

Regularized traces of D_1^2 type Sturm-Liouville operators with homogeneous delay

DRAGANA NEDIĆ[✉] 0000-0003-2572-1755,
ISMET KALČO[✉] 0000-0002-9154-7428,
GORAN CVJETKOVIĆ[✉] 0009-0006-5380-2869

ABSTRACT. The paper considers a special case of the operator $D^2 = D^2(h, H, q, \alpha)$, $h, H \in R$, $q(x) \in L_2[0, \pi]$, operator D_1^2 with boundary conditions $y(0) = 0$, $y'(\pi) + Hy(\pi) = 0$, ($h = \infty$, $H \in R \setminus \{0\}$). Presentation of characteristic functions in two ways from the direct problem, through transformations of multiple integral and the introduction of transition functions, as well as the of characteristic functions via infinite products according to Hadamard's factorization theorem enabled the calculation of the first regularized traces.

1. INTRODUCTION

The field of mathematical analysis that studies the spectrum, the set of eigenvalues and corresponding eigenvectors of linear operators defined of infinite Hilbert spaces, is called spectral theory. It is effective in solving a very broad class of problems in mathematics, physics, mechanics, electronics and other natural sciences. The most intensive development of this mathematical discipline is represented by inverse spectral problems. Significant results have been achieved in this area as evidenced by, numerous works published on the subject. We mention some of them [1–8].

For the special case of the operator

$$D^2 = D^2(h, H, q, \alpha), \quad q(x) \in L_2[0, \pi], \quad \alpha \in (0, 1),$$

the operator D_1^2 ($h = \infty$, $H \in R \setminus \{0\}$) characteristic functions are presented in two ways: from the direct problem through transformations of multiple integrals and by introducing transition functions, as well as representing functions using infinite products according to Hadamard's factorization theorem. This demonstrations that the asymptotics of obtained from infinite

2020 *Mathematics Subject Classification*. Primary: 47B02; Secondary: 39A06.

Key words and phrases. Regularized traces, Homogeneous delay, Characteristic function.

Full paper. Received 23 Mar 2026, accepted 20 Jul 2026, available online 2 Oct 2026.

products are the same as those from the direct problem, which enables the asymptotics of the eigenvalues of the operator. Based on the representation of characteristic function in the two mentioned ways and detailed explanations, the first regularized traces calculated.

2. THE FIRST REGULARIZED TRACES OF THE OPERATOR D_1^2

2.1. Characteristic functions of the operator D_1^2 . For the observed operator D_1^2 defined by a differential equation

$$(1) \quad -y''(x) + q(x)y(\alpha x) = \lambda y(x), \quad q(x) \in L_2[0, \pi], \quad \alpha \in (0, 1)$$

and boundary conditions

$$(2) \quad y(0) = 0,$$

$$(3) \quad y'(\pi) + Hy(\pi) = 0, \quad h = \infty, \quad H \in R \setminus \{0\},$$

characteristic functions were constructed, and by introducing transition functions, asymptotic decomposition of zeros of characteristic functions and eigenvalues of operator was enabled [2]. By applying the method of variation of constants, the method of successive approximations and using the given boundary conditions (2), (3) from equation (1), the characteristic function of the operator D_1^2 was obtained [1–5]

$$(4) \quad \begin{aligned} F(z) = & z \cos \pi z + H \sin \pi z + a_{cs}(z) + \frac{H}{z} a_{s2}(z) + \frac{1}{z} a_{cs2}(z) + \\ & + \frac{H}{z^2} a_{s3}(z) + \frac{1}{z^2} a_{cs3}(z) + \frac{H}{z^3} a_{s4}(z) + \\ & + \sum_{k=4}^{\infty} \frac{1}{z^{k-1}} a_{cs^k}(z) + \sum_{k=4}^{\infty} \frac{H}{z^k} a_{s^{k+1}}(z), \end{aligned}$$

where

$$\begin{aligned} a_{s2}(z) &= \int_0^\pi q(t_1) \sin z(\pi - t_1) \cdot \sin z\alpha t_1 \, dt_1, \\ a_{cs}(z) &= \int_0^\pi q(t_1) \cos z(\pi - t_1) \cdot \sin z\alpha t_1 \, dt_1, \\ a_{s^k}(z) &= \int_0^\pi q(t_1) \sin z(\pi - t_1) \cdot a_{s^{k-1}}(\alpha t_1, z) \, dt_1, \quad k = 3, 4, \dots, \\ a_{cs^k}(z) &= \int_0^\pi q(t_1) \cos z(\pi - t_1) \cdot a_{s^k}(\alpha t_1, z) \, dt_1, \quad k = 2, 3, \dots, \end{aligned}$$

where the labels were used

$$\begin{aligned} a_{s2}(z) &= a_{s2}(\pi, z), & a_{cs}(z) &= a_{cs}(\pi, z), \\ a_{s^k}(z) &= a_{s^k}(\pi, z), & a_{cs^k}(z) &= a_{cs^k}(\pi, z). \end{aligned}$$

By defining the so-called transition functions $\tilde{q}_1(\theta)$, an additional transformation of the characteristic functions is enabled, opening the possibility of obtaining a more complete asymptotic decomposition of function zeros

$$\tilde{q}_1(\theta) = \begin{cases} 0, & \theta \in \left(\frac{1+\alpha}{2}\pi, \pi\right]; \\ \frac{1}{1+\alpha}q\left(\frac{2\theta}{1+\alpha}\right), & \theta \in \left(\frac{1-\alpha}{2}\pi, \frac{1+\alpha}{2}\pi\right]; \\ \frac{1}{1+\alpha}q\left(\frac{2\theta}{1+\alpha}\right) + (-1)^j \frac{1}{1-\alpha}q\left(\frac{2\theta}{1-\alpha}\right), & \theta \in \left[0, \frac{1-\alpha}{2}\pi\right]. \end{cases}$$

By means of $\tilde{q}_1(\theta)$ it is obtained

$$\begin{aligned} a_{cs}(z) &= - \int_0^\pi \tilde{q}_1(\theta) \sin z(\pi - 2\theta) d\theta = -\tilde{b}^{(1)}(z), \\ a_{s^2}(z) &= \int_0^\pi \tilde{q}_1(\theta) \cos z(\pi - 2\theta) d\theta = \tilde{a}^{(1)}(z). \end{aligned}$$

Using functions $Q_{21}(\theta, q(\theta)) \in L_2[0, \pi]$, where

$$\tilde{Q}_{21}(\theta, q(\theta)) = \frac{1}{2} \left(\tilde{Q}_2^{(1)}(\theta, q(\theta)) + \tilde{Q}_2^{(2)}(\theta, q(\theta)) - \tilde{Q}_2^{(3)}(\theta, q(\theta)) - \tilde{Q}_2^{(4)}(\theta, q(\theta)) \right),$$

and

$$\begin{aligned} \tilde{Q}_2^{(1)}(\theta, q(\theta)) &= \begin{cases} 0, & \theta \in \left(\frac{1+\alpha^2}{2}\pi, \pi\right]; \\ \int_{\frac{2\theta}{1+\alpha^2}}^{\frac{\pi}{1+\alpha^2}} q(t_1)q\left(\frac{2\theta}{1+\alpha} - \frac{1-\alpha}{1+\alpha}t_1\right) \frac{dt_1}{1+\alpha}, & \theta \in \left(\frac{1-\alpha}{2}\pi, \frac{1+\alpha^2}{2}\pi\right]; \\ \int_{\frac{2\theta}{1+\alpha^2}}^{\frac{1-\alpha}{2}} q(t_1)q\left(\frac{2\theta}{1+\alpha} - \frac{1-\alpha}{1+\alpha}t_1\right) \frac{dt_1}{1+\alpha}, & \theta \in \left[0, \frac{1-\alpha}{2}\pi\right]; \end{cases} \\ \tilde{Q}_2^{(2)}(\theta, q(\theta)) &= \begin{cases} 0, & \theta \in \left(\frac{1+\alpha}{2}\pi, \pi\right]; \\ \int_{\frac{2\theta}{1+\alpha}}^{\frac{\pi}{1+\alpha}} q(t_1)q\left(t_1 - \frac{2\theta}{1+\alpha}\right) \frac{dt_1}{1+\alpha}, & \theta \in \left(\frac{1-\alpha^2}{2}\pi, \frac{1+\alpha}{2}\pi\right]; \\ \int_{\frac{2\theta}{1+\alpha}}^{\frac{1-\alpha^2}{2}} q(t_1)q\left(t_1 - \frac{2\theta}{1+\alpha}\right) \frac{dt_1}{1+\alpha}, & \theta \in \left[0, \frac{1-\alpha^2}{2}\pi\right]; \end{cases} \\ \tilde{Q}_2^{(3)}(\theta, q(\theta)) &= \begin{cases} 0, & \theta \in \left(\frac{1-\alpha^2}{2}\pi, \pi\right]; \\ \int_{\frac{2\theta}{1-\alpha^2}}^{\frac{\pi}{1-\alpha^2}} q(t_1)q\left(\frac{2\theta}{1-\alpha} - t_1\right) \frac{dt_1}{1-\alpha}, & \theta \in \left(\frac{1-\alpha}{2}\pi, \frac{1+\alpha^2}{2}\pi\right]; \\ \int_{\frac{2\theta}{1-\alpha^2}}^{\frac{1-\alpha}{2}} q(t_1)q\left(\frac{2\theta}{1-\alpha} - t_1\right) \frac{dt_1}{1-\alpha}, & \theta \in \left[0, \frac{1-\alpha}{2}\pi\right]; \end{cases} \\ \tilde{Q}_2^{(4)}(\theta, q(\theta)) &= \begin{cases} 0, & \theta \in \left(\frac{1+\alpha}{2}\pi, \pi\right]; \\ \int_{\frac{2\theta}{1+\alpha}}^{\frac{\pi}{1+\alpha}} q(t_1)q\left(\frac{1+\alpha}{1-\alpha}t_1 - \frac{2\theta}{1-\alpha}\right) \frac{dt_1}{1-\alpha}, & \theta \in \left(\frac{1+\alpha^2}{2}\pi, \frac{1+\alpha}{2}\pi\right]; \\ \int_{\frac{2\theta}{1+\alpha}}^{\frac{1+\alpha^2}{2}} q(t_1)q\left(\frac{1+\alpha}{1-\alpha}t_1 - \frac{2\theta}{1-\alpha}\right) \frac{dt_1}{1-\alpha}, & \theta \in \left[0, \frac{1+\alpha^2}{2}\pi\right]; \end{cases} \end{aligned}$$

functions are presented

$$a_{cs^2}(z) = \int_0^\pi \tilde{Q}_{21}(\theta, q(\theta)) \cos z(\pi - 2\theta) d\theta = a^{(2,1)}(z),$$

$$a_{s^3}(z) = \int_0^\pi \tilde{Q}_{21}(\theta, q(\theta)) \sin z(\pi - 2\theta) d\theta = b^{(2,1)}(z).$$

Next, the functions

$$a_{cs^3}(z) = - \int_0^\pi \tilde{Q}_{31}(\theta, q(\theta)) \sin z(\pi - 2\theta) d\theta = -b^{(3,1)}(z),$$

$$a_{s^4}(z) = \int_0^\pi \tilde{Q}_{31}(\theta, q(\theta)) \cos z(\pi - 2\theta) d\theta = a^{(3,1)}(z),$$

were obtained using

$$\begin{aligned} \tilde{Q}_{31}(\theta, q(\theta)) = & \frac{1}{4} \left(-\tilde{Q}_3^{(1)}(\theta, q(\theta)) - \tilde{Q}_3^{(2)}(\theta, q(\theta)) + \tilde{Q}_3^{(3)}(\theta, q(\theta)) + \right. \\ & + \tilde{Q}_3^{(4)}(\theta, q(\theta)) - \tilde{Q}_3^{(5)}(\theta, q(\theta)) - \tilde{Q}_3^{(6)}(\theta, q(\theta)) + \\ & \left. + \tilde{Q}_3^{(7)}(\theta, q(\theta)) + \tilde{Q}_3^{(8)}(\theta, q(\theta)) \right), \quad \tilde{Q}_{31} \in L_2[0, \pi]. \end{aligned}$$

We will give examples of functions $\tilde{Q}_3^{(2)}(\theta, q(\theta))$ and $\tilde{Q}_3^{(8)}(\theta, q(\theta))$:

$$\begin{aligned} \tilde{Q}_3^{(2)}(\theta, q(\theta)) = & \begin{cases} 0, & \theta \in \left[\frac{1+\alpha^2}{2}\pi, \pi\right]; \\ \int_{\frac{1-\alpha^2}{1+\alpha^2}}^\pi \frac{1}{1+\alpha} q(t_1) \int_{\frac{2\theta}{1+\alpha} - \frac{1-\alpha}{1+\alpha} t_1}^{\alpha t_1} q(t_2) q\left(\frac{1-\alpha}{1+\alpha} t_1 + t_2 - \frac{2\theta}{1+\alpha}\right) dt_2 dt_1, & \theta \in \left[\frac{1-\alpha^3}{2}\pi, \frac{1+\alpha^2}{2}\pi\right]; \\ \int_{\frac{1-\alpha^3}{1+\alpha^2}}^\pi \frac{1}{1+\alpha} q(t_1) \int_{\frac{2\theta}{1+\alpha} - \frac{1-\alpha}{1+\alpha} t_1}^{\frac{2\theta}{1+\alpha^2} - \frac{1}{1+\alpha} t_1} q(t_2) q\left(\frac{1-\alpha}{1+\alpha} t_1 + t_2 - \frac{2\theta}{1+\alpha}\right) dt_2 dt_1 + \\ + \int_{\frac{1-\alpha^3}{1+\alpha^2}}^{\frac{2\theta}{1+\alpha^2}} \frac{1}{1+\alpha} q(t_1) \int_{\frac{2\theta}{1+\alpha} - \frac{1-\alpha}{1+\alpha} t_1}^{\alpha t_1} q(t_2) q\left(\frac{1-\alpha}{1+\alpha} t_1 + t_2 - \frac{2\theta}{1+\alpha}\right) dt_2 dt_1, & \theta \in \left[\frac{1-\alpha}{2}\pi, \frac{1-\alpha^3}{2}\pi\right]; \\ \int_{\frac{1-\alpha^3}{1+\alpha^2}}^{\frac{2\theta}{1+\alpha^2}} \frac{1}{1+\alpha} q(t_1) \int_{\frac{2\theta}{1+\alpha} - \frac{1-\alpha}{1+\alpha} t_1}^{\frac{2\theta}{1+\alpha^2} - \frac{1}{1+\alpha} t_1} q(t_2) q\left(\frac{1-\alpha}{1+\alpha} t_1 + t_2 - \frac{2\theta}{1+\alpha}\right) dt_2 dt_1 + \\ + \int_{\frac{1-\alpha^3}{1+\alpha^2}}^{\frac{2\theta}{1+\alpha^2}} \frac{1}{1+\alpha} q(t_1) \int_{\frac{2\theta}{1+\alpha} - \frac{1-\alpha}{1+\alpha} t_1}^{\alpha t_1} q(t_2) q\left(\frac{1-\alpha}{1+\alpha} t_1 + t_2 - \frac{2\theta}{1+\alpha}\right) dt_2 dt_1, & \theta \in \left(0, \frac{1-\alpha}{2}\pi\right]; \end{cases} \end{aligned}$$

$$\tilde{Q}_3^{(8)}(\theta, q(\theta)) = \begin{cases} 0, & \theta \in \left[\frac{1-\alpha^3}{2}\pi, \pi\right]; \\ \int \frac{\pi}{1-\alpha^3} \frac{1}{1-\alpha} q(t_1) \int \frac{\alpha t_1}{1-\alpha^2} - \frac{1}{1+\alpha} t_1 q(t_2) q\left(\frac{2\theta}{1-\alpha} - t_1 - t_2\right) dt_2 dt_1, & \theta \in \left[\frac{1-\alpha^2}{2}\pi, \frac{1-\alpha^3}{2}\pi\right]; \\ \int \frac{\pi}{1-\alpha^2} \frac{1}{1-\alpha} q(t_1) \int \frac{\frac{2\theta}{1-\alpha} - t_1}{1-\alpha^2} - \frac{1}{1+\alpha} t_1 q(t_2) q\left(\frac{2\theta}{1-\alpha} - t_1 - t_2\right) dt_2 dt_1 + \\ + \int \frac{\frac{2\theta}{1-\alpha^2}}{1-\alpha^3} \frac{1}{1-\alpha} q(t_1) \int \frac{\alpha t_1}{1-\alpha^2} - \frac{1}{1+\alpha} t_1 q(t_2) q\left(\frac{2\theta}{1-\alpha} - t_1 - t_2\right) dt_2 dt_1, & \theta \in \left[\frac{1-\alpha}{2}\pi, \frac{1-\alpha^2}{2}\pi\right]; \\ \int \frac{\frac{2\theta}{1-\alpha}}{1-\alpha^2} \frac{1}{1-\alpha} q(t_1) \int \frac{\frac{2\theta}{1-\alpha} - t_1}{1-\alpha^2} - \frac{1}{1+\alpha} t_1 q(t_2) q\left(\frac{2\theta}{1-\alpha} - t_1 - t_2\right) dt_2 dt_1 + \\ + \int \frac{\frac{2\theta}{1-\alpha^2}}{1-\alpha^3} \frac{1}{1-\alpha} q(t_1) \int \frac{\alpha t_1}{1-\alpha^2} - \frac{1}{1+\alpha} t_1 q(t_2) q\left(\frac{2\theta}{1-\alpha} - t_1 - t_2\right) dt_2 dt_1, & \theta \in \left(0, \frac{1-\alpha}{2}\pi\right]. \end{cases}$$

Other functions

$$\begin{aligned} & \tilde{Q}_3^{(1)}(\theta, q(\theta)), \quad \tilde{Q}_3^{(3)}(\theta, q(\theta)), \quad \tilde{Q}_3^{(4)}(\theta, q(\theta)), \\ & \tilde{Q}_3^{(5)}(\theta, q(\theta)), \quad \tilde{Q}_3^{(6)}(\theta, q(\theta)), \quad \tilde{Q}_3^{(7)}(\theta, q(\theta)) \end{aligned}$$

are obtained analogously [1, 2].

Based on the above transformations, an asymptotic representation of the characteristic functions was obtained [2]

$$\begin{aligned} (5) \quad F(z) &= z \cos \pi z + H \sin \pi z - \tilde{b}^{(1)}(z) + \frac{H}{z} \tilde{a}^{(1)}(z) + \frac{1}{z} a^{(2,1)}(z) + \\ &+ \frac{H}{z^2} b^{(2,1)}(z) - \frac{1}{z^2} b^{(3,1)}(z) + O\left(\frac{a^{(3,1)}(z)}{z^3}\right). \end{aligned}$$

The zeros of the function (5) have an asymptotic decomposition [2]

$$\begin{aligned} (6) \quad z_{n_1} &= n_1 + \frac{1}{n_1} \left(\frac{H}{\pi} - \frac{1}{\pi} \tilde{a}_{2n_1}^{(1)} \right) + \\ &+ \frac{1}{n_1^2} \left(\frac{1}{\pi} b_{2n_1}^{(2,1)} + \frac{2H}{\pi^2} \hat{b}_{2n_1}^{(1)} + \frac{1}{\pi} \tilde{a}_{2n_1}^{(1)} \cdot \tilde{b}_{2n_1}^{(1)} - \frac{2}{\pi^2} \tilde{a}_{2n_1}^{(1)} \cdot \hat{b}_{2n_1}^{(1)} \right) + \\ &+ O\left(\frac{a_{2n_1}^{(3,1)}}{n_1^3}\right) \end{aligned}$$

and eigenvalues asymptotic [2]

$$(7) \quad \begin{aligned} \lambda_{n_1} = & n_1^2 + \frac{2H}{\pi} - \frac{2}{\pi} \tilde{a}_{2n_1}^{(1)} + \\ & \frac{1}{n_1} \left(\frac{2}{\pi} b_{2n_1}^{(2,1)} + \frac{4H}{\pi^2} \hat{b}_{2n_1}^{(1)} + \frac{2}{\pi} \tilde{a}_{2n_1}^{(1)} \cdot \tilde{b}_{2n_1}^{(1)} - \frac{4}{\pi^2} \tilde{a}_{2n_1}^{(1)} \cdot \hat{b}_{2n_1}^{(1)} \right) + \\ & + O\left(\frac{a_{2n_1}^{(3,1)}}{n_1^2}\right), \quad n \rightarrow \infty, \quad n_1 = n + \frac{1}{2}. \end{aligned}$$

Here, the number sequences are given by

$$\begin{aligned} \tilde{a}_{2n_1}^{(1)} &= \int_0^\pi \tilde{q}_1(\theta) \cos 2n_1 \theta \, d\theta, & a_{2n_1}^{(3,1)} &= \int_0^\pi \tilde{Q}_{31}(\theta, q(\theta)) \cos 2n_1 \theta \, d\theta, \\ \tilde{b}_{2n_1}^{(1)} &= \int_0^\pi \tilde{q}_1(\theta) \sin 2n_1 \theta \, d\theta, & \hat{b}_{2n_1}^{(1)} &= \int_0^\pi \theta \tilde{q}_1(\theta) \sin 2n_1 \theta \, d\theta, \\ \tilde{b}_{2n_1}^{(2,1)} &= \int_0^\pi \tilde{Q}_{21}(\theta, q(\theta)) \sin 2n_1 \theta \, d\theta. \end{aligned}$$

2.2. Operator D_1^2 traces. The characteristic function $F(z)$ represented by the infinite product is given by the relation

$$(8) \quad \begin{aligned} F(z) = & z \cos \pi z + z \cos \pi z \cdot \sum_{n=0}^{\infty} \frac{\lambda_{n_1} - n_1^2}{n_1^2 - z^2} + \frac{z \cos \pi z}{2} \cdot \\ & \cdot \left[\left(\sum_{n=0}^{\infty} \frac{\lambda_{n_1} - n_1^2}{n_1^2 - z^2} \right)^2 - \sum_{n=0}^{\infty} \left(\frac{\lambda_{n_1} - n_1^2}{n_1^2 - z^2} \right)^2 \right] + z \cos \pi z \cdot \psi(z). \end{aligned}$$

The analytical expression of the function ψ is not given because it is negligible compared to the sums before it in relation (8).

The squares of the zeros of the entire function $F(z)$ of exponential type and unit degree of growth in z have asymptotics given by (7). We will transform the characteristic function (8) represented by the infinite product.

In the following, we use functions

$$\begin{aligned} \tilde{a}_0^{(1)} &= \int_0^\pi \tilde{q}_1(\theta) \, d\theta, & a_0^{(2,1)} &= \int_0^\pi \tilde{Q}_{21}(\theta, q(\theta)) \, d\theta, \\ \hat{a}_0^{(2,1)} &= \int_0^\pi \theta \tilde{Q}_{2,1}(\theta, q(\theta)) \, d\theta, \\ \hat{a}_{c0}^{(1)}(z) &= \int_0^\pi \theta \tilde{q}_1(\theta) \cos 2z\theta \, d\theta, & \tilde{b}_{s0}^{(1)}(z) &= \int_0^\pi \tilde{q}_1(\theta) \sin 2z\theta \, d\theta, \end{aligned}$$

$$\begin{aligned}
\hat{b}_{s0}^{(1)}(z) &= \int_0^\pi \theta \tilde{q}_1(\theta) \sin 2z\theta \, d\theta, & \tilde{a}^{(1)}(z) &= \int_0^\pi \tilde{q}_1(\theta) \cos z(\pi - 2\theta) \, d\theta, \\
\tilde{b}^{(1)}(z) &= \int_0^\pi \tilde{q}_1(\theta) \sin z(\pi - 2\theta) \, d\theta, & \hat{b}^{(1)}(z) &= \int_0^\pi \theta \tilde{q}_1(\theta) \sin z(\pi - 2\theta) \, d\theta, \\
\hat{\hat{a}}_0^{(1)} &= \int_0^\pi \theta \tilde{q}_1(\theta) \, d\theta, & b^{(2,1)}(z) &= \int_0^\pi \tilde{Q}_{21}(\theta, q(\theta)) \sin z(\pi - 2\theta) \, d\theta,
\end{aligned}$$

as well as sums

$$(9) \quad \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{n_1^2 - z^2} = \frac{1}{z} \tan \pi z;$$

$$(10) \quad -\frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\tilde{a}_{2n_1}^{(1)}}{n_1^2 - z^2} = -\frac{1}{z \cos \pi z} \tilde{b}^{(1)}(z);$$

$$(11) \quad \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{b_{2n_1}^{(2,1)}}{n_1(n_1^2 - z^2)} = \frac{1}{z^2 \cos \pi z} a^{(2,1)}(z) - \frac{1}{z^2} a_0^{(2,1)};$$

$$(12) \quad \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\tilde{b}_{2n_1}^{(1)}}{n_1(n_1^2 - z^2)} = \frac{1}{z^2 \cos \pi z} \tilde{a}^{(1)} - \frac{1}{z^2} \tilde{a}_0^{(1)};$$

$$(13) \quad \frac{4}{\pi^2} \sum_{n=0}^{\infty} \frac{\hat{\hat{b}}_{2n}^{(1)}}{n_1(n_1^2 - z^2)} = \frac{2}{\pi z^2 \cos \pi z} \hat{\hat{a}}^{(1)} - \frac{2}{\pi z^2} \hat{\hat{a}}_0^{(1)};$$

$$(14) \quad \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\tilde{a}_{2n_1}^{(1)} \tilde{b}_{2n_1}^{(1)}}{n_1(n_1^2 - z^2)} = \frac{\tilde{a}_{c0}^{(1)}(z) \tilde{a}^{(1)}(z)}{z^2 \cos \pi z} - \frac{1}{z^2} (\tilde{a}_0^{(1)})^2;$$

$$(15) \quad -\frac{4}{\pi^2} \sum_{n=0}^{\infty} \frac{\tilde{a}_{2n_1}^{(1)} \hat{\hat{b}}_{2n}^{(1)}}{n_1(n_1^2 - z^2)} = -\frac{2}{\pi z^2} \left(\frac{1}{\cos \pi z} \tilde{a}_{c0}^{(1)}(z) \hat{\hat{a}}^{(1)}(z) - \tilde{a}^{(1)} \hat{\hat{a}}_0^{(1)} \right);$$

$$(16) \quad \frac{4}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(n_1^2 - z^2)^2} = -\frac{1}{\pi z^3} \tan \pi z + \frac{1}{z^2} \cdot \frac{1}{\cos^2 \pi z};$$

$$\begin{aligned}
(17) \quad \frac{4}{\pi^2} \sum_{n=0}^{\infty} \frac{(\tilde{a}_{2n_1}^{(1)})^2}{(n_1^2 - z^2)^2} &= -\frac{16}{\pi z^3 \cos \pi z} \tilde{b}^{(1)}(z) \tilde{b}_{c0}^{(1)}(z) + \frac{16}{z^2 \cos \pi z} \tilde{a}^{(1)}(z) \tilde{a}_{c0}^{(1)}(z) - \\
&- \frac{32}{\pi z^2 \cos \pi z} \hat{\hat{a}}^{(1)}(z) \tilde{a}_{c0}^{(1)}(z) - \frac{32}{\pi z^2 \cos \pi z} \tilde{b}^{(1)}(z) \hat{\hat{b}}_{s0}^{(1)}(z) + \\
&+ \frac{16 \sin \pi z}{z^2 \cos \pi z} \tilde{b}^{(1)}(z) \tilde{a}_{c0}^{(1)}(z);
\end{aligned}$$

$$\begin{aligned}
(18) \quad -\frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{\tilde{a}_{2n_1}^{(1)}}{(n_1^2 - z^2)^2} &= -\frac{2}{\pi z^3 \cos \pi z} \tilde{b}^{(1)}(z) + \frac{2}{z^2 \cos^2 \pi z} \tilde{b}^{(1)}(z) - \\
&- \frac{2}{z^2 \cos \pi z} \tilde{a}^{(1)}(z) + \frac{4}{\pi z^2 \cos \pi z} \hat{\hat{a}}^{(1)}(z).
\end{aligned}$$

By analogy with Levitan $\sum_{n=0}^{\infty} \frac{\lambda_{n_1} - n_1^2}{n_1^2 - z^2}$ can be transformed as

$$\begin{aligned}
 \sum_{n=0}^{\infty} \frac{\lambda_{n_1} - n_1^2}{n_1^2 - z^2} &= \sum_{n=0}^{\infty} \frac{\Delta \lambda_{n_1}}{n_1^2 - z^2} + \frac{2H}{\pi} \sum_{n=0}^{\infty} \frac{1}{n_1^2 - z^2} - \\
 &\quad - \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\tilde{a}_{2n_1}^{(1)}}{n_1^2 - z^2} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{b_{2n_1}^{(1)}}{n_1(n_1^2 - z^2)} + \\
 (19) \quad &\quad + \frac{4H}{\pi^2} \sum_{n=0}^{\infty} \frac{\hat{b}_{2n_1}^{(1)}}{n_1(n_1^2 - z^2)} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\tilde{a}_{2n_1}^{(1)} \tilde{b}_{2n_1}^{(1)}}{n_1(n_1^2 - z^2)} - \\
 &\quad - \frac{4}{\pi^2} \sum_{n=0}^{\infty} \frac{\tilde{a}_{2n_1}^{(1)} \hat{b}_{2n_1}^{(1)}}{n_1(n_1^2 - z^2)}, \quad n_1 = n + \frac{1}{2}, \quad n \rightarrow \infty,
 \end{aligned}$$

where

$$\begin{aligned}
 \Delta \lambda_{n_1} &= \lambda_{n_1} - n_1^2 - \frac{2H}{\pi} + \frac{2}{\pi} \tilde{a}_{2n_1}^{(1)} - \frac{1}{n_1} \left(\frac{2}{\pi} b_{2n_1}^{(2,1)} + \frac{4H}{\pi^2} \hat{b}_{2n_1}^{(1)} + \right. \\
 (20) \quad &\quad \left. + \frac{2}{\pi} \tilde{a}_{2n_1}^{(1)} \tilde{b}_{2n_1}^{(1)} - \frac{4}{\pi^2} \tilde{a}_{2n_1}^{(1)} \hat{b}_{2n_1}^{(1)} \right) = O\left(\frac{a_{2n}^{(3,1)}}{n_1^2}\right), \quad n \rightarrow \infty,
 \end{aligned}$$

$$(21) \quad \sum_{n=0}^{\infty} \frac{\Delta \lambda_{n_1}}{n_1^2 - z^2} = -\frac{1}{z^2} S_1^* + \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{n_1^2 \Delta \lambda_{n_1}}{n_1^2 - z^2} = -\frac{1}{z^2} S_1^* + o\left(\frac{1}{z^2}\right).$$

From (19), (20), (21), as well as the sum (10)–(18), we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} \frac{\lambda_{n_1} - n_1^2}{n_1^2 - z^2} &= -\frac{1}{z^2} S_1^* + \frac{H}{z} \tan \pi z - \frac{1}{z \cos \pi z} \tilde{b}^{(1)}(z) + \\
 (22) \quad &\quad + \frac{1}{z^2 \cos \pi z} a^{(2,1)}(z) - \frac{1}{z^2} a_0^{(2,1)} + \frac{2H}{\pi z^2 \cos \pi z} \hat{a}^{(1)}(z) - \\
 &\quad - \frac{2H}{\pi z^2} \hat{a}_0^{(1)} + \frac{\tilde{a}_{c0}^{(1)}(z) \tilde{a}^{(1)}(z)}{z^2 \cos \pi z} - \frac{1}{z^2} (\tilde{a}_0^{(1)})^2 - \\
 &\quad - \frac{2}{\pi z^2} \left(\frac{1}{\cos \pi z} \tilde{a}_{c0}^{(1)}(z) \hat{a}^{(1)}(z) - \tilde{a}^{(1)}(z) \hat{a}_0^{(1)} \right),
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{\lambda_{n_1} - n_1^2}{n_1^2 - z^2} \right)^2 &= \frac{H^2}{2z^2} \tan^2 \pi z + \frac{1}{2z^2 \cos^2 \pi z} (\tilde{b}^{(1)}(z))^2 - \\
 (23) \quad &\quad - \frac{H \sin \pi z}{z^2 \cos^2 \pi z} \tilde{b}^{(1)}(z) + o\left(\frac{\tilde{a}_{c0}^{(1)}(z) \tilde{a}^{(1)}(z)}{z^2 \cos \pi z}\right),
 \end{aligned}$$

$$\begin{aligned}
(24) \quad & -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{\lambda_{n_1} - n_1^2}{n_1^2 - z^2} \right)^2 = \frac{H^2}{2\pi z^3} \tan \pi z + \frac{H^2}{2z^2 \cos^2 \pi z} + \\
& + \frac{8}{\pi z^3 \cos \pi z} \tilde{b}^{(1)}(z) \tilde{a}_{c0}^{(1)}(z) - \frac{8}{z^2 \cos \pi z} \tilde{a}^{(1)}(z) \tilde{a}_{c0}^{(1)}(z) + \\
& + \frac{16}{\pi z^2 \cos \pi z} \left(\hat{a}^{(1)}(z) \tilde{a}_{c0}^{(1)}(z) + \tilde{b}^{(1)}(z) \hat{b}_{s0}^{(1)}(z) \right) - \\
& - \frac{8 \sin \pi z}{z^2 \cos \pi z} \tilde{b}^{(1)}(z) \tilde{a}_{c0}^{(1)}(z) - \frac{H}{\pi z^3 \cos \pi z} \tilde{b}^{(1)}(z) - \\
& - \frac{H}{z^2 \cos^2 \pi z} \tilde{b}^{(1)}(z) + \frac{H}{z^2 \cos \pi z} \tilde{a}^{(1)}(z) - \frac{2H}{\pi z^2 \cos \pi z} \hat{a}^{(1)}(z).
\end{aligned}$$

From (8), (22), (23) and (24) we get

$$\begin{aligned}
(25) \quad & F(z) = z \cos \pi z + H \sin \pi z - \tilde{b}^{(1)}(z) + \frac{H}{z} \tilde{a}^{(1)}(z) + \frac{1}{z} a^{(2,1)}(z) + \\
& + \frac{\cos \pi z}{z} \left(-S_1^* - a_0^{(2,1)} - (\tilde{a}_0^{(1)})^2 - \frac{2H}{\pi} \hat{a}_0^{(1)} + \frac{2}{\pi} \tilde{a}_0^{(1)} \hat{a}_0^{(1)} \right) + \\
& + O\left(\frac{\tilde{a}_{c0}^{(1)}(z) \tilde{a}^{(1)}(z)}{z^2} \right).
\end{aligned}$$

If we equate the coefficients with $\frac{\cos \pi z}{z}$ in relations (5) and (25), we obtain that the first regularized trace of the operator is

$$S_1^* = -a_0^{(2,1)} - (\tilde{a}_0^{(1)})^2 - \frac{2H}{\pi} \hat{a}_0^{(1)} + \frac{2}{\pi} \tilde{a}_0^{(1)} \hat{a}_0^{(1)}.$$

This proves the basic result of the paper.

Theorem 1. *The first regularized trace of the operator D_1^2 is*

$$S_1^* = -a_0^{(2,1)} - (\tilde{a}_0^{(1)})^2 - \frac{2H}{\pi} \hat{a}_0^{(1)} + \frac{2}{\pi} \tilde{a}_0^{(1)} \hat{a}_0^{(1)},$$

where are functions

$$\tilde{a}_0^{(1)} = \int_0^\pi \tilde{q}_1(\theta) d\theta, \quad a_0^{(2,1)} = \int_0^\pi \tilde{Q}_{21}(\theta, q(\theta)) d\theta, \quad \hat{a}_0^{(1)} = \int_0^\pi \theta \tilde{q}_1(\theta) d\theta.$$

3. CONCLUSION

The first regularized traces were obtained based on the representation of characteristic functions in two ways, directly from the operator and via the infinite product. The paper represents a contribution to the spectral theory of operators.

REFERENCES

- [1] D. Nedić, E. Čatrnja, *Spectral properties of some differential operators of Sturm-Liouville type with homogeneous delay*, Hacettepe Journal of Mathematics and Statistics, 51 (3) (2022), 658-665.
- [2] D. Nedić, M. Pikula, *Karakteristične funkcije i asimptotika sopstvenih vrijednost operatora D_1^2 i D_2^2 tipa Šturm-Liuvila sa homogenim kašnjenjem*, Savremeni matematički problemi, 55 (2) (2018), 301-308.
- [3] M. Pikula, V. Vladičić, D. Nedić, *Inverse Sturm-Liouville problems with homogeneous delay*, Siberian Mathematical Journal, 55 (2) (2014), 301-308.
- [4] M. Pikula, E. Čatrnja, I. Kalčo, *Spectral problems for operators with deviating arguments*, Hacettepe Journal of Mathematics and Statistics, 47 (5) (2018), 1172-1183.
- [5] M. Pikula, D. Nedić, E. Čatrnja, *Partial invese spectral problems for the Sturm-Liouville operator with delay*, Sarajevo Journal of Mathematics, 16 (1) (2020), 41-54.
- [6] V. Vladičić, M. Pikula, *An inverse problems for Sturm-Liouville type differential equation with a constant delay*, Sarajevo Journal of Mathematics, 12 (1) (2016), 83-88.
- [7] B. Vojvodić, N. Pavlović Komazec, F.A. Çetinkaya, *Recovering differential operators with two retarded arguments*, Boletín de la Sociedad Matemática Mexicana, 28 (2022), Article ID: 68, 1-21.
- [8] I. Kalčo, F. Destović, D. Nedić, *Bilinear Delay*, International Journal of Recent Engineering Research and Development (IJRERD), 06 (09) (2021), 1-4.

DRAGANA NEDIĆ

UNIVERSITY OF EAST SARAJEVO,
 FACULTY OF TRANSPORT AND TRAFFIC ENGINEERING
 VOJVODE MIŠIĆA 52
 74000 DOBOJ
 BOSNIA AND HERZEGOVINA
E-mail address: dragana.nedic@sf.ues.rs.ba

ISMET KALČO

UNIVERSITY OF ZENICA
 FACULTY OF PHILOSOPHY
 ZMAJA OD BOSNE 56
 72000 ZENICA
 BOSNIA AND HERZEGOVINA
E-mail address: ismet.kalco@unze.ba

GORAN CVJETKOVIĆ

GYMNASIUM JOVAN DUČIĆ,
 HILANDARSKA 2
 74000 DOBOJ
 BOSNIA AND HERZEGOVINA
E-mail address: cvjetkovicgoran128@gmail.com