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BROUWER'S THEOREM - 90th NEXT

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1. Main facts and history

Perheps the most important of all fixed point theorems is the famous theorem of Brouwer[1] in 1909 which states that *every continuous mapping of the closed unit ball of the Euclidean space* $\mathbb{R}^n$ *into itself has a fixed point.*

Schauder's theorem in 1927 is a generalization of Brouwer's theorem to infinite dimensional normed linear spaces. **Schauder's theorem** states that *every continuous mapping of a compact convex subset of a normed linear space into itself has a fixed point.*

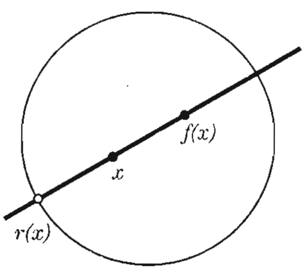

Figure 1

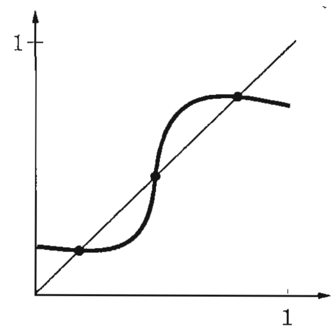

Figure 2

For proof of Brouwer's theorem the following facts are essential. Namely, if B is a closed ball of radius $R > 0$, and f a fixed - point free map, so that $fx \neq x$ for all $x \in B$, then we can construct a retraction $r : B \to \partial B$ as follows. For each $x \in B$, follow the directed line segment from $f(x)$ through x

[1]**Luitzen Egbertus Jan Brouwer** (1881-1966) – German-Dutch mathematician, is one of the greatest mathematicians. He was a professor at the University of Amsterdam, and he is supposed to be one of the founders of modern topology. Brouwer proved the topological invariance of the dimension of the Cartesian space. Furthermore he is famous for his contributions to the fundaments of Mathematics. But his most important result is the fixed point theorem mentioned above.

to its intersection with ∂B, and let the intersection point be $r(x)$, as in Figure 1, which is contradicts.

In general, pick a closed ball B containing a nonempty compact convex subset. Thus $x = f(r(x))$ and $x = r(x)$, i.e., $x = f(x)$. Figure 2 is the case for $f : [0,1] \to [0,1]$. From Brouwer's theorem the graph of f intersects the diagonal in Figure 2.

In recent years a great number of papers have presented generalizations of the well-known essential Brouwer principle. Some of these generalizations refer to result containing the Schauder fixed point theorem.

Brouwer's theorem is one of the oldest and best known results in topology. It was proved for $n = 3$ by L. Brouwer in 1909. An equivalent result was established earlier in 1904 by Bohl.

It was in 1910 Hadamard who gave (using the Kronecker index) the first proof for arbitrary n. Soon afterwards in 1912 Brouwer gave another proof using the simplicial approximation technique and notion of the degree. Other proofs depeding on various definitions of degree were also given by Alexander in 1922 and Birkhoff - Kellogg in 1922.

A simple and short proof of the Brouwer theorem based on combinatorial technique and Sperner's lemma in 1928, was given by Knaster - Kuratowski - Mazurkiewicz in 1929.

An extension of Brouwer's fixed-point theorem, for a class of noncontinuous maps of the n-ball was given by Stallings in 1959.

Recent, in 1998 (and former in 1991) an extension of Brouwer's theorem by introducing "A-variation" was given by Tasković. In this sense, central idea is, that every continuous, bounded variation mapping of complete metric space into itself has a fixed point.

The fact that the Brouwer's theorem admits an equivalent formulation in terms of homotopy and another one in terms of retraction was observed by K. Borsuk in 1931.

Among the proofs of this nonretraction result we mention that of M. Hirsch in 1963 (based on the relative simplicial approximation theorem), its analytical version by J. Milnor (based on Sard's lemma) and Alexandroff - - Pasinkoff in 1957 based on the Knaster - Kuratowski - Mazurkiewicz theorem. Another noteworthy analytic proof in 1971 (using the theory of differential forms) was given by S. Łojasiewicz.

The fact that the Brouwer theorem does not hold for arbitrary continuous maps of the unit ball in infinite dimensional Banach spaces was observed by several authors.

In 1935 Tychonoff (answering a question of S. Ulam) had shown that the unit sphere of l_2 is a retract of the unit ball. Leray in 1935 observed that the unit sphere of $C[0,1]$ is contractible. Kakutani in 1943 gave an example of a fixed point free homeomorphism of the unit ball of l_2 into itself. Klee in 1955 generalized the result of J. Dugundji and established: *A convex set of a metrizable locally convex space is a fixed-point space if and only if it is compact.*

Anather proofs of the Brouwer fixed-point theorem which do not use the mapping degree: Eilenberg and Steenrod in 1952; Dold in 1972 (homology theory); Burger in 1959; Kantorovič and Akilov in 1964 (combinatorial Sperner lemma); Kuhn in 1960; Riedrich in 1976 (cubic Sperner lemma); Dunford - - Schwartz in 1958 (Gauss integral theorem); Hirsch in 1963; Milnor in 1969 (Morse-Sard theorem and one-dimensional manifolds); Chow, Mallet-Paret and Yorke in 1978; Allgower and Georg in 1980 (constructive proof by using the parametrized Sard theorem), Milnor in 1978 (volume trick); Gröger in 1981 (volume trick); and Leinfelder and Simader in 1981. For all references further see [9].

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