already proved. By doing this we shall gain a new insight about our results.

Recall that $\omega_n^{(0)}(u) = n\Delta u_n$. If we do not look necessarily at the classical modulo of the sequence $\{u_n\}$ but rather the classical modulo of the $(C,1)$-mean of the sequence, then we have

$$\omega_n^{(0)}(\sigma^{(1)}(u)) = n\Delta\sigma_n^{(1)}(u) = n(\sigma_n^{(1)}(u) - \sigma_{n-1}^{(1)}(u)) = \\ = u_n - \sigma_n^{(1)}(u) = V_n^{(0)}(\Delta u).$$

Therefore, when convenient $\{\omega_n^{(0)}(\sigma^{(1)}(u))\}$ and $\{V_n^{(0)}(\Delta u)\}$ may be interchanged. By assuming that $\{\omega_n^{(0)}(\sigma^{(1)}(u))\}$ is slowly-oscillating, we obtain convergence of $\{u_n\}$ from the existence of the limit (1). For $\omega_n^{(0)}(\sigma^{(1)}(u)) = O(1)$, $n \to \infty$, we do not get convergence but subsequential convergence of $\{u_n\}$.

Now consider the modulo $\omega_n^{(1)}(\sigma^{(1)}(u))$,

$$\omega_n^{(1)}(\sigma^{(1)}(u)) = \omega_n^{(0)}(\sigma^{(1)}(u)) - \sigma_n^{(1)}(\omega^{(0)}(\sigma^{(1)}(u))) = \\ V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u).$$

By replacing $\sigma_n^{(1)}(\omega^{(1)}(u)) = O(1)$, $n \to \infty$ with $\omega_n^{(1)}(\sigma^{(1)}(u)) = O(1)$, $n \to \infty$, Theorem 10. can be restated as the following.

**Theorem 14.** *For a real $\{u_n\}$ such that $\Delta V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$, let the limit (1) exist. If $\{\omega_n^{(1)}(\sigma^{(1)}(u))\}$ is bounded, then $\{u_n\}$ is subsequentially convergent.*

In the above theorem, instead of assuming the existence of the limit (1), we could have assumed that $\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n$ exists and still obtain subsequential convergence of $\{u_n\}$.

Replacing $\omega_n^{(1)}(\sigma^{(1)}(u)) = O(1)$, $n \to \infty$ by moderately oscillatory $\{\omega_n^{(1)}(\sigma^{(1)}(u))\}$, we obtain a generalization of Theorem 14 which is a restatement of Theorem 11.

**Theorem 15.** *For a real $\{u_n\}$ such that $\Delta V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$, let the limit (1) exist. If $\{\omega_n^{(1)}(\sigma^{(1)}(u))\}$ is moderately oscillatory, then $\{u_n\}$ is subsequentially convergent.*

## 3. Regularly generated sequences

### 3.1 Introduction

In this section we will study regularly generated sequences first introduced in [19, 20]. Let $\mathcal{L}$ be any linear space and let $\mathcal{B}$ be a class of sequences $\{B_n\}$ from $\mathcal{L}$. The class $\mathcal{U}(\mathcal{B})$ consisting of sequences defined by

$$u_n = B_n + \sum_{k=1}^{n}\frac{B_k}{k}, \text{ for all } n$$

is the class of all regularly generated sequences $\{u_n\}$ by the class $\mathcal{B}$.

For instance, if $\mathcal{B}$ is the class of all bounded slowly-oscillating real sequences, then $\mathcal{U}(\mathcal{B})$ is the classical class of slowly-oscillating sequences and the sequence $\{u_n\} \in \mathcal{U}(\mathcal{B})$ is convergent if the limit (1) exists. In what follows, we shall sketch a rather brief proof of this classical result based on the corollary to Karamata's Hauptsatz.

Since $V_n^{(0)}(\Delta u) = O(1)$, $n \to \infty$, there exists $C \geq 0$ such that $V_n^{(0)}(\Delta u) \geq -C$ for all nonnegative integers $n$. The existence of the limit (1) implies that

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} V_n^{(0)}(\Delta u)x^n = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}(u_n - \sigma_n^{(1)}(u))x^n = 0. \tag{63}$$

Then by the corollary to Karamata's Hauptsatz $\sigma_n^{(1)}(V^{(0)}(\Delta u)) = V_n^{(1)}(\Delta u) = o(1)$, $n \to \infty$. Indeed, $V_n^{(0)}(\Delta u) \geq -C$ and (63) imply that

$$\lim_n \sigma_n^{(1)}(V^{(0)}(\Delta u)) = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} V_n^{(0)}(\Delta u)x^n = 0.$$

Since $\{V_n^{(0)}(\Delta u)\}$ is slowly-oscillating, from the identity

$$V_n^{(0)}(\Delta u) = V_n^{(1)}(\Delta u) + \frac{[\lambda n]+1}{[\lambda n]-n}(V_{[\lambda n]}^{(1)}(\Delta u) - V_n^{(1)}(\Delta u))$$
$$-\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}(V_j^{(0)}(\Delta u) - V_{j-1}^{(0)}(\Delta u))$$

it follows that $V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$. Hence

$$u_n - \sigma_n^{(1)}(u) = n\Delta\sigma_n^{(1)}(u) = V_n^{(0)}(\Delta u) = o(1), \quad n \to \infty,$$

By the corollary to the original Tauber theorem, we have

$$\lim_n \sigma_n^{(1)}(u) = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} u_n x^n,$$

and finally

$$\lim_n u_n = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} u_n x^n.$$

However, if $\mathcal{B}$ is the class of all slowly-oscillating sequences, the class $\mathcal{U}(\mathcal{B})$ is not a classical class, although it is easy to show that if the limit

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n \tag{64}$$

exists, then $\lim_n u_n = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n$. From the slow-oscillation of $\{V_n^{(0)}(\Delta u)\}$, we have $V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u) = O(1)$, $n\to\infty$, i.e.,

$$n\Delta V_n^{(1)}(\Delta u) = O(1), \quad n\to\infty. \tag{65}$$

The existence of the limit (64) implies that

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} V_n^{(1)}(\Delta u)x^n = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}(\sigma_n^{(1)}(u)-\sigma_n^{(2)}(u))x^n = 0. \tag{66}$$

In contrast to a similar proof of the previous result, we can not use one sided boundedness of $\{n\Delta V_n^{(1)}(\Delta u)\}$ although we have it, because we do not know whether or not the limit $\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}(V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u))x^n$ exists. Best way to avoid this is to apply the Littlewood Theorem. That is, from (65) and (66) by the Littlewood Theorem we have $V_n^{(1)}(\Delta u) = o(1)$, $n\to\infty$. Then $V_n^{(0)}(\Delta u) = O(1)$, $n\to\infty$, i.e., $u_n - \sigma_n^{(1)}(u) = n\Delta\sigma_n^{(1)}(u) = O(1)$, $n\to\infty$. It seems that we need to use the Littlewood Theorem one more time. However, this can be avoided in two ways. First, recalling

$$\sigma_n^{(1)}(u) - \sigma_n^{(2)}(u) = V_n^{(1)}(\Delta u) = o(1), \quad n\to\infty,$$

we have $\lim_n \sigma_n^{(2)}(u) = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n$ by the corollary to the original Tauber Theorem, which is more elementary than the Littlewood Theorem. In conclusion, $\lim_n \sigma_n^{(1)}(u) = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n$. The rest of the proof goes as in the previous result.

It is possible to give another proof based on Çanak's Theorem [4]. Since $\{V_n^{(0)}(\Delta u)\}$ is slowly-oscillating, so are $\{u_n\}$ and $\{\sigma_n^{(1)}(u)\}$. Using the generalized Littlewood Theorem, we have

$$\lim_n \sigma_n^{(1)}(u) = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n.$$

Since $\{u_n\}$ is slowly-oscillating and $(C,1)$-summable, using the identity

$$u_n = \sigma_n^{(1)}(u) + \frac{[\lambda n]+1}{[\lambda n]-n}(\sigma_{[\lambda n]}^{(1)}(u) - \sigma_n^{(1)}(u)) - \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}(u_j - u_{j-1})$$

we obtain $\lim_n u_n = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(1)}(u)x^n$.

Throughout this dissertation, we shall consider Tauberian conditions from which convergence or subsequential convergence are implied from the limit (1). To this end, we will continue to study the classes $\mathcal{U}(\mathcal{B})$ of all regularly generated sequences $\{u_n\}$ by the class $\mathcal{B}$ and obtain corresponding Tauberian theorems.

For instance, consider $\{u_n\}$ from the class $\mathcal{U}(\mathcal{B})$ where $\mathcal{B}$ is the class of all bounded sequences $\{B_n\}$. In this case, we obtain subsequential convergence of $\{u_n\}$.

**Theorem 16.** *For a real sequence $\{u_n\}$ such that $\Delta V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$, let the limit (64) exist. If $\{u_n\} \in \mathcal{U}(\mathcal{B})$ where $\mathcal{B}$ is the class of all bounded sequences $\{B_n\}$, then $\{u_n\}$ is subsequentially convergent.*

**Proof.** From $\{u_n\} \in \mathcal{U}(\mathcal{B})$, it follows that for some bounded sequences $\{B_n\}$ and all $n$

(67)
$$u_n = B_n + \sum_{k=1}^{n} \frac{B_k}{k}.$$

The representation (67) implies that

$$\Delta u_n = u_n - u_{n-1} = B_n + \sum_{k=1}^{n} \frac{B_k}{k} - B_{n-1} - \sum_{k=1}^{n-1} \frac{B_k}{k} = B_n - B_{n-1} + \frac{B_n}{n},$$

and

$$n\Delta u_n = n(B_n - B_{n-1}) + B_n = n\Delta B_n + B_n.$$

If we take $(C,1)$- means of the last equality, we obtain

$$V_n^{(0)}(\Delta u) = \frac{1}{n+1}\sum_{k=1}^{n} k\Delta B_k + \frac{1}{n+1}\sum_{k=1}^{n} B_k = V_n^{(0)}(\Delta B) + \sigma_n^{(1)}(B) = B_n.$$

Since $\{B_n\}$ is bounded, we have $V_n^{(0)}(\Delta u) = O(1)$, $n \to \infty$, and

(68)
$$u_n - \sigma_n^{(1)}(u) = V_n^{(0)}(\Delta u) = O(1), \quad n \to \infty,$$

i.e.,

(69)
$$n\Delta\sigma_n^{(1)}(u) = O(1), \quad n \to \infty.$$

Applying the Littlewood Theorem, the existence of the limit (64) and (69) yield

(70)
$$\lim_{n} \sigma_n^{(1)}(u) = \lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} \sigma_n^{(1)}(u) x^n.$$

From (68) and (70), we obtain $u_n = O(1)$, $n \to \infty$. The assumption

$$\Delta V_n^{(0)}(\Delta u) = o(1), \qquad n \to \infty$$

together with (70) imply that $\Delta u_n = o(1)$, $n \to \infty$. Since $u_n = O(1)$, $n \to \infty$ and $\Delta u_n = o(1)$, $n \to \infty$, finally we obtain that the sequence $\{u_n\}$ is subsequentially convergent.

Taking generators to be moderately oscillatory, we obtain a considerable generalization of the above theorem.

**Theorem 17.** *For a real sequence $\{u_n\}$ such that $\Delta V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$, let the limit (64) exist. If $\{u_n\} \in \mathcal{U}(\mathcal{B}')$ where $\mathcal{B}'$ is the class of all moderately oscillatory sequences $\{B_n\}$, then $\{u_n\}$ is subsequentially convergent.*

**Proof.** From $\{u_n\} \in \mathcal{U}(\mathcal{B}')$ it follows that for some moderately oscillatory sequences $\{B_n\}$ and all $n$

$$u_n = B_n + \sum_{k=1}^{n} \frac{B_k}{k}.$$

From the above representation we obtain $V_n^{(0)}(\Delta u) = \frac{1}{n+1} \sum_{k=1}^{n} k\Delta u_k = B_n$. Since $\{B_n\}$ is moderately oscillatory, $\{V_n^{(0)}(\Delta u)\}$ is moderately oscillatory, and

$$V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u) = O(1), \quad n \to \infty, \tag{71}$$

i.e.,

$$n\Delta V_n^{(1)}(\Delta u) = O(1), \quad n \to \infty. \tag{72}$$

The existence of the limit (64) implies that

$$\lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} V_n^{(1)}(\Delta u) x^n = \tag{73}$$

$$\lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} (\sigma_n^{(1)}(u) - \sigma_n^{(2)}(u)) x^n = 0.$$

Applying the Littlewood Theorem, (72) and (73) imply that

$$V_n^{(1)}(\Delta u) = o(1), \quad n \to \infty, \tag{74}$$

From (71) and (74) it follows that $V_n^{(0)}(\Delta u) = O(1)$, $n \to \infty$. The rest of the proof follows from the proof of Theorem 16.

## 3.2 3.2. Tauberian theorems for regularly generated sequences

In the introduction of this chapter, we defined regularly generated sequences and proved corresponding Tauberian theorems. In this section, we will continue our study of regularly generated sequences by extending the class $\mathcal{U}(\mathcal{B})$ that we defined earlier.

In the previous section we let $\mathcal{L}$ be any linear space, $\mathcal{B}$ be a class of sequences $\{B_n\}$ from $\mathcal{L}$, and the class $\mathcal{U}(\mathcal{B})$ consist of sequences

$$u_n = B_n + \sum_{k=1}^{n} \frac{B_k}{k}, \qquad \text{for all } n.$$

Now we define a new class $\mathcal{U}(\mathcal{B}^*)$ [19, 20] as follows:

**Definition 5.** *Let $\mathcal{B}^*$ be the class of all sequences $\{B_n^*\}$ such that for some $\{B_n\} \in \mathcal{B}$ and all $n$*

$$B_n^* = \sum_{k=1}^{n} \frac{B_k}{k}.$$

The class $\mathcal{U}(\mathcal{B}^*)$ is the class of all regularly generated sequences by $\mathcal{B}^*$ if for any $\{u_n\} \in \mathcal{U}(\mathcal{B}^*)$ there exists a sequence $\{B_n^*\} \in \mathcal{B}^*$ such that for all $n$

$$u_n = B_n^* + \sum_{k=1}^{n} \frac{B_k^*}{k} = \sum_{k=1}^{n} \frac{B_k}{k} + \sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{B_j}{j}}{k} \tag{75}$$

While the class $\mathcal{U}(\mathcal{B})$ describes the classical slowly-oscillating series for all bounded and slowly-oscillating sequences $\{B_n\}$, the class $\mathcal{U}(\mathcal{B}^*)$ is an extension of the class $\mathcal{U}(\mathcal{B})$. The next theorem is the corresponding Tauberian theorem for that extended class.

**Theorem 18.** *Let the limit (1) exist. If $\{u_n\} \in (\mathcal{B}^*)$, then*

$$\lim_n u_n = \lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} u_n x^n.$$

**Proof.** The representation (75) implies that

$$n\Delta u_n = n(u_n - u_{n-1}) = n\left(B_n^* + \sum_{k=1}^{n} \frac{B_k^*}{k} - B_{n-1}^* - \sum_{k=1}^{n-1} \frac{B_k^*}{k}\right) =$$

$$= n\left(B_n^* - B_{n-1}^* + \frac{B_n^*}{n}\right) = n\left(\sum_{k=1}^{n} \frac{B_k}{k} - \sum_{k=1}^{n-1} \frac{B_k}{k} + \frac{\sum_{k=1}^{n} \frac{B_k}{k}}{n}\right) =$$

$$= n\left(\frac{B_n}{n} + \frac{\sum_{k=1}^{n} \frac{B_k}{k}}{n}\right) = B_n + \sum_{k=1}^{n} \frac{B_k}{k}$$

and

$$V_n^{(0)}(\Delta u) = \sigma_n^{(1)}(B) + \sigma_n^{(1)}(B^*).$$

Then

$$n\Delta u_n - V_n^{(0)}(\Delta u) = B_n - \sigma_n^{(1)}(B) + B_n^* - \sigma_n^{(1)}(B^*) =$$

$$= B_n - \sigma_n^{(1)}(B) + \frac{1}{n+1}\sum_{k=0}^{n} k\Delta B_k^* = B_n - \sigma_n^{(1)}(B) + \frac{1}{n+1}\sum_{k=0}^{n} k\frac{B_k}{k} =$$

$$= B_n - \sigma_n^{(1)}(B) + \sigma_n^{(1)}(B) = B_n.$$

Since $\{B_n\}$ is bounded, we have

$$n\Delta u_n - V_n^{(0)}(\Delta u) = n\Delta V_n^{(0)}(\Delta u) = O(1), \quad n \to \infty. \tag{76}$$

The existence of the limit (1) implies that

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}(u_n - \sigma_n^{(1)}(u))x^n = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} V_n^{(0)}(\Delta u)x^n = 0. \tag{77}$$

Applying the Littlewood Theorem, from (76) and (77) we obtain $V_n^{(0)}(\Delta u) = o(1)$, $n \to \infty$. Hence

$$u_n - \sigma_n^{(1)}(u) = V_n^{(0)}(\Delta u) = o(1), \quad n \to \infty. \tag{78}$$

Finally by the original Tauber Theorem we have $\lim_n u_n = \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} u_n x^n$.

The class $\mathcal{U}(\mathcal{B})$ is contained in $\mathcal{U}(\mathcal{B}^*)$. Because, if $\mathcal{B}$ is the class of all bounded slowly-oscillating sequences then $\mathcal{B}^*$ is the class of all sequences $\{B_n^*\}$ such that for some $\{B_n\} \in \mathcal{B}$ and all $n$, $B_n^* = \sum_{k=1}^{n} \frac{B_k}{k}$. Since $\{u_n\} \in \mathcal{U}(\mathcal{B}^*)$ means that for some slowly-oscillating $\{B_n^*\}$

$$u_n = B_n^* + \sum_{k=1}^{n} \frac{B_k^*}{k},$$

comparing this with the classical case, i.e.,

$$u_n = B_n + \sum_{k=1}^{n} \frac{B_k}{k}$$

for some bounded slowly-oscillating sequence $\{B_n\}$, we have

$$\mathcal{U}(\mathcal{B}) \underset{\neq}{\subset} \mathcal{U}(\mathcal{B}^*).$$

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