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AN ALTMAN TYPE GENERALIZATION OF THE BRÉZIS–BROWDER ORDERING PRINCIPLE

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Abstract. By making use the ideas of M. Altman, we prove a more natural generalization of the famous ordering principle of H. Brézis and F.E. Browder.

1. Introduction

The following simple, but important ordering principle was first proved by Brézis and Browder [2]. (See also [9, p.163].)

Theorem 1. *Suppose that φ is an increasing real-valued function of an ordered set X such that:*

(a) *for each $x \in X$ there exists $y \in X$, with $x \leq y$, such that $\varphi(x) < \varphi(y)$;*

(b) *for any increasing sequence (x_n) in X, with $\sup_{n\in\mathbb{N}} \varphi(x_n) < +\infty$, the sequence (x_n) is bounded.*

Then, $\sup_{y\geq x} \varphi(y) = +\infty$ for all $x \in X$.

As a direct consequence of this theorem, the above authors have also established the following useful maximality principle.

Corollary 1. *If X is an ordered set such that each increasing sequence in X is bounded, and there exists a strictly increasing real-valued function of X which is bounded above, then X has a maximal element.*

The above Theorem 1, having in mind the function defined by $\Phi(x, y) = \varphi(x) - \varphi(y)$ for all $x, y \in X$, was to some extent generalized by Altman [1] in the following less satisfactory dual form. (See also [8, p.515].)

Theorem 2. *Let X be an ordered set, and suppose that Φ is a real-valued function of X^2 such that:*

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(1) $-\infty < \inf_{x \leq y} \Phi(x, y)$ *for all* $y \in X$;
(2) $\Phi(x, y) \leq 0$ *for all* $x, y \in X$ *with* $x \leq y$;
(3) $\Phi(x, y_2) \leq \Phi(x, y_1)$ *for all* $x, y_1, y_2 \in X$ *with* $y_1 \leq y_2$;
(4) *every decreasing sequence* (x_n) *in* X *is bounded below, and moreover* $\underline{\lim}_{n\to\infty} \Phi(x_{n+1}, x_n) = 0$.

Then, there exists a $y \in X$ *such that* $\Phi(x, y) = 0$ *for all* $x \in X$ *with* $x \leq y$.

As a direct consequence of this theorem, M. Altman has also stated

Corollary 2. *Suppose that the hypotheses of Theorem 1 are satisfied, with the assumption (2) replaced by the one that* $\Phi(x, y) < 0$ *for all* $x, y \in X$ *with* $x < y$. *Then,* X *has a minimal element.*

Actually, the above authors have proved that the family of all maximal (minimal) elements of X is cofinal upward (downward). However, this statement seems to be of no particular importance. Since, for each $x_o \in X$, we may restrict ourselves to the ordered set $X_0 = \{x \in X : x_o \leq x\}$ $\big(X_0 = \{x \in X : x \leq x_o\}\big)$.

Therefore, in the sequel, we shall only give some natural generalizations of Theorem 1 and Corollary 1. For this, we shall have in mind the function defined by $\Phi(x, y) = \varphi(y) - \varphi(x)$ for all $x, y \in X$. Moreover, we shall replace ordered sets by preordered ones. Applications of our generalizations will be presented elsewhere.

2. An ordering principle

A reflexive and transitive relation $\leq$ on a nonvoid set X is called a preorder on X, and the ordered pair $X(\leq) = (X, \leq)$ is called a preordered set. In the sequel, we shall simply write X in place of $X(\leq)$.

Now, by making use the ideas of H. Brézis, F.E. Browder and M. Altman, we can easily prove the following generalization of Theorem 1.

Theorem 1. *Let* X *be a preordered set, and suppose that* Φ *is a real-valued function of* X^2 *such that:*
(1) $0 \leq \Phi(x, y)$ *for all* $x, y \in X$ *with* $x \leq y$;
(2) *for each* $x \in X$ *there exists* $y \in X$, *with* $x \leq y$, *such that* $0 < \Phi(x, y)$;
(3) $\Phi(x_2, y) \leq \Phi(x_1, y)$ *for all* $x_1, x_2, y \in X$ *with* $x_1 \leq x_2$ *and* $x_2 \leq y$;
(4) *for any increasing sequence* (x_n) *in* X, *with* $\sup_{n\in\mathbb{N}} \Phi(x_1, x_n) < +\infty$, *the sequence* (x_n) *is bounded and* $\underline{\lim}_{n\to\infty} \Phi(x_n, x_{n+1}) = 0$.

Then, $\sup_{y\geq x} \Phi(x, y) = +\infty$ *for all* $x \in X$.

Proof. For each $x \in X$, define

$$\varphi(x) = \sup_{y \geq x} \Phi(x, y).$$

Then, by the condition (2), it is clear that φ is a positive function of X. Moreover, by using the property (3) and the transitivity of the relation $\leq$, we can easily see that φ is a decreasing function of X. Namely, if $x_1, x_2, y \in X$ such that $x_1 \leq x_2$ and $x_2 \leq y$, then we evidently have

$$\Phi(x_2, y) \leq \Phi(x_1, y) \leq \sup_{z \geq x_1} \Phi(x_1, z) = \varphi(x_1).$$

And hence, it is already quite obvious that

$$\varphi(x_2) = \sup_{y \geq x_2} \Phi(x_2, y) \leq \varphi(x_1).$$

Now, to prove the required assertion that $\varphi(x) = +\infty$ for all $x \in X$, assume on the contrary that there exists an $x_1 \in X$ such that $\varphi(x_1) < +\infty$. Then

$$\varphi(x_1) - 1 < \varphi(x_1) = \sup_{y \geq x_1} \Phi(x_1, y).$$

Therefore, there exists an $x_2 \in X$, with $x_1 \leq x_2$, such that

$$\varphi(x_1) - 1 < \Phi(x_1, x_2).$$

Moreover, we can also note that $\varphi(x_2) \leq \varphi(x_1) < +\infty$. Therefore,

$$\varphi(x_2) - 2^{-1} < \varphi(x_2) = \sup_{y \geq x_2} \Phi(x_2, y).$$

Thus, there exists an $x_3 \in X$, with $x_2 \leq x_3$, such that

$$\varphi(x_2) - 2^{-1} < \Phi(x_2, x_3).$$

Hence, by induction, it is clear that there exists an increasing sequence (x_n) in X such that

$$\varphi(x_n) - n^{-1} < \Phi(x_n, x_{n+1})$$

for all $n \in \mathbb{N}$. Moreover, we can also note that

$$\sup_{n \in \mathbb{N}} \Phi(x_1, x_n) \leq \sup_{y \geq x_1} \Phi(x_1, y) = \varphi(x_1) < +\infty.$$

Therefore, by the property (4), there exist an $x \in X$ such that $x_n \leq x$ for all $n \in \mathbb{N}$. Moreover, there exists a strictly increasing sequence (k_n) in $\mathbb{N}$ such that

$$\lim_{n \to \infty} \Phi(x_{k_n}, x_{k_n+1}) = 0.$$

Now, if $y \in X$ such that $x \leq y$, then by the property (1) and the above observations it is clear that

$$0 \leq \Phi(x, y) = \sup_{z \geq x} \Phi(x, z) = \varphi(x) \leq \varphi(x_{k_n}) <$$
$$\Phi(x_{k_n}, x_{k_n+1}) + k_n^{-1} \leq \Phi(x_{k_n}, x_{k_n+1}) + n^{-1}$$

for all $n \in \mathbb{N}$. Hence, by letting $n \to \infty$, we can infer that $0 \le \Phi(x, y) \le 0$, and thus $\Phi(x, y) = 0$. However, this is already a contradiction by the property (2). Therefore, the assertion of the theorem is true.

Now, to demonstrate the appropriateness of the above theorem, we can also prove

Corollary 3. *Suppose that φ is an increasing real-valued function of a preordered set X such that:*

(a) *for each $x \in X$ there exists $y \in X$, with $x \le y$, such that $\varphi(x) < \varphi(y)$;*
(b) *for any increasing sequence (x_n) in X, with $\sup_{n\in\mathbb{N}} \varphi(x_n) < +\infty$, the sequence (x_n) is bounded.*

Then, $\sup_{y\ge x} \varphi(y) = +\infty$ for all $x \in X$.

Proof. Define

$$\Phi(x, y) = \varphi(y) - \varphi(x)$$

for all $x, y \in X$. Then, by the increasingness of φ and the property (a), it is clear that the hypotheses (1) and (2) of Theorem 3 are satisfied.

Moreover, if $x_1, x_2, y \in X$ such that $x_1 \le x_2$, then also by the increasingness of φ, it is clear that

$$\Phi(x_2, y) = \varphi(y) - \varphi(x_2) \le \varphi(y) - \varphi(x_1) = \Phi(x_1, y).$$

Thus, in particular, the hypothesis (3) of Theorem 3 also holds.

On the other hand, if (x_n) is an increasing sequence in X such that $\sup_{n\in\mathbb{N}} \Phi(x_1, x_n) < +\infty$, then we evidently have

$$\sup_{n\in\mathbb{N}} \varphi(x_n) = \sup_{n\in\mathbb{N}} \big(\varphi(x_n) - \varphi(x_1) + \varphi(x_1)\big) =$$
$$\sup_{n\in\mathbb{N}} \big(\Phi(x_1, x_n) + \varphi(x_1)\big) \le \sup_{n\in\mathbb{N}} \Phi(x_1, x_n) + \varphi(x_1) < +\infty.$$

Therefore, by the property (b), there exists an $x \in X$, such that $x_n \le x$ for all $n \in \mathbb{N}$. Hence, by the increasingness of φ, it is clear that $\big(\varphi(x_n)\big)$ is an increasing sequence in $\mathbb{R}$ such that

$$\lim_{n\to\infty} \varphi(x_n) = \sup_{n\in\mathbb{N}} \varphi(x_n) \le \varphi(x).$$

Therefore,

$$\lim_{n\to\infty} \Phi(x_n, x_{n+1}) = \lim_{n\to\infty} \big(\varphi(x_{n+1}) - \varphi(x_n)\big) =$$
$$\lim_{n\to\infty} \varphi(x_{n+1}) - \lim_{n\to\infty} \varphi(x_n) = 0.$$

Thus, in particular, the hypothesis (4) of Theorem 3 also holds. Therefore, by the conclusion of Theorem 3, we necessarily have

$$+\infty = \sup_{y\ge x} \Phi(x, y) = \sup_{y\ge x} \big(\varphi(y) - \varphi(x)\big) \le \sup_{y\ge x} \varphi(y) - \varphi(x),$$

and hence $\sup\limits_{y\ge x} \varphi(y) = +\infty$ for all $x \in X$.

3. A maximality principle

An element x of a preordered set X is called **maximal** if $x \leq y$ implies $y \leq x$ for all $y \in X$. In particular, we say that x is a **strong maximal element** of X if $x \leq y$ implies $x = y$ for all $y \in X$.

Note that thus each strong maximal element of X is also a maximal element of X. Moreover, if in particular X is a partially ordered set, then the two notions coincide.

A possible extension of Zorn's lemma says that if each well-ordered subset of a preordered set X has an upper bound, then X has a maximal element. (See [7] and [8, p. 32].)

Therefore, it is of some interest to establish the following generalization of Corollary 1.

Theorem 1. *Let X be a preordered set such that there exists a real-valued function Φ of X^2 such that:*

(1) $\sup\limits_{y \geq x} \Phi(x, y) < +\infty$ *for some* $x \in X$*;*

(2) $0 < \Phi(x, y)$ *for all* $x, y \in X$ *with* $x < y$*;*

(3) $\Phi(x_2, y) \leq \Phi(x_1, y)$ *for all* $x_1, x_2, y \in X$ *with* $x_1 \leq x_2$ *and* $x_2 \leq y$*;*

(4) *for any increasing sequence* (x_n) *in* X*, with* $\sup_{n \in \mathbb{N}} \Phi(x_1, x_n) < +\infty$*, the sequence* (x_n) *is bounded and* $\underline{\lim}_{n \to \infty} \Phi(x_n, x_{n+1}) = 0$*.*

Then, X has a strong maximal element.

Proof. The condition (4) implies that $\Phi(x, x) = \underline{\lim}_{n \to \infty} \Phi(x, x) = 0$ for all $x \in X$. Therefore, the hypothesis (1) of Theorem 3 is also satisfied. However, the conclusion of Theorem 3 does not hold. Therefore, the hypothesis (2) of Theorem 3 cannot also hold. Thus, there exists an $x \in X$ such that $\Phi(x, y) = 0$ for all $y \in X$ with $x \leq y$.

Now, it remains only to show that x is a strong maximal element of X. For this, note that if this not the case, then there exists a $y \in X$, with $x \leq y$, such that $x \neq y$, and hence $x < y$. Then, by the above property of x, we necessarily have $\Phi(x, y) = 0$. Moreover, by the condition (2), we also have $0 < \Phi(x, y)$. And this contradiction proves the required maximality of x.

Now, to demonstrate the appropriateness of the above theorem, we can also prove

Corollary 4. *Let X be a preordered set such that there exists a strictly increasing real-valued function φ of X such that:*

(a) $\sup_{y \geq x} \varphi(y) < +\infty$ *for some* $x \in X$*;*

(b) *for any increasing sequence* (x_n) *in* X*, with* $\sup_{n \in \mathbb{N}} \varphi(x_n) < +\infty$*, the sequence* (x_n) *is bounded.*

Then, X has a strong maximal element.

Proof. Again, define $\Phi(x,y) = \varphi(y) - \varphi(x)$ for all $x, y \in X$. Then, from the proof of Corollary 3, it is clear that the hypotheses of Theorem 4 are satisfied. Therefore, the required assertion is also true.

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