

## Fixed point theorems for paired contractions in $b$ -metric spaces

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ABSTRACT. In this research, we establish new fixed-point theorems for mappings that satisfy the paired contraction property in  $b$ -metric spaces. Our results extend and generalize several well-known fixed-point theorems in the literature.

### 1. INTRODUCTION AND PRELIMINARIES

The field of fixed point theory has experienced significant dynamism over the past few decades, captivating researchers due to its inherent simplicity, accessibility, and wide-ranging applications. This area of study holds particular interest for its relevance in diverse fields such as locating solutions to integral equations, solving differential equations, optimization, numerical analysis, as well as its applications in chemistry, physics, computer science, and among others.

Numerous studies have explored fixed points and their wide-ranging applications across various fields of mathematics and science. To uncover new and compelling results, researchers typically follow two main strategies. The first approach involves altering the nature of the operators by either strengthening or relaxing certain constraints, leading to the development of interesting operators that remain effective even in cases, where traditional contraction principles fail to apply [12, 20, 22, 23, 27]. The second strategy focuses on changing the underlying abstract space or framework, encompassing structures such as metric spaces, symmetric and non-symmetric spaces, quasi-metric spaces,  $S$ -metric spaces,  $G$ -metric spaces, and others [4, 6, 8, 24].

The concept of  $b$ -metric spaces, also known as metric-type spaces, was initially introduced by I. A. Bakhtin in 1989 [2], and further developed by

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2020 *Mathematics Subject Classification*. Primary: 47H10; Secondary: 54H25.

*Key words and phrases*.  $b$ -metric spaces, Fixed point, Paired contraction.

*Full paper*. Received 30 Sep 2025, accepted 19 Jan 2026, available online 9 Feb 2026.

S. Czerwik in 1993 [13]. In recent years, increasing attention has been devoted to fixed point results in generalized metric structures, including various extensions of  $b$ -metric spaces. Several authors have investigated fixed point theorems under different contractive conditions in such settings, aiming to obtain more flexible and unifying results, we highlight a selection of them for reference [1, 5, 7, 15–19]. Motivated by the extensive developments in  $b$ -metric spaces and their generalizations, it is natural to investigate fixed point results under new contractive frameworks that can further relax classical assumptions. In this direction, the paired contraction property provides an effective tool for extending fixed point theory in  $b$ -metric spaces, as it allows the treatment of a broader class of mappings beyond standard contraction conditions. Consequently, the present work focuses on establishing new fixed point theorems for mappings satisfying the paired contraction property in the setting of  $b$ -metric spaces, thereby contributing to the ongoing development of fixed point theory in generalized metric frameworks.

**Definition 1** ([2, 13]). Let  $X$  be a nonempty set, and  $s \geq 1$  be a given real number. A function  $d : X \times X \rightarrow [0, +\infty)$  is said to be  $b$ -metric if it satisfies the following conditions for all  $x, y, z \in X$ :

- (a)  $d(x, y) = 0$  if and only if  $x = y$ ,
- (b)  $d(x, y) = d(y, x)$ ,
- (c)  $d(x, z) \leq s(d(x, y) + d(y, z))$ .

Then,  $(X, d)$  is called a  $b$ -metric space.

By definition, we note that every metric space is a  $b$ -metric space with  $s = 1$ , but the converse is not true in general, the following example illustrates this.

**Example 1** ([26]). Let  $X = \{0, 1, 2\}$ , define  $d : X \times X \rightarrow \mathbb{R}_+$  by

$$\begin{aligned} d(0, 0) &= d(1, 1) = d(2, 2) = 0, \\ d(1, 0) &= d(0, 1) = d(2, 1) = d(1, 2) = 1, \\ d(0, 2) &= d(2, 0) = m, \end{aligned}$$

where  $m$  is a given real number such that  $m \geq 2$ . It is easy to check that for all  $x, y, z \in X$ , we have

$$d(x, y) \leq \frac{m}{2}(d(x, z) + d(z, y)).$$

Therefore,  $(X, d)$  is a  $b$ -metric space with coefficient  $s = \frac{m}{2}$ . The ordinary triangle inequality does not hold if  $m > 2$ , and so  $(X, d)$  is not a metric space.

**Definition 2** ([14]). Let  $(X, d)$  be a  $b$ -metric space and  $\{x_n\}$  be a sequence in  $X$  and  $x \in X$ .

- (1)  $\{x_n\}$  is called convergent to a point  $x$  if and only if

$$\lim_{n \rightarrow \infty} d(x_n, x) = 0, \text{ as } n \rightarrow +\infty,$$

we can write  $\lim_{n \rightarrow \infty} x_n = x$ .

- (2)  $\{x_n\}$  is a Cauchy sequence if and only if

$$d(x_n, x_m) \rightarrow 0, \text{ as } n, m \rightarrow +\infty.$$

- (3) A  $b$ -metric space  $(X, d)$  is said to be complete if and only if every Cauchy sequence in  $X$  is convergent.

**Lemma 1** ([21]). Let  $(X, d)$  be a  $b$ -metric space with constant  $s \geq 1$ , suppose that  $\{x_n\}$  is a sequence in  $X$ . If there exists  $\gamma \in [0, 1)$  satisfying

$$d(x_{n+1}, x_n) \leq \gamma d(x_n, x_{n-1}),$$

for every  $n \in \mathbb{N}$ , then  $\{x_n\}$  a Cauchy sequence.

**Lemma 2** ([25]). Let  $(X, d)$  is a  $b$ -metric space with parameter  $s \geq 1$ , if  $\{x_n\}$  and  $\{y_n\}$  are two  $b$ -convergent sequences with the limit  $x$  and  $y$  respectively. Then, we have

$$\frac{1}{s^2}d(x, y) \leq \liminf_{n \rightarrow \infty} d(x_n, y_n) \leq \limsup_{n \rightarrow \infty} d(x_n, y_n) \leq s^2d(x, y).$$

In particular limits are same, i.e.,  $x = y$ , then we have  $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$ . Moreover, for any  $z \in X$ , we have

$$\frac{1}{s}d(x, z) \leq \liminf_{n \rightarrow \infty} d(x_n, z) \leq \limsup_{n \rightarrow \infty} d(x_n, z) \leq sd(x, z).$$

On the other hand, in 2024, D. Chand and Y. Rohen. [9] established a new type of contraction, known as paired contraction, to study fixed points in metric spaces.

**Definition 3** ([9]). If  $(X, d)$  is a metric space with a cardinality  $|X| \geq 3$ . The self-mapping  $T: X \rightarrow X$  is referred to as a Paired contraction mapping if there is a constant  $\lambda \in [0, 1)$  so that the inequality

$$d(Tx, Ty) + d(Ty, Tz) \leq \lambda (d(x, y) + d(y, z))$$

holds for all pairwise distinct  $x, y, z \in X$ .

**Theorem 1** ([9]). Let  $(X, d)$  be a complete metric space with a cardinality  $|X| \geq 3$ , and consider the mapping  $T : X \rightarrow X$  satisfying the following two conditions

- (a)  $T$  is a Paired Contraction mapping.  
 (b) There are no periodic elements of prime period two for  $T$ .

Then,  $T$  has at least one fixed point. The number of fixed points is at most two.

Where it is important that the distinct points  $x, y, z$  are pairwise distinct. Failure of this condition leads to the definition of Banach's contraction mapping [3]. This was followed by generalizations of this new type of contraction [10, 11].

It is our purpose in this paper to prove fixed-point theorems for mappings that possess the paired contraction property in  $b$ -metric spaces. Examples are furnished to illustrate the validity of our results.

## 2. MAIN RESULTS

In this section, we define the paired contraction condition in the setting of a  $b$ -metric space and establish a corresponding fixed point result. This generalizes the existing results by relaxing the standard triangle inequality, making the theorem applicable to a wider range of mappings and spaces where traditional methods may fail.

**Definition 4.** Let  $(X, d)$  be a  $b$ -metric space with a cardinality  $|X| \geq 3$ . The self-mapping  $T: X \rightarrow X$  is referred to as a Paired contraction mapping in  $b$ -metric space if there is a constant  $\lambda \in [0, \frac{1}{s})$  so that the inequality

$$(1) \quad d(Tx, Ty) + d(Ty, Tz) \leq \lambda(d(x, y) + d(y, z)),$$

holds for all pairwise distinct  $x, y, z \in X$ .

**Theorem 2.** Let  $(X, d)$  be a complete  $b$ -metric space with a cardinality  $|X| \geq 3$ , and consider the mapping  $T: X \rightarrow X$  satisfy the following two conditions

- (a)  $T$  is a Paired Contraction mapping in  $b$ -metric space.
- (b) There are no periodic elements of prime period 2 for  $T$ .

Then,  $T$  has a fixed point. The number of fixed points is not more than two.

*Proof.* Let  $T$  be a mapping with no periodic elements of prime period two. Consider  $x_0$  be an initial point in  $X$  such that  $Tx_0 = x_1, Tx_1 = x_2, \dots, Tx_n = x_{n+1}$ . The sequence is formed  $x_0 = x, x_1 = Tx_0$  and  $x_n = Tx_{n-1} = T^n x_0$ .

Now, in order to prove that all points of  $x_n$  are distinct, we assume that none of the points  $x_n$  are fixed points of the mapping  $T$  for all  $n = 0, 1, 2, \dots$ , we can deduce that  $x_n \neq Tx_n = x_{n+1}$ . Since,  $T$  had no periodic point with prime period two implies  $x_{n+1} = T(T(x_{n-1})) \neq x_{n-1}$ . Therefore,  $x_{n-1}, x_n$  and  $x_{n+1}$  are pairwise distinct. Taking  $x = x_{n-1}, y = x_n$  and  $z = x_{n+1}$  in (1), we get

$$\begin{aligned} d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1}) &\leq \lambda(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \\ &\leq \lambda^2(d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n)) \\ &\vdots \\ &\leq \lambda^n(d(x_0, x_1) + d(x_1, x_2)). \end{aligned}$$

Further, we set  $d_0 = d(x_0, x_1) + d(x_1, x_2)$ ,  $d_1 = d(x_1, x_2) + d(x_2, x_3)$ ,  $\dots$ ,  $d_n = d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})$ , then

$$(2) \quad d_n \leq \lambda d_{n-1} \leq \lambda^2 d_{n-1} \leq \dots \leq \lambda^n d_0.$$

Assume that there is a smallest natural number  $j \geq 3$  for  $x_j = x_i$  with  $i$  satisfying  $0 \leq i \leq j - 2$ . In this situation, we find that  $x_{j+1} = x_{i+1}$  and  $x_{j+2} = x_{i+2}$ . Then,

$$\begin{aligned} d_i &= d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) \\ &= d(x_j, x_{j+1}) + d(x_{j+1}, x_{j+2}) \\ &= d_j, \end{aligned}$$

which contradicts to equation (2). Then, no such values of  $i$  and  $j$  can be found.

Now we shall show that  $\{x_n\}$  is a Cauchy sequence. From the preceding arguments, it is evident that

$$(3) \quad d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) = d_n \leq \lambda^n d_0.$$

Let  $p \in \mathbb{N}$ ,  $p > 2$ , applying the triangle inequality repeatedly and considering equation (3), we can write the following

$$\begin{aligned} d(x_n, x_{n+p}) &\leq s(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+p})) \\ &\leq sd(x_n, x_{n+1}) + s^2(d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+p})) \\ &\vdots \\ &\leq sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + \dots + s^pd(x_{n+p-1}, x_{n+p}) \\ &\leq s\lambda^n d_0 + s^2\lambda^{n+1}d_0 + \dots + s^p\lambda^{n+p-1}d_0 \\ &= s\lambda^n(1 + s\lambda + \dots + s^{p-1}\lambda^{p-1})d_0 \\ &= s\lambda^n \left( \frac{1}{1 - s\lambda} \right) d_0. \end{aligned}$$

Previously we have that  $\lambda \in [0, \frac{1}{s})$ . Thus, as we let  $n \rightarrow \infty$ , we find that  $d(x_n, x_{n+p}) \rightarrow 0$  for every  $p > 0$ . It follows, sequence is a Cauchy sequence. As well as that  $(X, d)$  is complete, we can conclude that  $\{x_n\}$  has a limit  $x^* \in X$ .

From the above we know that every three consecutive elements of  $\{x_n\}$  consist of pairwise distinct elements. If  $x^* \neq x_i$  for any  $i \in \{1, 2, 3, \dots\}$ , this implies that the inequality (1) is satisfied for any three elements  $x^*$ ,  $x_{n-1}$  and  $x_n$ . Now, for the smallest possible index  $i \in \{1, 2, 3, \dots\}$  such that  $x^* = x_i$ . Let  $m > i$  be such that  $x^* = x_m$ , which means that the sequence  $\{x_n\}$  is cyclic starting from  $i$  and making it non-Cauchy. Then, for  $n - 1 > i$ , the points  $x^*$ ,  $x_{n-1}$  and  $x_n$  are pairwise distinct.

Next, we show that  $x^*$  is a fixed point of  $T$ . If there is  $x_i = x^*$  for all  $i \in \{1, 2, 3, \dots\}$ , then for  $n - 1 > k$  and by the triangle inequality with

inequality (1), this yields

$$\begin{aligned} d(x^*, Tx^*) &\leq s(d(x^*, x_n) + d(x_n, Tx^*)) \\ &= s(d(x^*, x_n) + d(Tx_{n-1}, Tx^*)) \\ &\leq sd(x^*, x_n) + s(d(Tx_{n-1}, Tx^*) + d(Tx^*, Tx_n)) \\ &\leq sd(x^*, x_n) + s\lambda(d(x_{n-1}, x^*) + d(x^*, x_n)). \end{aligned}$$

Taking the limit as  $n \rightarrow \infty$  we obtain

$$d(x^*, Tx^*) \leq 0,$$

then  $x^*$  is a fixed point of  $T$ .

Suppose that there is a minimum of three distinct fixed points of  $T$ ,  $x^*$ ,  $y^*$  and  $z^*$ , we can write  $Tx^* = x^*$ ,  $Ty^* = y^*$  and  $Tz^* = z^*$ . Applying (1) we have

$$d(Tx^*, Ty^*) + d(Ty^*, Tz^*) \leq \lambda(d(x^*, y^*) + d(y^*, z^*)),$$

which gives

$$(1 - \lambda)(d(x^*, y^*) + d(y^*, z^*)) \leq 0,$$

which is a contradiction. Therefore,  $T$  has at most two fixed points.  $\square$

**Definition 5.** Let  $(X, d)$  be a  $b$ -metric space with a cardinality  $|X| \geq 3$ . The self-mapping  $T: X \rightarrow X$  is referred to as a Paired-Kannan contraction mapping in  $b$ -metric space if there is a constant  $\lambda \in [0, \frac{1}{2})$ , so that the inequality

$$(4) \quad d(Tx, Ty) + d(Ty, Tz) \leq \lambda(d(x, Tx) + d(y, Ty) + d(z, Tz)),$$

holds for all pairwise distinct  $x, y, z \in X$ .

**Example 2.** Let  $X = \{0, \frac{1}{4}, \frac{1}{2}, 1\}$  and  $d : X \times X \rightarrow \mathbb{R}^+$  defined by:

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ |x - y| + 1, & \text{if } x, y \in \{0, \frac{1}{4}, \frac{1}{2}\}; \\ 3, & \text{if } x, y \in \{0, 1\}; \\ 5, & \text{Otherwise.} \end{cases}$$

Then  $(X, d)$  is a complete  $b$ -metric space with  $s = \frac{20}{17}$ , but it is not a metric space, since

$$5 = d(1, \frac{1}{4}) > d(1, 0) + d(0, \frac{1}{4}) = \frac{17}{4}.$$

Define  $T : X \rightarrow X$  by

$$Tx = \begin{cases} 0, & \text{if } x \in \{0, \frac{1}{4}, \frac{1}{2}\}; \\ 1, & \text{if } x = 1. \end{cases}$$

To shorten, we take

$$\begin{aligned} L(x, y, z) &= d(Tx, Ty) + d(Ty, Tz), \\ R(x, y, z) &= d(x, y) + d(y, z). \end{aligned}$$

TABLE 1. The values of  $L(x, y, z)$  and  $R(x, y, z)$ .

$(x, y, z)$	$L(x, y, z)$	$R(x, y, z)$
$(0, \frac{1}{2}, \frac{1}{4})$	0	$\frac{11}{4}$
$(\frac{1}{2}, 0, \frac{1}{4})$	0	$\frac{11}{4}$
$(0, \frac{1}{4}, \frac{1}{2})$	0	$\frac{5}{2}$
$(0, \frac{1}{2}, 1)$	3	$\frac{13}{2}$
$(\frac{1}{2}, 0, 1)$	3	$\frac{9}{2}$
$(0, \frac{1}{4}, 1)$	3	$\frac{25}{4}$
$(\frac{1}{4}, 0, 1)$	3	$\frac{17}{4}$
$(\frac{1}{4}, \frac{1}{2}, 1)$	3	$\frac{25}{4}$
$(\frac{1}{2}, \frac{1}{4}, 1)$	3	$\frac{25}{4}$
$(\frac{1}{2}, 1, \frac{1}{4})$	6	10
$(0, 1, \frac{1}{2})$	6	8
$(0, 1, \frac{1}{4})$	6	8

Then, from the Table 1, we notice that in all cases the inequality (1) satisfying

$$L(x, y, z) \leq \frac{6}{8}R(x, y, z)$$

for all three distinct points  $x, y, z \in X$ , so both the conditions of Theorem 2 is satisfied and we can say that the inequality (1) holds for any  $\frac{6}{8} \leq \lambda < \frac{17}{20}$ . Hence,  $T$  has two fixed points.

**Theorem 3.** *Let  $(X, d)$  be a complete  $b$ -metric space with a cardinality  $|X| \geq 3$ , and consider the mapping  $T : X \rightarrow X$  satisfy the following two conditions*

- (a)  *$T$  is a Paired-Kannan Contraction mapping in  $b$ -metric space.*
- (b) *There are no periodic elements of prime period 2 for  $T$ .*

*Then  $T$  has a fixed point. The number of fixed points is not more than two.*

*Proof.* Consider  $x_0$  be an initial point in  $X$  and set the sequence  $\{x_n\}$  such that  $x_0 = x$ ,  $x_1 = Tx_0$  and  $x_n = Tx_{n-1} = T^n x_0$ .

Same as we did in Theorem 2, assuming that none of the points  $x_n$  are fixed points of the mapping  $T$  for all  $n = 0, 1, 2, \dots$ , we can deduce that  $x_n \neq Tx_n = x_{n+1}$ . Since,  $T$  had no periodic point with prime period two implies  $x_{n+1} = T(T(x_{n-1})) \neq x_{n-1}$ . Therefore,  $x_{n-1}, x_n$  and  $x_{n+1}$  are pairwise distinct. Taking  $x = x_{n-1}$ ,  $y = x_n$  and  $z = x_{n+1}$  in (4), we get

$$\begin{aligned} & d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1}) \\ & \leq \lambda(d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})). \end{aligned}$$

That implies

$$\begin{aligned} & d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \\ & \leq \lambda(d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})), \end{aligned}$$

which yields

$$(1 - \lambda)[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \leq \lambda d(x_{n-1}, x_n).$$

On the other hand, we have

$$(1 - \lambda)d(x_n, x_{n+1}) \leq (1 - \lambda)[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})],$$

which gives

$$(1 - \lambda)d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n).$$

Let  $\theta = \frac{\lambda}{(1-\lambda)}$ . As we know that  $\lambda \in [0, \frac{1}{2})$ , we obtain  $\theta \in [0, 1)$ . Hence,

$$d(x_n, x_{n+1}) \leq \theta d(x_{n-1}, x_n).$$

Applying Lemma 1, it follows that  $\{x_n\}$  is a  $b$ -Cauchy sequence. As well as that  $(X, d)$  is complete, we can conclude that  $\{x_n\}$  has a limit  $x^* \in X$ .

Note that every three consecutive elements of  $\{x_n\}$  consist of pairwise distinct elements. If  $x^* \neq x_i$  for any  $i \in \{1, 2, 3, \dots\}$ , this implies that the inequality (1) is satisfied for any three elements  $x^*, x_{n-1}$  and  $x_n$ . For the smallest possible index  $i \in \{1, 2, 3, \dots\}$  such that  $x^* = x_i$ . Let  $m > i$  be such that  $x^* = x_m$ , which means that the sequence  $\{x_n\}$  is cyclic starting from  $i$  and making it non-Cauchy. Then, for  $n - 1 > i$ , the points  $x^*, x_{n-1}$  and  $x_n$  are pairwise distinct.

Next, we show that  $x^*$  is a fixed point of  $T$ . If there is  $x_i = x^*$  for all  $i \in \{1, 2, 3, \dots\}$ , then for  $n - 1 > k$  and by the triangle inequality with inequality (1), yields

$$\begin{aligned} d(x^*, Tx^*) & \leq s[d(x^*, x_n) + d(x_n, Tx^*)] \\ & = s[d(x^*, x_n) + d(Tx_{n-1}, Tx^*)] \\ & \leq sd(x^*, x_n) + s[d(Tx_{n-1}, Tx^*) + d(Tx^*, Tx_n)] \\ & \leq sd(x^*, x_n) + s\lambda[d(x_{n-1}, Tx_{n-1}) + d(x^*, Tx^*) + d(x_n, Tx_n)] \end{aligned}$$

$$\begin{aligned} \Rightarrow (1 - s\lambda)d(x^*, Tx^*) &\leq sd(x^*, x_n) + s\lambda[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] \\ \Rightarrow d(x^*, Tx^*) &\leq \frac{sd(x^*, x_n) + s\lambda[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)]}{(1 - s\lambda)}. \end{aligned}$$

Taking the limit as  $n \rightarrow \infty$  we obtain

$$d(x^*, Tx^*) \leq 0,$$

then  $x^*$  is a fixed point of  $T$ .

Suppose that there is a minimum of three distinct fixed points of  $T$ ,  $x^*$ ,  $y^*$  and  $z^*$ , we can write  $Tx^* = x^*$ ,  $Ty^* = y^*$  and  $Tz^* = z^*$ . Applying (1), we have

$$d(Tx^*, Ty^*) + d(Ty^*, Tz^*) \leq \lambda[d(x^*, Tx^*) + d(y^*, Ty^*) + d(z^*, Tz^*)],$$

which gives

$$d(x^*, y^*) + d(y^*, z^*) \leq 0,$$

which is a contradiction. Therefore,  $T$  has at most two fixed points.  $\square$

**Example 3.** Let  $X = \{0, \frac{1}{2}, 1\}$ , equipped with the distance function  $d$  defined by

$$d(0, 0) = d(\frac{1}{2}, \frac{1}{2}) = d(1, 1) = 0, \quad d(0, 1) = 1, \quad d(\frac{1}{2}, 1) = 3, \quad d(0, \frac{1}{2}) = 6.$$

Then  $(X, d)$  is a complete  $b$ -metric space with  $s = \frac{3}{2}$ , but it is not a metric space, since

$$d(0, \frac{1}{2}) = 6 > 4 = d(0, 1) + d(1, \frac{1}{2}).$$

Define  $T : X \rightarrow X$  by

$$Tx = \begin{cases} 0, & \text{if } x \in \{0, \frac{1}{2}\}; \\ 1, & \text{if } x = 1. \end{cases}$$

As in previous example, we find that the left side

$$L(x, y, z) = 1 \quad \text{or} \quad L(x, y, z) = 2,$$

but the right side in all cases we find  $R(x, y, z) = 6$  of the inequality (4). It can be noted in all cases that

$$L(x, y, z) \leq \frac{1}{3}R(x, y, z).$$

So, the two conditions of Theorem 3 is satisfied and we can say that the inequality (4) holds for any  $\frac{1}{3} \leq \lambda < \frac{1}{2}$ . Hence,  $T$  has two fixed points.

**Remark 1.** If we put  $s = 1$  in Theorem 3, then we get Theorem 3.2 of D. Chand [10].

### 3. CONCLUSION

In this paper, we have reiterated the definition and properties of  $b$ -metric spaces. We then established several fixed point theorems for mappings satisfying the paired contraction condition in the setting of  $b$ -metric spaces. The proofs were presented in a direct and systematic manner, relying on the structure of the underlying space. Furthermore, several illustrative examples illustrate the viability and usefulness of our results. Our results on fixed point conditions add further insight into fixed point theory and contractive principles regarding generalized metric spaces. Future work could be undertaken to broaden this concept into other classes of generalized spaces, or to explore other types of contractive principles within generalized metric spaces.

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