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Some new identities for the sum of generalized Fibonacci numbers and generalized Lucas numbers

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Abstract. Let U_n and V_n denote the n-th generalized Fibonacci and generalized Lucas numbers, respectively. In this paper, we calculate the n-th powers of some square matrices by diagonalizing them with their eigenvalues and eigenvectors. We find some terms of these matrices as generalized Fibonacci numbers and generalized Lucas numbers. Using matrix powers and the binomial theorem, we give some new identities. Finally, we show again that the known identity $(k^2 + 4t)(-t)^{n-1} = V_{n+1}V_{n-1} - V_n^2$ where is satisfied with the help of the obtained matrices.

1. Introduction

The sequence of numbers $0, 1, 1, 2, 3, 5, 8, \ldots$, introduced in the book *Liber Abaci*, has attracted the attention of many mathematicians. Each term of this sequence is obtained as the sum of the two preceding terms, and it is known as the Fibonacci sequence. The n-th Fibonacci number, denoted by F_n, is defined recursively by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2,$$

with initial conditions $F_0 = 0$ and $F_1 = 1$.

Another closely related sequence, obtained by choosing different initial conditions, is the Lucas sequence. It is given by $2, 1, 3, 4, 7, 11, \ldots$ and satisfies the same recurrence relation

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2,$$

with initial conditions $L_0 = 2$ and $L_1 = 1$.

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Let n be an integer, k and t be integers different from the zero with $k^2+4t \neq 0$. A generalized Fibonacci sequence (OEIS A015441), $(U_n(k,t))$, is defined by

$$U_0(k,t)=0, U_1(k,t)=1 \quad \text{and} \quad U_n(k,t)=kU_{n-1}(k,t)+tU_{n-2}(k,t)$$

for $n \geq 2$ and a generalized Lucas sequence (OEIS A075117), $(V_n(k,t))$, is defined by

$$V_0(k,t)=2, V_1(k,t)=k \quad \text{and} \quad V_n(k,t)=kV_{n-1}(k,t)+tV_{n-2}(k,t)$$

for $n \geq 2$. Moreover, the characteristic equation for the generalized Fibonacci and generalized Lucas sequences is $x^2-kx-t=0$ and its roots are

$$\alpha=\frac{k+\sqrt{k^2+4t}}{2} \quad \text{and} \quad \beta=\frac{k-\sqrt{k^2+4t}}{2},$$

respectively. Furthermore, for $n \in \mathbb{N}$, $U_{-n}(k,t)$ and $V_{-n}(k,t)$ are defined as

$$U_{-n}(k,t)=-(-t)^{-n}U_n(k,t) \quad \text{and} \quad V_{-n}(k,t)=(-t)^{-n}V_n(k,t). \tag{1}$$

From now on, instead of $U_n(k,t)$ and $V_n(k,t)$, we will use U_n and V_n, respectively.

$$U_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$$

and

$$V_n=\alpha^n+\beta^n \tag{2}$$

are called Binet-like formulas for the generalized Fibonacci and generalized Lucas sequences. If $k=t=1$ is taken, then these sequences are the Fibonacci sequence (F_n) and the Lucas sequence (L_n), respectively. For an integer n, the following equations are well known:

$$V_n=U_{n+1}+tU_{n-1}, \tag{3}$$

$$(k^2+4t)U_n=V_{n+1}+tV_{n-1}, \tag{4}$$

and

$$U_{n+1}U_{n-1}-U_n^2=-(-t)^{n-1}. \tag{5}$$

There are many identities in the literature about Fibonacci and Lucas sequences. Mathematical induction, matrices, and Binet formulas are usually used to prove these identities. The most well-known of the matrices used to obtain identities is $Q=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$, also called Fibonacci Q matrix. In [1], the author show that the n-th power of the Fibonacci Q matrix is $Q^n=\begin{pmatrix}F_{n+1} & F_n\\ F_n & F_{n-1}\end{pmatrix}$. In [2], the author show that the n-th power of the $R=\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$ is $R^n=\begin{pmatrix}F_{n-1} & F_n\\ F_n & F_{n+1}\end{pmatrix}$. The Cassini's identity, first given by Robert Simson in 1753, is $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$. It is obtained using the equality $|Q|^n = |Q^n|$ and later $|R|^n = |R^n|$. This identity can also be found in [3]. In [4] and [5], the authors find the most general form of the Cassini identity $U_{n+1}U_{n-1} - U_n^2 = -(-t)^{n-1}$ with the help of determinant of $\begin{pmatrix} U_{n+1} & tU_n \\ U_n & tU_{n-1} \end{pmatrix}$, which is the n-th power of $\begin{pmatrix} k & t \\ 1 & 0 \end{pmatrix}$. In [6], Melham and Shannon consider the matrices $M = \begin{pmatrix} k & t \\ 1 & 0 \end{pmatrix}$, $M_{k,m} = \begin{pmatrix} U_{k+m} & -(-t)^m U_k \\ U_k & -(-t)^m U_{k-m} \end{pmatrix}$, and $N_{k,m} = \begin{pmatrix} V_{k+m} & -(-t)^m V_k \\ V_k & -(-t)^m V_{k-m} \end{pmatrix}$. They give the matrices M^n, $M_{k,m}^n$ and $N_{k,m}^n$. By using these matrices, they find summation identities involving terms U_n and V_n. In [7], Cerda-Morales consider the matrix $M = \begin{pmatrix} k & t \\ 1 & 0 \end{pmatrix}$ and define the matrix $V_{(k,t)} = \begin{pmatrix} k^2 + 2t & kt \\ k & 2t \end{pmatrix}$, which is used to obtain identities related to generalized Fibonacci and generalized Lucas sequences. In [4], Şiar and Keskin state that the matrices $\begin{pmatrix} k & t \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ t & k \end{pmatrix}$ are special cases of 2×2 matrices satisfying the equation $X^2 = kX + tI$. They also define X matrix as $X = \begin{pmatrix} a & b \\ c & k-a \end{pmatrix}$ with $|X| = -t$. They show that the n-th power of this matrix is $X^n = \begin{pmatrix} aU_n + tU_{n-1} & bU_n \\ cU_n & U_{n+1} - aU_n \end{pmatrix}$ and they introduce the matrix $S = \begin{pmatrix} \frac{k}{2} & \frac{k^2+4t}{2} \\ \frac{1}{2} & \frac{k}{2} \end{pmatrix}$ and showed that $S^n = \begin{pmatrix} \frac{V_n}{2} & \frac{(k^2+4t)U_n}{2} \\ \frac{U_n}{2} & \frac{V_n}{2} \end{pmatrix}$. Afterward, they utilize matrices X and S to generate identities related to generalized Fibonacci and generalized Lucas numbers. In [8], Kawale and Lahurikar define the matrix $M(a,b) = \begin{pmatrix} a & b \\ \frac{1+a-a^2}{b} & 1-a \end{pmatrix}$, $b \neq 0$. They show that $a = b = 1$ reduces the matrix $M(a,b)$ to the Q matrix. If $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, then taking $a = \alpha$ and $b = 1$ results in the matrix $\begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix}$ and taking $a = \beta$ and $b = -1$ results in the matrix $\begin{pmatrix} \beta & -1 \\ 0 & \alpha \end{pmatrix}$. These matrices have been used to obtain identities related to Fibonacci and Lucas numbers. In [9], Kalman and Mena consider the matrix $\begin{pmatrix} 0 & 1 \\ t & k \end{pmatrix}$ to obtain identities related to generalized Fibonacci and generalized Lucas numbers. For more information on identities involving the Fibonacci sequence and its generalizations, see the references [5, 10–16]. In this study, we give the n-th powers of matrices $\begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix}$,

$\begin{pmatrix}\beta & -1\\ 0 & \alpha\end{pmatrix}$, $\begin{pmatrix}k & t\\ 1 & 0\end{pmatrix}$, $\begin{pmatrix}0 & t\\ 1 & -k\end{pmatrix}$, and $\begin{pmatrix}2t & -kt\\ -k & k^2+2t\end{pmatrix}$, $\begin{pmatrix}k^2+2t & kt\\ k & 2t\end{pmatrix}$. Also, with the help of these matrices, we find new identities about the sums containing some generalized Fibonacci and generalized Lucas numbers. Finally, we obtain the known identity $(k^2+4t)(-t)^{n-1} = V_{n+1}V_{n-1} - V_n^2$ with the help of the obtained matrices. But we find it by a different method such as recurrence relations, matrix representations, and algebraic techniques. In [17], we give some other identities by using similar matrices used above. This study can be regarded as a continuation of the study [17].

2. Main theorems

Theorem 1. *Let n be a natural number. Then*

X	X^n
$\begin{pmatrix}\alpha & 1\\ 0 & \beta\end{pmatrix}$	$\begin{pmatrix}\alpha^n & U_n\\ 0 & \beta^n\end{pmatrix}$
$\begin{pmatrix}\beta & -1\\ 0 & \alpha\end{pmatrix}$	$\begin{pmatrix}\beta^n & -U_n\\ 0 & \alpha^n\end{pmatrix}$
$\begin{pmatrix}\sqrt{k^2+4t} & 2\\ 0 & -\sqrt{k^2+4t}\end{pmatrix}$	$\begin{cases}(k^2+4t)^{n/2}I, & \text{for even } n;\\ (k^2+4t)^{(n-1)/2}X, & \text{for odd } n;\end{cases}$

where I denotes the identity matrix.

Proof. Let $X = \begin{pmatrix}\alpha & 1\\ 0 & \beta\end{pmatrix}$. The characteristic equation of matrix X is $\lambda^2 - (\alpha+\beta)\lambda + \alpha\beta = 0$. Then, it can be easily seen that its roots are α and β. Since α and β are distinct eigenvalues of X, X can be diagonalized. The eigenvectors corresponding to the eigenvalue α are obtained from the equation $\begin{pmatrix}0 & 1\\ 0 & \beta-\alpha\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = 0$. So, we find eigenvectors as $\begin{pmatrix}a\\ 0\end{pmatrix}$, where $a \neq 0$. When $a = 1$, we get $\begin{pmatrix}1\\ 0\end{pmatrix}$ as an eigenvector. For the eigenvectors related to β, from $\begin{pmatrix}\alpha-\beta & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = 0$, we get eigenvectors as $\begin{pmatrix}b\\ (\beta-\alpha)b\end{pmatrix}$, where $b \neq 0$. Thus, for $b = 1$, we obtain the eigenvector $\begin{pmatrix}1\\ \beta-\alpha\end{pmatrix}$. We can choose $P = \begin{pmatrix}1 & 1\\ 0 & \beta-\alpha\end{pmatrix}$ and $J = \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$. Since $|P| = \beta-\alpha \neq 0$, we get $P^{-1} = \frac{1}{\beta-\alpha}\begin{pmatrix}\beta-\alpha & -1\\ 0 & 1\end{pmatrix}$. Since $X = PJP^{-1}$, we have $X^n = PJ^nP^{-1}$

and so we obtain $X^n = \begin{pmatrix} 1 & 1 \\ 0 & \beta-\alpha \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}^n \cdot \frac{1}{\beta-\alpha} \begin{pmatrix} \beta-\alpha & -1 \\ 0 & 1 \end{pmatrix}$, i.e., $X^n = \begin{pmatrix} \alpha^n & U_n \\ 0 & \beta^n \end{pmatrix}$. A similar proof can be done for other matrices. □

Details regarding the matrix $\begin{pmatrix} k & t \\ 1 & 0 \end{pmatrix}$ can be found in references [4, 14].

Theorem 2. *Let n be a nonnegative integers. Then*

A	A^n
$\begin{pmatrix} k & t \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} U_{n+1} & tU_n \\ U_n & tU_{n-1} \end{pmatrix}$
$\begin{pmatrix} 0 & t \\ 1 & -k \end{pmatrix}$	$\begin{cases} \begin{pmatrix} tU_{n-1} & -tU_n \\ -U_n & U_{n+1} \end{pmatrix}, & \text{for even } n; \\ \begin{pmatrix} -tU_{n-1} & tU_n \\ U_n & -U_{n+1} \end{pmatrix}, & \text{for odd } n; \end{cases}$
$\begin{pmatrix} 2t & -kt \\ -k & k^2+2t \end{pmatrix}$	$\begin{cases} (k^2+4t)^{\frac{n}{2}} \begin{pmatrix} tU_{n-1} & -tU_n \\ -U_n & U_{n+1} \end{pmatrix}, & \text{for even } n; \\ (k^2+4t)^{\frac{n-1}{2}} \begin{pmatrix} tV_{n-1} & -tV_n \\ -V_n & V_{n+1} \end{pmatrix}, & \text{for odd } n; \end{cases}$
$\begin{pmatrix} k^2+2t & kt \\ k & 2t \end{pmatrix}$	$\begin{cases} (k^2+4t)^{\frac{n}{2}} \begin{pmatrix} U_{n+1} & tU_n \\ U_n & tU_{n-1} \end{pmatrix}, & \text{for even } n; \\ (k^2+4t)^{\frac{n-1}{2}} \begin{pmatrix} V_{n+1} & tV_n \\ V_n & tV_{n-1} \end{pmatrix}, & \text{for odd } n. \end{cases}$

Proof. Let $X = \begin{pmatrix} k & t \\ 1 & 0 \end{pmatrix}$. The characteristic equation of matrix X is $\lambda^2 - k\lambda - t = 0$. Then, it can be easily seen that its roots are α and β. Since α and β are distinct eigenvalues of X, X can be diagonalized. The eigenvectors corresponding to the eigenvalue α are obtained from the equation $\begin{pmatrix} k-\alpha & t \\ 1 & -\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$. So, we find eigenvectors as $\begin{pmatrix} \alpha \\ 1 \end{pmatrix}$. For the eigenvectors related to β, from $\begin{pmatrix} k-\beta & t \\ 1 & -\beta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$, we get eigenvectors as $\begin{pmatrix} \beta \\ 1 \end{pmatrix}$. We can choose $P = \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. Since $|P| = \alpha-\beta \neq 0$, we get $P^{-1} = \frac{1}{\alpha-\beta} \begin{pmatrix} 1 & -\beta \\ -1 & \alpha \end{pmatrix}$. Since $X = PJP^{-1}$, we have

$X^n = PJ^nP^{-1}$ and so we obtain $X^n = \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}^n \cdot \frac{1}{\alpha-\beta} \begin{pmatrix} 1 & -\beta \\ -1 & \alpha \end{pmatrix}$, i.e., $X^n = \begin{pmatrix} U_{n+1} & tU_n \\ U_n & tU_{n-1} \end{pmatrix}$. Now, let $Y = \begin{pmatrix} 0 & t \\ 1 & -k \end{pmatrix}$. Then the eigenvalues of Y are

$$\frac{-k-\sqrt{k^2+4t}}{2} = -\alpha, \qquad \frac{-k+\sqrt{k^2+4t}}{2} = -\beta,$$

and corresponding eigenvectors are

$$v_{-\alpha} = \begin{pmatrix} t \\ -\alpha \end{pmatrix}, \quad v_{-\beta} = \begin{pmatrix} t \\ -\beta \end{pmatrix}.$$

Then we get

$$P = \begin{pmatrix} t & t \\ -\alpha & -\beta \end{pmatrix}, \quad J = \begin{pmatrix} -\alpha & 0 \\ 0 & -\beta \end{pmatrix}.$$

Since $Y = PJP^{-1}$, we have $Y^n = PJ^nP^{-1}$. We compute

$$J^n = \begin{pmatrix} (-\alpha)^n & 0 \\ 0 & (-\beta)^n \end{pmatrix}, \quad P^{-1} = \frac{1}{t(\alpha-\beta)} \begin{pmatrix} -\beta & -t \\ \alpha & t \end{pmatrix}.$$

Thus, for all $n \geq 1$,

$$Y^n = \begin{pmatrix} t & t \\ -\alpha & -\beta \end{pmatrix} \begin{pmatrix} (-1)^n\alpha^n & 0 \\ 0 & (-1)^n\beta^n \end{pmatrix} \frac{1}{t(\alpha-\beta)} \begin{pmatrix} -\beta & -t \\ \alpha & t \end{pmatrix}$$

$$= \frac{(-1)^n}{t(\alpha-\beta)} \begin{pmatrix} t^2(\alpha^{n-1}-\beta^{n-1}) & -t^2(\alpha^n-\beta^n) \\ -t(\alpha^n-\beta^n) & t(\alpha^{n+1}-\beta^{n+1}) \end{pmatrix}.$$

The proof ends with the multiplication and simplification operations.
A similar proof can be done for other matrices. □

Theorem 3. *Let n be a natural number. The following identities hold true:*

$$(k^2+4t)^{\frac{2n+1}{2}} = \sum_{j=0}^{2n+1} \binom{2n+1}{j} t^j \alpha^{2n+1-2j}, \tag{6}$$

$$(k^2+4t)^{\frac{2n+1}{2}} = -\sum_{j=0}^{2n+1} \binom{2n+1}{j} t^j \beta^{2n+1-2j}, \tag{7}$$

$$2\left(k^2+4t\right)^n = \sum_{j=0}^{2n+1} \binom{2n+1}{j} t^j U_{2n+1-2j}. \tag{8}$$

Proof. Let $X = \begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix}$ and $Y = \begin{pmatrix} \beta & -1 \\ 0 & \alpha \end{pmatrix}$. Then, we obtain

$$XY = YX = -tI, \tag{9}$$

$$X - Y = \begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix} - \begin{pmatrix} \beta & -1 \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} \sqrt{k^2+4t} & 2 \\ 0 & -\sqrt{k^2+4t} \end{pmatrix}, \tag{10}$$

and

$$(X - Y)^2 = \begin{pmatrix} k^2+4t & 0 \\ 0 & k^2+4t \end{pmatrix} = (k^2+4t)I, \tag{11}$$

where I denotes the identity matrix. Moreover, we find that

$$X^n = \begin{pmatrix} \alpha^n & U_n \\ 0 & \beta^n \end{pmatrix} \tag{12}$$

by using Theorem 1. Therefore, from (11), we have

$$(X - Y)^{2n+1} = (X - Y)^{2n}(X - Y) = (k^2+4t)^n(X - Y). \tag{13}$$

Moreover,

$$\begin{aligned} (X - Y)^{2n+1} &= \sum_{j=0}^{2n+1} \binom{2n+1}{j} X^{2n+1-j}(-Y)^j \\ &= \sum_{j=0}^{2n+1} \binom{2n+1}{j} X^{2n+1-2j}(-XY)^j. \end{aligned} \tag{14}$$

Therefore, from (13) and (10), we obtain

$$(k^2+4t)^n(X - Y) = \begin{pmatrix} (k^2+4t)^n\sqrt{k^2+4t} & 2(k^2+4t)^n \\ 0 & -(k^2+4t)^n\sqrt{k^2+4t} \end{pmatrix}.$$

Also, from (13), (9) and (14), we get

$$\begin{aligned} &(k^2+4t)^n(X - Y) \\ &= \begin{pmatrix} \sum_{j=0}^{2n+1} \binom{2n+1}{j} t^j \alpha^{2n+1-2j} & \sum_{j=0}^{2n+1} \binom{2n+1}{j} t^j U_{2n+1-2j} \\ 0 & \sum_{j=0}^{2n+1} \binom{2n+1}{j} t^j \beta^{2n+1-2j} \end{pmatrix}. \end{aligned}$$

From these equalities, we obtain (6), (7) and (8). □

Theorem 4. *Let m and n be natural numbers. The following statements are true:*

$$V_m^n = \sum_{j=0}^{n} \binom{n}{j} \alpha^{mn-2mj}(-t)^{mj}, \tag{15}$$

$$V_m^n = \sum_{j=0}^{n} \binom{n}{j} \beta^{mn-2mj}(-t)^{mj}, \tag{16}$$

$$0 = \sum_{j=0}^{n} \binom{n}{j} U_{mn-2mj}(-t)^{mj}, \tag{17}$$

$$2V_m^n = \sum_{j=0}^{n} \binom{n}{j} V_{mn-2mj}(-t)^{mj}. \tag{18}$$

Proof. Let $X = \begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix}$ and $Y = \begin{pmatrix} \beta & -1 \\ 0 & \alpha \end{pmatrix}$. Then, we have

$$XY = YX = -tI, \tag{19}$$

where I denotes the identity matrix. By using Theorem 1, we obtain

$$\begin{aligned} X^m &= \begin{pmatrix} \alpha^m & U_m \\ 0 & \beta^m \end{pmatrix}, \\ Y^m &= \begin{pmatrix} \beta^m & -U_m \\ 0 & \alpha^m \end{pmatrix}, \end{aligned} \tag{20}$$

and so

$$\begin{aligned} X^m + Y^m &= \begin{pmatrix} \alpha^m & U_m \\ 0 & \beta^m \end{pmatrix} + \begin{pmatrix} \beta^m & -U_m \\ 0 & \alpha^m \end{pmatrix} \\ &= \begin{pmatrix} V_m & 0 \\ 0 & V_m \end{pmatrix} = V_m I. \end{aligned} \tag{21}$$

Moreover, by using (19), (20) and (21), we get

$$\begin{aligned} V_m^n I = (V_m I)^n = (X^m + Y^m)^n &= \sum_{j=0}^{n} \binom{n}{j} (X^m)^{n-j} (Y^m)^j \\ &= \begin{pmatrix} \sum_{j=0}^{n} \binom{n}{j} \alpha^{mn-2mj}(-t)^{mj} & \sum_{j=0}^{n} \binom{n}{j} U_{mn-2mj}(-t)^{mj} \\ 0 & \sum_{j=0}^{n} \binom{n}{j} \beta^{mn-2mj}(-t)^{mj} \end{pmatrix}. \end{aligned} \tag{22}$$

From this equality, we obtain the equations (15), (16) and (17). Additionally, by using (15), (16) and (2), we obtain (18). □

When $m = 1$ is taken in Theorem 4, we find $V_1 = k$. Then, the following result can be given easily.

Corollary 1. *Let n be a natural number. Then*

$$k^n = \sum_{j=0}^{n} \binom{n}{j} \alpha^{n-2j}(-t)^j = \sum_{j=0}^{n} \binom{n}{j} \beta^{n-2j}(-t)^j,$$

$$0 = \sum_{j=0}^{n} \binom{n}{j} U_{n-2j}(-t)^j,$$

and

$$2k^n = \sum_{j=0}^{n} \binom{n}{j} V_{n-2j}(-t)^j.$$

Theorem 5. *Let m and n be natural numbers. The following statements are true:*
If n is an even natural number, then

$$U_m^n (k^2 + 4t)^{\frac{n}{2}} = \sum_{j=0}^{n} \binom{n}{j} \alpha^{mn-2mj}(-t)^{mj}(-1)^j, \tag{23}$$

$$U_m^n (k^2 + 4t)^{\frac{n}{2}} = \sum_{j=0}^{n} \binom{n}{j} \beta^{mn-2mj}(-t)^{mj}(-1)^j, \tag{24}$$

$$0 = \sum_{j=0}^{n} \binom{n}{j} U_{mn-2mj}(-t)^{mj}(-1)^j, \tag{25}$$

$$2U_m^n (k^2 + 4t)^{\frac{n}{2}} = \sum_{j=0}^{n} \binom{n}{j} V_{mn-2mj}(-t)^{mj}(-1)^j. \tag{26}$$

If n is an odd natural number, then

$$U_m^n (k^2 + 4t)^{\frac{n}{2}} = \sum_{j=0}^{n} \binom{n}{j} \alpha^{mn-2mj}(-t)^{mj}(-1)^j, \tag{27}$$

$$U_m^n (k^2 + 4t)^{\frac{n}{2}} = -\sum_{j=0}^{n} \binom{n}{j} \beta^{mn-2mj}(-t)^{mj}(-1)^j, \tag{28}$$

$$2U_m^n (k^2 + 4t)^{\frac{n-1}{2}} = \sum_{j=0}^{n} \binom{n}{j} U_{mn-2mj}(-t)^{mj}(-1)^j, \tag{29}$$

$$0 = \sum_{j=0}^{n} \binom{n}{j} (-t)^{mj}(-1)^j V_{mn-2mj}. \tag{30}$$

Proof. Let $X = \begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix}$ and $Y = \begin{pmatrix} \beta & -1 \\ 0 & \alpha \end{pmatrix}$. Then, we find

$$\begin{aligned} X - Y &= \begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix} - \begin{pmatrix} \beta & -1 \\ 0 & \alpha \end{pmatrix} \\ &= \begin{pmatrix} \left(k^2+4t\right)^{\frac{1}{2}} & 2 \\ 0 & -\left(k^2+4t\right)^{\frac{1}{2}} \end{pmatrix}, \end{aligned} \tag{31}$$

$$XY = YX = -tI, \tag{32}$$

where I denotes the identity matrix. By using Theorem 1, we have

$$X^m = \begin{pmatrix} \alpha^m & U_m \\ 0 & \beta^m \end{pmatrix}, \tag{33}$$

$$Y^m = \begin{pmatrix} \beta^m & -U_m \\ 0 & \alpha^m \end{pmatrix}, \tag{34}$$

$$(X-Y)^n = \begin{cases} (k^2+4t)^{n/2} I, & \text{for even } n; \\ (k^2+4t)^{(n-1)/2}(X-Y), & \text{for odd } n; \end{cases} \tag{35}$$

and by using (31), (33) and (34), we get

$$\begin{aligned} X^m - Y^m &= \begin{pmatrix} \alpha^m & U_m \\ 0 & \beta^m \end{pmatrix} - \begin{pmatrix} \beta^m & -U_m \\ 0 & \alpha^m \end{pmatrix} \\ &= U_m \begin{pmatrix} (k^2+4t)^{\frac{1}{2}} & 2 \\ 0 & -(k^2+4t)^{\frac{1}{2}} \end{pmatrix} = U_m(X-Y). \end{aligned} \tag{36}$$

On the other hand, by using (32) and (36), we find that

$$\begin{aligned} U_m^n (X-Y)^n &= (U_m(X-Y))^n = (X^m - Y^m)^{\ n} \\ &= \sum_{j=0}^{n} \binom{n}{j} X^{mn-2mj}(-t)^{mj}(-1)^j. \end{aligned} \tag{37}$$

Thus, if n is an even natural number, then

$$U_m^n (X-Y)^n = (k^2+4t)^{\frac{n}{2}} {U_m}^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{aligned} &U_m^n (X-Y)^n \\ &= \begin{pmatrix} \sum_{j=0}^{n} \binom{n}{j} \alpha^{mn-2mj}(-t)^{mj}(-1)^j & \sum_{j=0}^{n} \binom{n}{j} U_{mn-2mj}(-t)^{mj}(-1)^j \\ 0 & \sum_{j=0}^{n} \binom{n}{j} \beta^{mn-2mj}(-t)^{mj}(-1)^j \end{pmatrix} \end{aligned}$$

by (35) and (37). From these equalities, we get the equations (23), (24) and (25). Additionally, by summing the equations (23) and (24), we can give

(26). If n is an odd natural number, then

$$U_m^n(X-Y)^n = (k^2+4t)^{\frac{n-1}{2}} U_m^n \begin{pmatrix} (k^2+4t)^{\frac{1}{2}} & 2 \\ 0 & -(k^2+4t)^{\frac{1}{2}} \end{pmatrix},$$

$$U_m^n(X-Y)^n$$
$$= \begin{pmatrix} \sum_{j=0}^{n} \binom{n}{j} \alpha^{mn-2mj}(-t)^{mj}(-1)^j & \sum_{j=0}^{n} \binom{n}{j} U_{mn-2mj}(-t)^{mj}(-1)^j \\ 0 & \sum_{j=0}^{n} \binom{n}{j} \beta^{mn-2mj}(-t)^{mj}(-1)^j \end{pmatrix}$$

by (35) and (37). From this matrix equation, we obtain the equations (27), (28) and (29). Also, by subtracting the equations (28) from (27), we obtain (30). □

Theorem 6. *Let m and n be natural numbers. The following are true:*

(38) $$\alpha^{mn} = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \beta^{mn-mj} V_m^j,$$

(39) $$\beta^{mn} = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \alpha^{mn-mj} V_m^j,$$

(40) $$U_{mn} = -\sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} U_{mn-mj} V_m^j,$$

(41) $$V_{mn} = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} V_{mn-mj} V_m^j.$$

Proof. Let $X = \begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix}$ and $Y = \begin{pmatrix} \beta & -1 \\ 0 & \alpha \end{pmatrix}$. By using Theorem 1, it can be seen that

(42) $$X^m = \begin{pmatrix} \alpha^m & U_m \\ 0 & \beta^m \end{pmatrix},$$

(43) $$X^{mn} = \begin{pmatrix} \alpha^{mn} & U_{mn} \\ 0 & \beta^{mn} \end{pmatrix},$$

(44) $$Y^m = \begin{pmatrix} \beta^m & -U_m \\ 0 & \alpha^m \end{pmatrix},$$

(45) $$X^m + Y^m = \begin{pmatrix} \alpha^m & U_m \\ 0 & \beta^m \end{pmatrix} + \begin{pmatrix} \beta^m & -U_m \\ 0 & \alpha^m \end{pmatrix} \begin{pmatrix} V_m & 0 \\ 0 & V_m \end{pmatrix} = V_m I,$$

where I denotes the identity matrix. From this, we get

$$\begin{aligned} X^{mn} &= (X^m)^n = (-Y^m + V_m I)^n = \sum_{j=0}^{n} \binom{n}{j} (-Y^m)^{n-j} V_m^j \\ &= \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} Y^{mn-mj} V_m^j. \end{aligned} \tag{46}$$

So, by using (44) and (46), we find that

$$X^{mn} = \begin{pmatrix} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \beta^{mn-mj} V_m^j & -\sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} U_{mn-mj} V_m^j \\ 0 & \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \alpha^{mn-mj} V_m^j \end{pmatrix}.$$

From these equalities, (38), (39) and (40) can be seen easily. Also, by adding equations (38) and (39), we find (41). □

Theorem 7. *The following statements are true:*
If n is an even natural number, then

$$tU_{n-1} = \sum_{j=0}^{n} \binom{n}{j} (-k)^j U_{n-j+1}, \tag{47}$$

$$U_{n+1} = t \sum_{j=0}^{n} \binom{n}{j} (-k)^j U_{n-j-1}, \tag{48}$$

$$U_n = -\sum_{j=0}^{n} \binom{n}{j} (-k)^j U_{n-j}, \tag{49}$$

$$V_n = \sum_{j=0}^{n} \binom{n}{j} (-k)^j V_{n-j}. \tag{50}$$

If n is an odd natural number, then

$$U_{n+1} = -t \sum_{j=0}^{n} \binom{n}{j} (-k)^j U_{n-j-1}, \tag{51}$$

$$U_n = \sum_{j=0}^{n} \binom{n}{j} (-k)^j U_{n-j}, \tag{52}$$

$$-tU_{n-1} = \sum_{j=0}^{n} \binom{n}{j} (-k)^j U_{n-j+1}, \tag{53}$$

$$V_n = -\sum_{j=0}^{n} \binom{n}{j} (-k)^j V_{n-j}. \tag{54}$$

Proof. Let $A = \begin{pmatrix} k & t \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & t \\ 1 & -k \end{pmatrix}$. Then

$$A - B = \begin{pmatrix} k & t \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & t \\ 1 & -k \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = kI$$

and so

$$B^n = (A - kI)^n = \sum_{j=0}^{n} \binom{n}{j} A^{n-j}(-k)^j, \tag{55}$$

where I denotes the identity matrix. By using Theorem 2, we have

$$A^{n-j} = \begin{pmatrix} U_{n-j+1} & tU_{n-j} \\ U_{n-j} & tU_{n-j-1} \end{pmatrix} \tag{56}$$

and if n is an even natural number, then

$$B^n = \begin{pmatrix} tU_{n-1} & -tU_n \\ -U_n & U_{n+1} \end{pmatrix}. \tag{57}$$

Hence, from (55), (56) and (57), it follows that

$$\begin{aligned} B^n &= \begin{pmatrix} tU_{n-1} & -tU_n \\ -U_n & U_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{j=0}^{n} \binom{n}{j} (-k)^j U_{n-j+1} & t\sum_{j=0}^{n} \binom{n}{j} (-k)^j U_{n-j} \\ \sum_{j=0}^{n} \binom{n}{j} (-k)^j U_{n-j} & t\sum_{j=0}^{n} \binom{n}{j} (-k)^j U_{n-j-1} \end{pmatrix}. \end{aligned}$$

From these equalities, we obtain the equations (47), (48) and (49). Also, (47) and (48) are added side by side, we find (50) from (3). On the other hand, according to Theorem 2, if n is an odd natural number, then

$$B^n = \begin{pmatrix} -tU_{n-1} & tU_n \\ U_n & -U_{n+1} \end{pmatrix}. \tag{58}$$

Then, from (55), (56) and (58), we see that

$$\begin{aligned} B^n &= \begin{pmatrix} -tU_{n-1} & tU_n \\ U_n & -U_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{j=0}^{n} \binom{n}{j} (-k)^j U_{n-j+1} & t\sum_{j=0}^{n} \binom{n}{j} (-k)^j U_{n-j} \\ \sum_{j=0}^{n} \binom{n}{j} (-k)^j U_{n-j} & t\sum_{j=0}^{n} \binom{n}{j} (-k)^j U_{n-j-1} \end{pmatrix}. \end{aligned}$$

From this equality, we find the equations (51), (52) and (53). Also, by using (51) and (53), we can give (54). □

Theorem 8. *The following statements are true:*
If n is an even natural number, then

$$U_{n-1}(k^2+4t)^{\frac{n}{2}} = \sum_{j=0}^{n} \binom{n}{j} t^j U_{2n-2j-1}, \tag{59}$$

$$U_{n}(k^2+4t)^{\frac{n}{2}} = \sum_{j=0}^{n} \binom{n}{j} t^j U_{2n-2j}, \tag{60}$$

$$U_{n+1}(k^2+4t)^{\frac{n}{2}} = \sum_{j=0}^{n} \binom{n}{j} t^j U_{2n-2j+1}, \tag{61}$$

$$V_{n}(k^2+4t)^{\frac{n}{2}} = \sum_{j=0}^{n} \binom{n}{j} t^j V_{2n-2j}. \tag{62}$$

If n is an odd natural number, then

$$V_{n-1}(k^2+4t)^{\frac{n-1}{2}} = \sum_{j=0}^{n} \binom{n}{j} t^j U_{2n-2j-1}, \tag{63}$$

$$V_{n}(k^2+4t)^{\frac{n-1}{2}} = \sum_{j=0}^{n} \binom{n}{j} t^j U_{2n-2j}, \tag{64}$$

$$V_{n+1}(k^2+4t)^{\frac{n-1}{2}} = \sum_{j=0}^{n} \binom{n}{j} t^j U_{2n-2j+1}, \tag{65}$$

$$U_{n}(k^2+4t)^{\frac{n+1}{2}} = \sum_{j=0}^{n} \binom{n}{j} t^j V_{2n-2j}. \tag{66}$$

Proof. Let $A = \begin{pmatrix} 2t & -kt \\ -k & k^2+2t \end{pmatrix}$ and $B = \begin{pmatrix} 0 & t \\ 1 & -k \end{pmatrix}$. Clearly,

$$A = \begin{pmatrix} 2t & -kt \\ -k & k^2+2t \end{pmatrix} = \begin{pmatrix} 0 & t \\ 1 & -k \end{pmatrix}\begin{pmatrix} 0 & t \\ 1 & -k \end{pmatrix} + t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B^2 + tI$$

and so

$$A^n = (B^2+tI)^n = \sum_{j=0}^{n} \binom{n}{j} B^{2n-2j} t^j, \tag{67}$$

where I denotes the identity matrix. Here, according to Theorem 2, if n is an even natural number, then

$$A^n = \begin{pmatrix} t(k^2+4t)^{\frac{n}{2}} U_{n-1} & -t(k^2+4t)^{\frac{n}{2}} U_{n} \\ -(k^2+4t)^{\frac{n}{2}} U_{n} & (k^2+4t)^{\frac{n}{2}} U_{n+1} \end{pmatrix} \tag{68}$$

and

$$B^{2n-2j} = \begin{pmatrix} tU_{2n-2j-1} & -tU_{2n-2j} \\ -U_{2n-2j} & U_{2n-2j+1} \end{pmatrix}. \tag{69}$$

By using (67), (68) and (69), we find that

$$A^n = \begin{pmatrix} t(k^2+4t)^{\frac{n}{2}}U_{n-1} & -t(k^2+4t)^{\frac{n}{2}}U_n \\ -(k^2+4t)^{\frac{n}{2}}U_n & (k^2+4t)^{\frac{n}{2}}U_{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=0}^{n}\binom{n}{j}t^{j+1}U_{2n-2j-1} & -\sum_{j=0}^{n}\binom{n}{j}t^{j+1}U_{2n-2j} \\ -\sum_{j=0}^{n}\binom{n}{j}t^{j}U_{2n-2j} & \sum_{j=0}^{n}\binom{n}{j}t^{j}U_{2n-2j+1} \end{pmatrix}.$$

From this equality, we have the equations (59), (60) and (61). Also, by using these equations and (3), we find (62). Similarly, according to Theorem 2, if n is an odd natural number, then

$$A^n = \begin{pmatrix} t(k^2+4t)^{\frac{n-1}{2}}V_{n-1} & -t(k^2+4t)^{\frac{n-1}{2}}V_n \\ -(k^2+4t)^{\frac{n-1}{2}}V_n & (k^2+4t)^{\frac{n-1}{2}}V_{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=0}^{n}\binom{n}{j}t^{j+1}U_{2n-2j-1} & -\sum_{j=0}^{n}\binom{n}{j}t^{j+1}U_{2n-2j} \\ -\sum_{j=0}^{n}\binom{n}{j}t^{j}U_{2n-2j} & \sum_{j=0}^{n}\binom{n}{j}t^{j}U_{2n-2j+1} \end{pmatrix}.$$

From this equality, equations (63), (64) and (65) are seen easily. By using (3) and (4), we obtain the equation (66). □

Theorem 9. *If n is an even natural number, then*

$$(70)\qquad (k^2+4t)^{\frac{n}{2}}U_{n-1} = \sum_{j=0}^{n}\binom{n}{j}t^{n-j}U_{2j-1},$$

$$(71)\qquad (k^2+4t)^{\frac{n}{2}}U_{n} = \sum_{j=0}^{n}\binom{n}{j}t^{n-j}U_{2j},$$

$$(72)\qquad (k^2+4t)^{\frac{n}{2}}U_{n+1} = \sum_{j=0}^{n}\binom{n}{j}t^{n-j}U_{2j+1},$$

$$(73)\qquad (k^2+4t)^{\frac{n}{2}}V_{n} = \sum_{j=0}^{n}\binom{n}{j}t^{n-j}V_{2j}.$$

If n is an odd natural number, then

$$(74)\qquad (k^2+4t)^{\frac{n-1}{2}}V_{n-1} = \sum_{j=0}^{n}\binom{n}{j}t^{n-j}U_{2j-1},$$

$$(75)\qquad (k^2+4t)^{\frac{n-1}{2}}V_{n} = \sum_{j=0}^{n}\binom{n}{j}t^{n-j}U_{2j},$$

$$(76)\qquad (k^2+4t)^{\frac{n-1}{2}}V_{n+1} = \sum_{j=0}^{n}\binom{n}{j}t^{n-j}U_{2j+1},$$

$$(77)\qquad (k^2+4t)^{\frac{n+1}{2}}U_{n} = \sum_{j=0}^{n}\binom{n}{j}t^{n-j}V_{2j}.$$

Proof. Let $A = \begin{pmatrix} 2t & -kt \\ -k & k^2+2t \end{pmatrix}$ and $B = \begin{pmatrix} 0 & t \\ 1 & -k \end{pmatrix}$. Then,

$$A = \begin{pmatrix} 2t & -kt \\ -k & k^2+2t \end{pmatrix} = \begin{pmatrix} 0 & t \\ 1 & -k \end{pmatrix}\begin{pmatrix} 0 & t \\ 1 & -k \end{pmatrix} + t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B^2 + tI$$

and so

$$A^n = (tI + B^2)^n = \sum_{j=0}^{n} \binom{n}{j} t^{n-j} B^{2j}. \tag{78}$$

Therefore, according to Theorem 2 and (78), if n is an even natural number, then

$$\begin{aligned} A^n &= \begin{pmatrix} t(k^2+4t)^{\frac{n}{2}} U_{n-1} & -t(k^2+4t)^{\frac{n}{2}} U_n \\ -(k^2+4t)^{\frac{n}{2}} U_n & (k^2+4t)^{\frac{n}{2}} U_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{j=0}^{n} \binom{n}{j} t^{n-j+1} U_{2j-1} & -\sum_{j=0}^{n} \binom{n}{j} t^{n-j+1} U_{2j} \\ -\sum_{j=0}^{n} \binom{n}{j} t^{n-j} U_{2j} & \sum_{j=0}^{n} \binom{n}{j} t^{n-j} U_{2j+1} \end{pmatrix}. \end{aligned}$$

From this equation, we obtain (70), (71) and (72). Also, by using (3), we get the equation (73). Similarly, proof can be made if n is odd. □

The following theorem can be proved using a method similar to the previous one by taking $A = \begin{pmatrix} 2t & -kt \\ -k & k^2+2t \end{pmatrix}$ and $B = \begin{pmatrix} k & t \\ 1 & 0 \end{pmatrix}$.

Theorem 10. *If n is an even natural number, then*

$$(k^2+4t)^{\frac{n}{2}} tU_{n-1} = \sum_{j=0}^{n} \binom{n}{j} (-k)^{n-j} (k^2+2t)^j U_{n-j+1},$$

$$(k^2+4t)^{\frac{n}{2}} U_n = -\sum_{j=0}^{n} \binom{n}{j} (-k)^{n-j} (k^2+2t)^j U_{n-j},$$

$$(k^2+4t)^{\frac{n}{2}} U_{n+1} = t\sum_{j=0}^{n} \binom{n}{j} (-k)^{n-j} (k^2+2t)^j U_{n-j-1},$$

$$(k^2+4t)^{\frac{n}{2}} V_n = \sum_{j=0}^{n} \binom{n}{j} (-k)^{n-j} (k^2+2t)^j V_{n-j}.$$

If n is an odd natural number, then

$$(k^2+4t)^{\frac{n-1}{2}} tV_{n-1} = \sum_{j=0}^{n} \binom{n}{j} (-k)^{n-j} (k^2+2t)^j U_{n-j+1},$$

$$-(k^2+4t)^{\frac{n-1}{2}} V_n = \sum_{j=0}^{n} \binom{n}{j} (-k)^{n-j} (k^2+2t)^j U_{n-j},$$

$$(k^2+4t)^{\frac{n-1}{2}}V_{n+1} = t\sum_{j=0}^{n}\binom{n}{j}(-k)^{n-j}(k^2+2t)^jU_{n-j-1},$$

$$(k^2+4t)^{\frac{n+1}{2}}U_n = \sum_{j=0}^{n}\binom{n}{j}(-k)^{n-j}(k^2+2t)^jV_{n-j}.$$

Theorem 11. *If n is an even natural number, then*

$$(k^2+4t)^{\frac{n}{2}}U_{n+1} = \sum_{j=0}^{n}\binom{n}{j}k^{n-j}(2t)^jU_{n-j+1}, \tag{79}$$

$$(k^2+4t)^{\frac{n}{2}}U_n = \sum_{j=0}^{n}\binom{n}{j}k^{n-j}(2t)^jU_{n-j}, \tag{80}$$

$$(k^2+4t)^{\frac{n}{2}}U_{n-1} = \sum_{j=0}^{n}\binom{n}{j}k^{n-j}(2t)^jU_{n-j-1}, \tag{81}$$

$$(k^2+4t)^{\frac{n}{2}}V_n = \sum_{j=0}^{n}\binom{n}{j}k^{n-j}(2t)^jV_{n-j}. \tag{82}$$

If n is an odd natural number, then

$$(k^2+4t)^{\frac{n-1}{2}}V_{n+1} = \sum_{j=0}^{n}\binom{n}{j}k^{n-j}(2t)^jU_{n-j+1}, \tag{83}$$

$$(k^2+4t)^{\frac{n-1}{2}}V_n = \sum_{j=0}^{n}\binom{n}{j}k^{n-j}(2t)^jU_{n-j}, \tag{84}$$

$$(k^2+4t)^{\frac{n-1}{2}}V_{n-1} = \sum_{j=0}^{n}\binom{n}{j}k^{n-j}(2t)^jU_{n-j-1}, \tag{85}$$

$$(k^2+4t)^{\frac{n+1}{2}}U_n = \sum_{j=0}^{n}\binom{n}{j}k^{n-j}(2t)^jV_{n-j}. \tag{86}$$

Proof. Let $A = \begin{pmatrix} k & t \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} k^2+2t & kt \\ k & 2t \end{pmatrix}$. Then

$$B = \begin{pmatrix} k^2+2t & kt \\ k & 2t \end{pmatrix} = k\begin{pmatrix} k & t \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 2t & 0 \\ 0 & 2t \end{pmatrix} = kA + 2tI$$

and so

$$B^n = (kA+2tI)^n = \sum_{j=0}^{n}\binom{n}{j}k^{n-j}A^{n-j}(2t)^j, \tag{87}$$

where I denotes the identity matrix. From Theorem 2, it is clear that

$$A^n = \begin{pmatrix} U_{n+1} & tU_n \\ U_n & tU_{n-1} \end{pmatrix}. \tag{88}$$

Then, from the equations (87) and (88), it follows that

$$B^n = \begin{pmatrix} \sum_{j=0}^{n} \binom{n}{j} k^{n-j}(2t)^j U_{n-j+1} & t\sum_{j=0}^{n} \binom{n}{j} k^{n-j}(2t)^j U_{n-j} \\ \sum_{j=0}^{n} \binom{n}{j} k^{n-j}(2t)^j U_{n-j} & t\sum_{j=0}^{n} \binom{n}{j} k^{n-j}(2t)^j U_{n-j-1} \end{pmatrix}. \tag{89}$$

Therefore, according to Theorem 2, if n is an even natural number, then we find that

$$B^n = \begin{pmatrix} (k^2+4t)^{\frac{n}{2}} U_{n+1} & t(k^2+4t)^{\frac{n}{2}} U_n \\ U_n (k^2+4t)^{\frac{n}{2}} & (k^2+4t)^{\frac{n}{2}} tU_{n-1} \end{pmatrix}. \tag{90}$$

From (89) and (90), we get the equations (79), (80) and (81). Also, by using (3), we get (82). On the other hand, according to Theorem 2, if n is an odd natural number, then we obtain

$$B^n = \begin{pmatrix} (k^2+4t)^{\frac{n-1}{2}} V_{n+1} & t(k^2+4t)^{\frac{n-1}{2}} V_n \\ (k^2+4t)^{\frac{n-1}{2}} V_n & t(k^2+4t)^{\frac{n-1}{2}} V_{n-1} \end{pmatrix}. \tag{91}$$

From (89) and (91), we obtain the equations (83), (84) and (85). Also, by using (3) and (4), we get (86). □

Theorem 12. *Let n be a natural number. Then the following are true:*

$$k^n = \sum_{j=0}^{n} \binom{n}{j} (-t)^j U_{n-2j+1} = t\sum_{j=0}^{n} \binom{n}{j} (-t)^j U_{n-2j-1}.$$

Proof. Let $A = \begin{pmatrix} k & t \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & t \\ 1 & -k \end{pmatrix}$. Then, $AB = BA = tI$,

$$A - B = \begin{pmatrix} k & t \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & t \\ 1 & -k \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = kI, \tag{92}$$

and so

$$(A-B)^n = k^n I = \sum_{j=0}^{n} \binom{n}{j} A^{n-j}(-B)^j = \sum_{j=0}^{n} \binom{n}{j} A^{n-2j}(-t)^j, \tag{93}$$

where I denotes the identity matrix. From Theorem 2, the equations (92) and (93),

$$(A-B)^n = k^n I = \begin{pmatrix} \sum_{j=0}^{n} \binom{n}{j} (-t)^j U_{n-2j+1} & t\sum_{j=0}^{n} \binom{n}{j} (-t)^j U_{n-2j} \\ \sum_{j=0}^{n} \binom{n}{j} (-t)^j U_{n-2j} & t\sum_{j=0}^{n} \binom{n}{j} (-t)^j U_{n-2j-1} \end{pmatrix}$$

is obtanied. □

We will reproduce the following known theorem using different matrices.

Theorem 13. *Let k and t be integers different from the zero. Then*

$$(k^2 + 4t)(-t)^{n-1} = V_{n+1}V_{n-1} - V_n^2.$$

Proof. Let $A = \begin{pmatrix} k & t \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} k & 2t \\ 2 & -k \end{pmatrix}$. Then $|B| = -(k^2 + 4t)$. From Theorem 2, we have $A^n = \begin{pmatrix} U_{n+1} & tU_n \\ U_n & tU_{n-1} \end{pmatrix}$ and so

$$|A^n| = (tU_{n+1}U_{n-1} - tU_n^2). \tag{94}$$

Moreover,

$$BA^n = \begin{pmatrix} k & 2t \\ 2 & -k \end{pmatrix} \begin{pmatrix} U_{n+1} & tU_n \\ U_n & tU_{n-1} \end{pmatrix} = \begin{pmatrix} V_{n+1} & tV_n \\ V_n & tV_{n-1} \end{pmatrix}.$$

By using (5), (94) and taking the determinant of both sides, we find

$$|BA^n| = |B| \cdot |A^n| = -(k^2 + 4t)(tU_{n+1}U_{n-1} - tU_n^2) = -(k^2 + 4t)(-t)^n \tag{95}$$

and

$$|BA^n| = \left| \begin{pmatrix} V_{n+1} & tV_n \\ V_n & tV_{n-1} \end{pmatrix} \right| = t(V_{n+1}V_{n-1} - V_n^2). \tag{96}$$

By using (95) and (96), we obtain $(k^2 + 4t)(-t)^{n-1} = V_{n+1}V_{n-1} - V_n^2$. □

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