

Estimation of coefficients for bi-Bazilevič functions with bounded boundary rotation through the application of the Faber polynomial

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ABSTRACT. In this article, we introduce a new subclass, referred to as $\mathcal{B}_\Sigma(\vartheta, \lambda; \delta)$, of bi-Bazilevič type with bounded boundary rotation. Through the application of Faber polynomials, we investigate the coefficient estimate $|g_m|$ for the functions in this subclass. Additionally, we derive the initial estimates for the coefficients $|g_2|$ and $|g_3|$, and we calculate the Fekete-Szegő functional $|g_3 - \Re g_2^2|$ for this subclass. The results presented in this paper would improve some recent works of several earlier authors.

1. INTRODUCTION

We denote the class of functions g that are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ as $\mathcal{A}$, which is normalized by the equation:

$$(1) \quad g(z) = z + \sum_{m=2}^{\infty} g_m z^m.$$

Define $\mathcal{S}$ as the collection of functions $g \in \mathcal{A}$ that exhibit univalence in the region $\mathbb{U}$. A function $g \in \mathcal{A}$ is classified as starlike if it meets the condition given by the inequality

$$\Re \left(\frac{zg'(z)}{g(z)} \right) > 0.$$

The set of all starlike functions $g \in \mathcal{A}$ is represented by the notation $\mathcal{S}^*$. In the context of $0 \leq \lambda < 1$, we define $\mathcal{R}(\delta)$ as the group of analytic functions $g \in \mathcal{A}$ such that

$$\Re (g'(z)) > \lambda, \quad z \in \mathbb{U}.$$

It is widely recognized that every function g belonging to the class $\mathcal{S}$ possesses an inverse g^{-1} , which fulfills the conditions $g^{-1}(g(z)) = z$ for z in $\mathbb{D}$ and

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$g(g^{-1}(w)) = w$ for $|w| < 1/4$, as stated by the Koebe One Quarter theorem. Specifically, the inverse function g^{-1} is represented by the equation:

$$(2) \quad \chi(w) = g^{-1}(w) = w - g_2 w^2 + (2g_2^2 - g_3)w^3 - (5g_2^3 - 5g_2 g_3 + g_4)w^4 + \cdots.$$

A function g within the class $\mathcal{S}$ is defined as bi-univalent in the domain $\mathbb{U}$ if its inverse $\chi = g^{-1}$ is also univalent in $\mathbb{U}$. Let Σ be the designation for the class of bi-univalent functions existing in $\mathbb{U}$, according to (1). The concept of bi-univalent analytic functions was initially introduced and examined by Lewin [17], who established that $|g_2| < 1.51$. This result was later refined by Brannan and Clunie [7], who showed that $|g_2| \leq \sqrt{2}$, and subsequently, Netanyahu demonstrated that $|g_2| \leq 4/3$. Additionally, Brannan and Taha [8], explored specific subclasses of bi-univalent functions and provided estimates for their initial coefficients. In their investigations, Brannan and Taha [8] and Taha [26] examined specific subclasses of Σ and uncovered non-sharp estimates for $|g_2|$ and $|g_3|$. Recently, researchers such as Srivastava et al. [24], Frasin and Aouf [10], and Ali et al. [4] have derived estimates for the first two coefficients of certain subclasses of bi-univalent functions. However, knowledge regarding the higher coefficients of bi-univalent functions remains limited. Ali et al. [4] have indicated that establishing bounds for $|g_m|$ (where $m \in \mathbb{N} \setminus \{1, 2\}$) for bi-univalent functions is still an open problem.

For parameters λ and δ such that $0 \leq \delta < 1$ and $0 \leq \lambda < 1$, a function g is considered to be of Bazilevič type, denoted as $\mathcal{B}(\lambda; \delta)$, if it satisfies the condition

$$\Re \left(\left(\frac{z}{g(z)} \right)^{1-\delta} g'(z) \right) > \lambda, \quad z \in \mathbb{U}.$$

The introduction of the class $\mathcal{B}(\lambda; \delta)$ can be traced back to the work of Bazilevič [6] in 1955.

Remark 1. The following classes of univalent functions are obtained, comprising functions $g \in \mathcal{S}$ for particular values of δ .

- (i) When $\delta = 0$, the class $\mathcal{B}(\lambda; \delta)$ aligns with the class $\mathcal{S}^*(\lambda)$, defined as the class of starlike functions of order λ , as originally presented by Robertson [22].
- (ii) When $\delta = 1$, the class $\mathcal{B}(\lambda; \delta)$ aligns with the class $\mathcal{R}(\lambda)$, defined as the class of functions whose derivative has positive real part of order λ , as originally presented by Macgregor [19].

The Faber polynomials, as introduced by Faber [9], are significant in numerous fields of mathematical sciences, particularly in geometric function theory. Recent studies [11] and [12] that utilize Faber polynomial expansions for meromorphic bi-univalent functions have inspired us to extend this approach to classes of analytic bi-univalent functions. There is a limited number of publications addressing the general coefficient bounds $|g_m|$ for analytic bi-univalent functions defined by (1) through Faber polynomial expansions (refer to [14–16]

for examples). Hamidi and Jahangiri [14] explored the class of analytic bi-close-to-convex functions, while Jahangiri and Hamidi [16] examined the class established by Frasin and Aouf [10]. Additionally, Jahangiri et al. [15] investigated the class of analytic bi-univalent functions characterized by positive real-part derivatives.

Let us examine the function $g \in \mathcal{S}$, defined as in (1). The inverse map of g , denoted as $\chi = g^{-1}$, can be expressed using the Faber polynomial expansion (refer to [1, 3]), which is given by

$$\chi(w) = w + \sum_{m=2}^{\infty} \mathcal{R}_{m-1}^{-m}(g_2, g_3, \dots, g_m),$$

where

$$\begin{aligned} \mathcal{R}_{m-1}^{-m} &= \frac{(m)!}{(1-2m)!(m-1)!} g_2^{m-1} + \frac{(-m)!}{(2(1-m))!(-3+m)!} g_2^{m-3} g_3 \\ &+ \frac{(-m)!}{(3-2m)!(m-4)!} g_2^{m-4} g_4 \\ &+ \frac{(-m)!}{(2(2-m))!(m-5)!} g_2^{m-5} [g_5 + (2-m)g_3^2] \\ &+ \frac{(-m)!}{(5-2m)!(m-6)!} g_2^{m-6} [g_6 + (5-2m)g_2g_4] + \sum_{k=1}^{\infty} g^{m-k} \mathcal{D}_k, \end{aligned}$$

and $\mathcal{D}_k$ for $k \in [7, m]$ in a homogeneous polynomial in the variable $g_2, g_3, \dots, g_n$. When m is set to 2, 3, and 4, we arrive at the following conclusions

$$\begin{aligned} \mathcal{R}_1^{-2} &= -2g_2, \\ \mathcal{R}_2^{-3} &= (2g_2^2 - g_3), \end{aligned}$$

and

$$\mathcal{R}_3^{-4} = -4(5g_2^3 - 5g_2g_3 + g_4).$$

In a general context [2], for any $p \in \mathcal{N}$, $\mathcal{D}_m^p$ is defined as follows:

$$\mathcal{D}_m^p = pg_m + \frac{p(p-1)}{2} \mathcal{C}_m^2 + \frac{p!}{(p-3)!3!} \mathcal{C}_m^3 + \dots + \frac{p!}{(p-m)!m!} \mathcal{C}_m^m,$$

where

$$\mathcal{C}_m^p = \mathcal{C}_m^p(g_2, g_3, \dots),$$

and by [27],

$$\mathcal{C}_m^n(g_2, g_3, \dots, g_m) = \sum \frac{n!}{j_1! \dots j_m!} g_1^{j_1} \dots g_m^{j_m},$$

as long as $g_1 = 1$, the summation encompasses all non-negative integers j_1, j_2 , etc., that meet the requirement

$$\begin{aligned} j_1 + j_2 + \dots + j_m &= n \\ j_1 + 2j_2 + \dots + mj_m &= m. \end{aligned}$$

Consider $\vartheta \geq 2$ and $0 \leq \lambda < 1$. The set of functions denoted as $\mathcal{P}_\vartheta(\lambda)$ consists of normalized, analytic functions ψ for which $\psi(0) = 1$. These functions must fulfill the following condition:

$$\int_0^{2\pi} \left| \frac{\Re(\psi(z)) - \lambda}{1 - \lambda} \right| d\theta \leq \vartheta\pi$$

where $z = re^{i\theta} \in \mathbb{U}$. This class was first introduced by Padmanabhan and Parvatham [21] in their 1975 work.

The subsequent lemmas were crucial for proving our main results.

Lemma 1 ([3]). *If a function g is presented in the form (1), we can utilize the Faber polynomial expansions to obtain*

$$\begin{aligned} \left(\frac{g(z)}{z} \right)^q \frac{zg'(z)}{g(z)} &= 1 - \sum_{m=2}^{\infty} \mathcal{X}_{m-1}^{m+q-1}(g_2, g_3, \dots, g_m) \\ &= 1 + \sum_{m=2}^{\infty} \left(1 + \frac{m-1}{q} \right) \mathcal{R}_{m-1}^q(g_2, g_3, \dots, g_m). \end{aligned}$$

Lemma 2 ([5]). *If a function $\psi \in \mathcal{P}_\vartheta(\lambda)$ is expressed as*

$$\psi(z) = 1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \dots,$$

then for every $m \geq 1$, it holds that

$$|\psi_m| \leq \vartheta(1 - \lambda).$$

This result is sharp.

In this article, we introduce a new subclass, referred to as $\mathcal{B}_\Sigma(\vartheta, \lambda; \delta)$, of bi-Bazilevič type with bounded boundary rotation. Through the application of Faber polynomials, we investigate the coefficient estimate $|g_m|$ for the functions in this subclass. Additionally, we derive the initial estimates for the coefficients $|g_2|$ and $|g_3|$, and we calculate the Fekete-Szegő functional $|g_3 - \Re g_2^2|$ for this subclass. The results presented in this paper would improve some recent works of several earlier authors.

2. MAIN RESULTS

Definition 1. For parameters λ , δ and ϑ such that $0 \leq \delta < 1$, $0 \leq \lambda < 1$ and $2 \leq \vartheta \leq 4$, a function g is considered to be of bi-Bazilevič type with bounded boundary rotation, denoted as $\mathcal{B}_\Sigma(\vartheta, \lambda; \delta)$, if it satisfies the following conditions

$$\left(\frac{z}{g(z)} \right)^{1-\delta} g'(z) \in \mathcal{P}_\vartheta(\lambda)$$

and

$$\left(\frac{w}{\chi(w)} \right)^{1-\delta} \chi'(w) \in \mathcal{P}_\vartheta(\lambda).$$

Remark 2. The following classes of bi-univalent functions are obtained, comprising functions $g \in \Sigma$ for particular values of δ and ϑ .

- (i) When $\delta \rightarrow 1^-$ is selected in Definition 1, the class $\mathcal{B}_\Sigma(\vartheta, \lambda; \delta)$ is effectively transformed into the class $\mathcal{H}_\Sigma(\vartheta, \lambda)$, as presented and explored by Sharma et al. [23].
- (ii) When $\delta \rightarrow 1^-$ and $\vartheta = 2$ are selected in Definition 1, the class $\mathcal{B}_\Sigma(\vartheta, \lambda; \delta)$ is effectively transformed into the class $\mathcal{H}_\Sigma(\lambda)$, as presented and explored by Srivastava et al. [24].
- (iii) When $\delta = 0$ is selected in Definition 1, the class $\mathcal{B}_\Sigma(\vartheta, \lambda; \delta)$ is effectively transformed into the class $\mathcal{R}_\Sigma(\vartheta, \lambda)$, as articulated by Li et al. [18] and reviewed by Sharma et al. [23].
- (iv) When $\delta = 0$ and $\vartheta = 2$ are selected in Definition 1, the class $\mathcal{B}_\Sigma(\vartheta, \lambda; \delta)$ is effectively transformed into the class $\mathcal{S}_\Sigma^*(\lambda)$, as presented and explored by Brannan and Taha [8].

Theorem 1. *Let g be a function in the space $\mathcal{B}_\Sigma(\vartheta, \lambda; \delta)$ for $0 \leq \lambda < 1$, $0 \leq \delta < 1$ and $2 \leq \vartheta \leq 4$, and assume it is bi-Bazilevič type with bounded boundary rotation in the domain $\mathbb{U}$. If the coefficients $g_k = 0$ for all $2 \leq k \leq m-1$, then the inequality*

$$|g_m| \leq \frac{\vartheta(1-\lambda)}{(m-1)+\delta}; \quad m \geq 3,$$

holds true.

Proof. If g is a function in $\mathcal{B}_\Sigma(\vartheta, \lambda; \delta)$ and its inverse $\chi = g^{-1}$ is likewise in $\mathcal{B}_\Sigma(\vartheta, \lambda; \delta)$, then there exist two functions, $\psi(z)$ and $d(w)$, that are classified under $\mathcal{P}_\vartheta(\lambda)$. Consequently, it follows that

$$(3) \quad \left(\frac{z}{g(z)} \right)^{1-\delta} g'(z) = \psi(z)$$

and

$$(4) \quad \left(\frac{w}{\chi(w)} \right)^{1-\delta} \chi'(w) = d(w),$$

where

$$(5) \quad \psi(z) = 1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \dots$$

and

$$(6) \quad d(w) = 1 + d_1 w + d_2 w^2 + d_3 w^3 + \dots$$

Leveraging the Faber polynomial expansions provided in Lemma 1, equations (3) and (4) can be reformulated as follows:

$$(7) \quad \left(\frac{z}{g(z)} \right)^{1-\delta} g'(z) = 1 + \sum_{m=2}^{\infty} \left(1 + \frac{m-1}{\delta} \right) \mathcal{R}_{m-1}^{\delta}(g_2, g_3, \dots, g_m) z^{m-1}$$

and

$$(8) \quad \left(\frac{w}{\chi(w)} \right)^{1-\delta} \chi'(w) = 1 + \sum_{m=2}^{\infty} \left(1 + \frac{m-1}{\delta} \right) \mathcal{R}_{m-1}^{\delta}(G_2, G_3, \dots, G_m) w^{m-1},$$

where

$$G_m = \frac{1}{m} \mathcal{R}_{m-1}^{-m}(g_2, g_3, \dots, g_m).$$

Alternatively,

$$(9) \quad \left(\frac{z}{g(z)} \right)^{1-\delta} g'(z) = 1 + \sum_{m=1}^{\infty} \mathcal{D}_m^1(\psi_1, \psi_2, \dots, \psi_m)$$

and

$$(10) \quad \left(\frac{w}{\chi(w)} \right)^{1-\delta} \chi'(w) = 1 + \sum_{m=1}^{\infty} \mathcal{D}_m^1(d_1, d_2, \dots, d_m).$$

Upon comparing the coefficients from equations (7) and (9) for any $m \in \mathbb{N} \setminus \{1\}$, we arrive at the following result:

$$(11) \quad \left(1 + \frac{m-1}{\delta} \right) \mathcal{R}_{m-1}^{\delta}(g_2, g_3, \dots, g_m) = \mathcal{D}_m^1(\psi_1, \psi_2, \dots, \psi_m).$$

Similarly, upon comparing the coefficients from equations (8) and (10) for any $m \in \mathbb{N} \setminus \{1\}$, we arrive at the following result:

$$(12) \quad \left(1 + \frac{m-1}{\delta} \right) \mathcal{R}_{m-1}^{\delta}(G_2, G_3, \dots, G_m) = \mathcal{D}_m^1(d_1, d_2, \dots, d_m).$$

In the case of $m = 2$, we can extract from equations (11) and (12) the following expressions:

$$(13) \quad (1 + \delta)g_2 = \psi_1$$

and

$$(14) \quad -(1 + \delta)g_2 = d_1.$$

To determine g_2 , we can express it as follows:

$$(15) \quad g_2 = \frac{\psi_1}{1 + \delta} = -\frac{d_1}{1 + \delta}.$$

By referencing lemma 2, in the context of equation (15), we derive:

$$|g_2| \leq \frac{\vartheta(1 - \lambda)}{1 + \delta}.$$

In the scenario where $g_m = 0$ and $0 \leq k \leq m - 1$, we observe that $G_m = g_m$.

Therefore, utilizing equations (11) and (12), we obtain:

$$(16) \quad (\delta + (m - 1))g_m = \psi_m$$

and

$$(17) \quad -(\delta + (m - 1))g_m = d_m.$$

To determine g_m , we apply Lemma 2. Utilizing either equation (16) or (17), we arrive at the following result:

$$(18) \quad |g_m| \leq \frac{\vartheta(1 - \lambda)}{\delta + (m - 1)}.$$

Hence, we have established that Theorem 1 is valid. $\square$

Theorem 2. *Let g be a function in the space $\mathcal{B}_\Sigma(\vartheta, \lambda; \delta)$ for $0 \leq \lambda < 1$, $0 \leq \delta < 1$ and $2 \leq \vartheta \leq 4$, and assume it is bi-Bazilevič type with bounded boundary rotation in the domain $\mathbb{U}$. Then*

$$(19) \quad |g_2| \leq \min \left\{ \frac{\vartheta(1 - \lambda)}{2 + \delta}, \sqrt{\frac{2\vartheta(1 - \lambda)}{(\delta + 1)(\delta + 2)}} \right\},$$

$$(20) \quad |g_3| \leq \frac{2\vartheta(1 - \lambda)}{(\delta + 1)(\delta + 2)}.$$

There exists a real number $\aleph$ such that we have

$$(21) \quad |g_3 - \aleph g_2^2| \leq \begin{cases} \frac{2\vartheta(1 - \lambda)(1 - \aleph)}{(\delta + 1)(\delta + 2)}, & \text{if } \aleph < \frac{1 - \delta}{2}, \\ \frac{\vartheta(1 - \lambda)}{\delta + 2}, & \text{if } \frac{1 - \delta}{2} \leq \aleph \leq \frac{\delta + 3}{2}, \\ \frac{2\vartheta(1 - \lambda)(\aleph - 1)}{(\delta + 1)(\delta + 2)}, & \text{if } \aleph > \frac{\delta + 3}{2}. \end{cases}$$

Proof. The analysis of equation (11) for $m = 2$ and $m = 3$ reveals that

$$(22) \quad (\delta + 1)g_2 = \psi_1$$

and

$$(23) \quad (\delta + 2)g_3 + \frac{(\delta - 1)(\delta + 2)}{2}g_2^2 = \psi_2.$$

The analysis of equation (12) for $m = 2$ and $m = 3$ reveals that

$$(24) \quad -(\delta + 1)g_2 = d_1$$

and

$$(25) \quad \frac{(\delta + 2)(\delta + 3)}{2}g_2^2 - (\delta + 2)g_3 = d_2.$$

To find g_2 , we can utilize either equation (22) or (24) in conjunction with Lemma 2, which yields

$$(26) \quad |g_2| \leq \frac{\vartheta(1-\lambda)}{2+\delta}.$$

In contrast, the addition of equations (23) and (25), which yields

$$(27) \quad g_2^2 = \frac{\psi_2 + d_2}{(\delta+1)(\delta+2)}.$$

Through the use of Lemma 2 in equation (27), we arrive at

$$(28) \quad |g_2|^2 \leq \frac{2\vartheta(1-\lambda)}{(\delta+1)(\delta+2)}.$$

According to equations (26) and (28), the upper bounds for $|g_2|$ is presented in (19). By subtracting (23) from (25), we arrive at

$$(29) \quad g_3 = \frac{\psi_2 + d_2}{2(\delta+2)} + g_2^2.$$

By analyzing equations (27) and (29), we can conclude that

$$(30) \quad g_3 = \frac{\psi_2(\delta+3) + d_2(1-\delta)}{2(\delta+1)(\delta+2)}.$$

Through the implementation of Lemma 2 in equation (30), we can ascertain that the upper bounds for $|g_3|$ are indicated in (20). By forming any real number $\aleph$ and utilizing equations (27) and (29), we can derive

$$(31) \quad g_3 - \aleph g_2^2 = \frac{\psi_2(\delta+3-2\aleph) + d_2(1-\delta-2\aleph)}{2(\delta+1)(\delta+2)}.$$

Through the use of Lemma 2 in equation (31), we arrive at

$$(32) \quad |g_3 - \aleph g_2^2| \leq \frac{\vartheta(1-\lambda)[|\delta+3-2\aleph| + |1-\delta-2\aleph|]}{2(\delta+1)(\delta+2)}.$$

Equation (32) establishes the upper bounds for $g_3 - \aleph g_2^2$, which are presented in (21). Hence, we have established that Theorem 2 is valid. $\square$

3. OUTCOMES AND COROLLARIES

When $\delta \rightarrow 1^-$, we can deduce the ensuing corollary from Theorem 1.

Corollary 1. *Let g be a function in the space $\mathcal{H}_\Sigma(\vartheta, \lambda)$ for $0 \leq \lambda < 1$ and $2 \leq \vartheta \leq 4$. If the coefficients $g_k = 0$ for all $2 \leq k \leq m-1$, then the inequality*

$$|g_m| \leq \frac{\vartheta(1-\lambda)}{m}; \quad m \geq 3,$$

holds true.

When $\delta \rightarrow 1^-$ and $\vartheta = 2$, we can deduce the following corollary from Theorem 1.

Corollary 2. *Let g be a function in the space $\mathcal{H}_\Sigma(\lambda)$ for $0 \leq \lambda < 1$. If the coefficients $g_k = 0$ for all $2 \leq k \leq m-1$, then the inequality*

$$|g_m| \leq \frac{2(1-\lambda)}{m}; \quad m \geq 3,$$

holds true.

When $\delta = 0$, we can deduce the ensuing corollary from Theorem 1.

Corollary 3. *Let g be a function in the space $\mathcal{R}_\Sigma(\vartheta, \lambda)$ for $0 \leq \lambda < 1$ and $2 \leq \vartheta \leq 4$. If the coefficients $g_k = 0$ for all $2 \leq k \leq m-1$, then the inequality*

$$|g_m| \leq \frac{\vartheta(1-\lambda)}{m-1}; \quad m \geq 3,$$

holds true.

When $\delta = 0$ and $\vartheta = 2$, we can deduce the following corollary from Theorem 1.

Corollary 4. *Let g be a function in the space $\mathcal{S}_\Sigma^*(\lambda)$ for $0 \leq \lambda < 1$. If the coefficients $g_k = 0$ for all $2 \leq k \leq m-1$, then the inequality*

$$|g_m| \leq \frac{2(1-\lambda)}{m-1}; \quad m \geq 3,$$

holds true.

When $\delta \rightarrow 1^-$, we can deduce the ensuing corollary from Theorem 2.

Corollary 5. *Let g be a function in the space $\mathcal{H}_\Sigma(\vartheta, \lambda)$ for $0 \leq \lambda < 1$ and $2 \leq \vartheta \leq 4$. Then*

$$|g_2| \leq \sqrt{\frac{\vartheta(1-\lambda)}{3}},$$

$$|g_3| \leq \frac{\vartheta(1-\lambda)}{3}.$$

There exists a real number $\aleph$ such that we have

$$|g_3 - \aleph g_2^2| \leq \begin{cases} \frac{\vartheta(1-\lambda)(1-\aleph)}{3}, & \text{if } \aleph < 0, \\ \frac{\vartheta(1-\lambda)}{3}, & \text{if } 0 \leq \aleph \leq 2, \\ \frac{\vartheta(1-\lambda)(\aleph-1)}{3}, & \text{if } \aleph > 2. \end{cases}$$

When $\delta \rightarrow 1^-$ and $\vartheta = 2$, we can deduce the following corollary from Theorem 2.

Corollary 6. *Let g be a function in the space $\mathcal{H}_\Sigma(\vartheta)$ for $0 \leq \lambda < 1$. Then*

$$|g_2| \leq \sqrt{\frac{2(1-\lambda)}{3}},$$

$$|g_3| \leq \frac{2(1-\lambda)}{3}.$$

There exists a real number $\aleph$ such that we have

$$|g_3 - \aleph g_2^2| \leq \begin{cases} \frac{2(1-\lambda)(1-\aleph)}{3}, & \text{if } \aleph < 0, \\ \frac{2(1-\lambda)}{3}, & \text{if } 0 \leq \aleph \leq 2, \\ \frac{2(1-\lambda)(\aleph-1)}{3}, & \text{if } \aleph > 2. \end{cases}$$

When $\delta = 0$, we can deduce the ensuing corollary from Theorem 2.

Corollary 7. *Let g be a function in the space $\mathcal{R}_\Sigma(\vartheta, \lambda)$ for $0 \leq \lambda < 1$ and $2 \leq \vartheta \leq 4$. Then*

$$|g_2| \leq \min \left\{ \frac{\vartheta(1-\lambda)}{2}, \sqrt{\vartheta(1-\lambda)} \right\},$$

$$|g_3| \leq \vartheta(1-\lambda).$$

There exists a real number $\aleph$ such that we have

$$|g_3 - \aleph g_2^2| \leq \begin{cases} \vartheta(1-\lambda)(1-\aleph), & \text{if } \aleph < \frac{1}{2}, \\ \frac{\vartheta(1-\lambda)}{2}, & \text{if } \frac{1}{2} \leq \aleph \leq \frac{3}{2}, \\ \vartheta(1-\lambda)(\aleph-1), & \text{if } \aleph > \frac{3}{2}. \end{cases}$$

When $\delta = 0$ and $\vartheta = 2$, we can deduce the following corollary from Theorem 2.

Corollary 8. *Let g be a function in the space $\mathcal{S}_\Sigma^*(\lambda)$ for $2 \leq \vartheta \leq 4$. Then*

$$|g_2| \leq \min \left\{ 1-\lambda, \sqrt{2(1-\lambda)} \right\},$$

$$|g_3| \leq 2(1-\lambda).$$

There exists a real number $\aleph$ such that we have

$$|g_3 - \aleph g_2^2| \leq \begin{cases} 2(1-\lambda)(1-\aleph), & \text{if } \aleph < \frac{1}{2}, \\ 1-\lambda, & \text{if } \frac{1}{2} \leq \aleph \leq \frac{3}{2}, \\ 2(1-\lambda)(\aleph-1), & \text{if } \aleph > \frac{3}{2}. \end{cases}$$

- Remark 3.** (i) Corollary 1 and Corollary 3, provides verification for the results established by Tang [25] and Corollary 2 and Corollary 4, provides verification for the results established by Hamidi and Jahangiri [13] and Jahangiri et. al. [15].
- (ii) Corollary 5 and Corollary 7, provides verification for the results established by Sharma et. al. [23].
- (iii) Corollary 6, it affirms the restriction on $|a_2|$ and improves the restriction on $|a_3|$ that was obtained by Srivastava et. al. [24].
- (iv) Corollary 8, provides verification for the results established by Brannan and Taha [8].

CONCLUDING REMARKS AND OBSERVATIONS

In this article, using the Faber polynomial expansions, we found general coefficient bounds for newly defined subclasses of bi-univalent functions with bounded boundary rotation. We also derived interesting and familiar Fekete–Szegő inequality for these new classes. In addition, we gave interesting remarks and corollaries for certain choices of parameters. Similar coefficient problems may also be analyzed for other function classes existing in the literature. There is a strong connection between the investigation of m -fold symmetric bi-univalent functions with bounded boundary rotation and the study of bi-Bazilevič functions, as both frameworks utilize Faber polynomial expansions to obtain coefficient estimates under geometric constraints. Both studies are connected through the application of Faber polynomial techniques to derive coefficient estimates for subclasses of bi-univalent functions associated with bounded boundary rotation. For more details see [20]. The study also suggests that by employing q -calculus for values of $0 < q < 1$, along with orthogonal polynomials and different kinds of operators.

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