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On a new Laplace transform formula for shift properties on time scale

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Abstract. We give a new formula of the Laplace transform expressed using the delta derivative for the generalized time scales. This formula

$$F(z)_{\mathbb{T}} = \mathscr{L}_{\mathbb{T}}\{f(t)\}(z) = \int_0^{\infty} z e_{\ominus(z-1)}(\sigma(t), 0) f(t) \Delta t$$

combines the theory of classical Laplace and $\mathscr{Z}$ transforms. By choosing the time scale to be a set of real numbers, the classical Laplace transformation is obtained in a modified form, and if the time scale is chosen to be a set of integers, the classical $\mathscr{Z}$ transformation is obtained. Formulas for the Laplace transform of elementary functions, as well as formulas for real and complex shifting properties, are derived. These formulas are introduced for specific functions.

1. Introduction

Stefan Hilger introduced the theory of time scales in 1988 [16] with the aim to unify continuous and discrete analysis. The time scale is an arbitrary nonempty closed subset of real numbers. If we choose the time scale to be the set of real numbers, then, for example, the theory of dynamic equations on time scales yields results related to ordinary differential equations. If we choose the time scale to be the set of integers, the same general result yields results for difference equations. Having in mind that there are many other time scales besides these two examples, the theory of time scales provides a more general result. In this way, the results obtained on time scales represent extension of results on continuous and discrete analysis. First we give basic definitions and results related to the theory of time scales.

Stefan Hilger introduced the Laplace transform on time scales in [18], for $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = h\mathbb{Z}$, $(h > 0)$, with the aim to unify the definition of the classical Laplace transform defined on $\mathbb{R}$ and the classical $\mathscr{Z}$-transform on $\mathbb{Z}$. In [10] Martin Bohner and Allan Peterson defined the Laplace transform

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in arbitrary time scale, in a different way then Hilger. In Section 2 we provide the definition of Laplace transform that was given by Bohner and Peterson [10]. At this point we want to note that by taking $\mathbb{T} = \mathbb{R}$, from their form of the Laplace transform we get the classical Laplace transform

$$F(z) = \mathscr{L}\{f(t)\}(z) = \int_0^\infty e^{-zt} f(t)\, dt,$$

while by taking $\mathbb{T} = \mathbb{Z}$, we get the $\mathscr{Z}$-transform in the form

$$F(z) = \mathscr{L}\{f[n]\}(z) = \sum_{n=0}^{\infty} f[n]\, z^{-(n+1)} = \frac{\mathscr{Z}\{f[n]\}(z+1)}{z+1},$$

where

$$\mathscr{Z}\{f[n]\}(z) = \sum_{n=0}^{\infty} f[n]\, z^{-n}$$

is the classical form of the $\mathscr{Z}$-transform.

In Section 2 we will define Laplace transform in time scales in a slightly different way, so that by taking $\mathbb{T} = \mathbb{Z}$ we will get the classical $\mathscr{Z}$-transform, and by taking $\mathbb{T} = \mathbb{R}$ we get the following

$$F(z) = \int_0^\infty z e^{-(z-1)t} f(t)\, dt = z\mathscr{L}\{f(t)\}(z-1).$$

This new formula for the Laplace transform maintains the unification. On the other hand, when taking specific time scales ($\mathbb{Z}$ or $\mathbb{R}$), the new formula gives simpler connections to the classical Laplace and $\mathscr{Z}$-transform. Our motivation for introducing alternative definition of Laplace transform is to obtain formulas for shift properties that are simpler and more convenient for applications.

In the paper [19] the authors proved a result related to the first and second translation theorems for the Laplace transform on time scale. In that paper, the formula for the Laplace transform on time scale, which was set by Bohner and Peterson in [10], was used. The result presented in this paper is a continuation of results presented in [19]. We present a new formula of the Laplace transform on time scale, i.e.,

$$F(z)_{\mathbb{T}} = \mathscr{L}_{\mathbb{T}}\{f(t)\}(z) = \int_0^\infty z e_{\ominus(z-1)}(\sigma(t), 0)\, f(t)\, \Delta t.$$

Thus, all results given in [19] are here derived based on a new formula.

1.1. Time Scale Calculus

For basic definitions and results related to the theory of time scales see [19]. Here we provide few additional results that are needed.

Definition 1 ([8])**.** A function $f : \mathbb{T} \to \mathbb{R}$, is called regulated provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 2 ([8])**.** A function $f : \mathbb{T} \to \mathbb{R}$, is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$, will be denoted by $\mathbb{C}_{rd} = \mathbb{C}_{rd}(\mathbb{T})$.

Theorem 1 ([9])**.** *If $a, b, c \in \mathbb{T}$, $\alpha \in R$ and $f, g \in \mathbb{C}_{rd}$, then*

(1) $\int_a^b (\alpha f(t) + \beta g(t))\,\Delta t = \alpha \int_a^b f(t)\,\Delta t + \beta \int_a^b g(t)\,\Delta t$;

(2) $\int_a^b f(t)\,\Delta t = -\int_b^a f(t)\,\Delta t$;

(3) $\int_a^b f(t)\,\Delta t = \int_a^c f(t)\,\Delta t + \int_c^b f(t)\,\Delta t$;

(4) $\int_a^b f(\sigma(t))\,g^\Delta(t)\,\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)\,g(t)\,\Delta t$;

(5) $\int_a^b f(t)\,g^\Delta(t)\,\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)\,g(\sigma(t))\,\Delta t$;

(6) $\int_a^a f(t)\,\Delta t = 0$.

Lemma 1 ([8])**.** *If $p, q \in \Re$, then the function $p \oplus q$ and the function $\ominus p$ defined by*

$$(p \oplus q)(t) = p(t) + q(t) + \mu(t)\,p(t)\,q(t)\,, \tag{1}$$

$$(\ominus p)(t) = \frac{-p(t)}{1 + \mu(t)\,p(t)}, \quad \textit{for all } t \in \mathbb{T},\ p, q \in \Re. \tag{2}$$

Now, we give generalized monomials and polynomial series. Let us define recursively functions $h_k, g_k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $k \in \mathbb{N} \cup \{0\}$, as follows:

$$h_0(t, s) = 1, \quad h_{k+1}(t, s) = \int_s^t h_k(\tau, s)\,\Delta\tau, \quad \text{for all } s, t \in \mathbb{T}; \tag{3}$$

$$g_0(t, s) = 1, \quad g_{k+1}(t, s) = \int_s^t g_k(\sigma(\tau), s)\,\Delta\tau, \quad \text{for all } s, t \in \mathbb{T}. \tag{4}$$

If we let $h_k^\Delta(t, s)$, i.e., $g_k^\Delta(t, s)$, denote for each fixed s the derivative of $h_k(t, s)$, i.e., $g_k(t, s)$, with respect to t, then

$$h_k^\Delta(t, s) = h_{k-1}(t, s)\,, \quad \text{i.e.,} \quad g_k^\Delta(t, s) = g_{k-1}(\sigma(t), s)\,. \tag{5}$$

Lemma 2 ([9])**.** *If $p \in \Re$ and $r, s, t \in \mathbb{T}$ then:*

(1) $e_p(t, t) = 1$, *and* $e_0(t, s) = 1$;
(2) $e_p(\sigma(t), s) = (1 + \mu(t)\,p(t))\,e_p(t, s)$;
(3) $e_p(t, r)\,e_p(r, s) = e_p(t, s)$;
(4) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
(5) $e_p(t, s)\,e_q(t, s) = e_{p \oplus q}(t, s)$;
(6) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p \ominus q}(t, s)$;
(7) $e_p^\Delta(t, s) = p(t)\,e_p(t, s)$.

2. Main results

Bohner and Peterson defined the Laplace transform on time scales in the following form, in [10].

Definition 3 ([10])**.** For $f : \mathbb{T} \to \mathbb{R}$, the time scale or generalized Laplace transform of f, denote by $\mathscr{L}\{f(t)\}(z)$, is given by

$$F(z) = \mathscr{L}\{f(t)\}(z) = \int_0^\infty f(t)\, e^\sigma_{\ominus z}(t, 0)\, \Delta t. \tag{6}$$

By taking $\mathbb{T} = \mathbb{R}$, from this form of the Laplace transform it is the classical Laplace transform

$$F(z) = \mathscr{L}\{f(t)\}(z) = \int_0^\infty e^{-zt} f(t)\, dt,$$

while by taking $\mathbb{T} = \mathbb{Z}$, it is the $\mathscr{Z}$-transform in the form

$$F(z) = \mathscr{L}\{f[n]\}(z) = \sum_{n=0}^{\infty} f[n]\, z^{-(n+1)} = \frac{\mathscr{Z}\{f[n]\}(z+1)}{z+1},$$

where

$$\mathscr{Z}\{f[n]\}(z) = \sum_{n=0}^{\infty} f[n]\, z^{-n}$$

is the classical form of the $\mathscr{Z}$-transform. Here we define the Laplace transform in a slighltly different way.

Definition 4. For $f : \mathbb{T} \to \mathbb{R}$, the time scale or generalized Laplace transform of f, denoted by $\mathscr{L}_\mathbb{T}\{f(t)\}(z)$, is given by

$$F(z)_\mathbb{T} = \mathscr{L}_\mathbb{T}\{f(t)\}(z) = \int_0^\infty z e_{\ominus(z-1)}(\sigma(t), 0)\, f(t)\, \Delta t. \tag{7}$$

For $\mathbb{T} = \mathbb{Z}$ this formula becomes the clasical $\mathscr{Z}$-transformation, while for $\mathbb{T} = \mathbb{R}$ it becomes

$$F(z) = \int_0^\infty z e^{-(z-1)t} f(t)\, dt = z\mathscr{L}_\mathbb{T}\{f(t)\}(z-1).$$

3.1. Some properties of generalized Laplace transform on Time Scales

In this subsection we give some results related to the generalized Laplace transform in the sense of the newly formulated definition $\mathscr{L}_\mathbb{T}$, including the generalized Laplace transform of some elementary functions defined on time scales.

Theorem 2. *Let $\alpha \in \mathbb{C}$ and $1 + \alpha\mu(t) \neq 0$ for $t \in \mathbb{T}_0$. Then*

$$\mathscr{L}_\mathbb{T}\{e_\alpha(t, 0)\}(z) = \frac{z}{z - 1 - \alpha}, \quad t \in \mathbb{T}_0 \tag{8}$$

provided $\lim_{t\to\infty} e_{\alpha\ominus(z-1)}(t, 0) = 0$.

Proof. Using Definition 4 and property (ii) from Lemma 2 we have

$$\begin{aligned}\mathscr{L}_{\mathbb{T}}\left\{e_{\alpha}\left(t,0\right)\right\}\left(z\right) &= \int_0^{\infty} z e_{\alpha}\left(t,0\right) e_{\ominus(z-1)}\left(\sigma\left(t\right),0\right)\Delta t \\ &= \int_0^{\infty} z\left(1+\mu\left(t\right)\left(z-1\right)\right) e_{\alpha}\left(t,0\right) e_{\ominus(z-1)}\left(t,0\right)\Delta t.\end{aligned}$$

Using property (vii) from Lemma 2, we get

$$\begin{aligned}\mathscr{L}_{\mathbb{T}}\left\{e_{\alpha}\left(t,0\right)\right\}\left(z\right) &= \frac{z}{\alpha-z+1}\int_0^{\infty}\frac{\alpha-z+1}{1+\mu\left(t\right)\left(z-1\right)} e_{\alpha\ominus(z-1)}\left(t,0\right)\Delta t \\ &= \frac{z}{\alpha-z+1}\int_0^{\infty} e^{\Delta}_{\alpha\ominus(z-1)}\left(t,0\right)\Delta t.\end{aligned}$$

Letting t tend to infinity, and using the assumption in Theorem and property (i) from Lemma 2, we obtain

$$\begin{aligned}\mathscr{L}_{\mathbb{T}}\left\{e_{\alpha}\left(t,0\right)\right\}\left(z\right) &= \frac{z}{\alpha-z+1}\; e_{\alpha\ominus(z-1)}\left(t,0\right)\Big|_{t=0}^{t\to\infty} \\ &= \frac{z}{\alpha-z+1}\left\{\lim_{t\to\infty} e_{\alpha\ominus(z-1)}\left(t,0\right)-1\right\} \\ &= \frac{z}{z-1-\alpha},\end{aligned}$$

which completes the proof. □

Remark 1. For expression $\mathscr{L}_{\mathbb{T}}\left\{e_{a-1}\left(t,0\right)\right\}\left(z\right)$ holds:

$$\mathscr{L}_{\mathbb{T}}\left\{e_{a-1}\left(t,0\right)\right\}\left(z\right) = \begin{cases}\mathscr{L}\left\{e^{(a-1)t}\right\}\left(z\right), & \text{if } \mathbb{T}=\mathbb{R}; \\ \mathscr{Z}\left\{a^{t}\right\}\left(z\right), & \text{if } \mathbb{T}=\mathbb{Z}.\end{cases}$$

For $\alpha=0$, from (8), we get

$$\mathscr{L}_{\mathbb{T}}\left\{1\right\}\left(z\right) = \frac{z}{z-1}. \tag{9}$$

Corollary 1. *We have that*

(1) $\mathscr{L}_{\mathbb{T}}\left\{\sin_{\alpha}\left(t,0\right)\right\}\left(z\right) = \frac{z\alpha}{(z-1)^2+\alpha^2}$;

(2) $\mathscr{L}_{\mathbb{T}}\left\{\cos_{\alpha}\left(t,0\right)\right\}\left(z\right) = \frac{z(z-1)}{(z-1)^2+\alpha^2}$,

provided that $\lim_{t\to\infty} e_{i\alpha\ominus(z-1)}\left(t,0\right) = \lim_{t\to\infty} e_{-i\alpha\ominus(z-1)}\left(t,0\right) = 0.$

Theorem 3. *Let $f:\mathbb{T}_0\to\mathbb{C}$ be such that f^{Δ} is regulated. Then*

$$\mathscr{L}_{\mathbb{T}}\left\{f^{\Delta}\left(t\right)\right\}\left(z\right) = -zf\left(0\right)+z\left(z-1\right)\mathscr{L}_{\mathbb{T}}\left\{f\left(t\right)\right\}\left(z\right) \tag{10}$$

for those regressive $z\in\mathbb{C}$ satisfying

$$\lim_{t\to\infty}\left\{f\left(t\right)e_{\ominus(z-1)}\left(t,0\right)\right\} = 0. \tag{11}$$

Proof. Integration by parts (Theorem 1) yields

$$\begin{aligned} &\mathscr{L}_{\mathbb{T}}\left\{f^{\Delta}(t)\right\}(z) \\ &= \int_0^\infty z e_{\ominus(z-1)}(\sigma(t),0) f^{\Delta}(t)\,\Delta t \\ &= z\left\{\lim_{t\to\infty} e_{\ominus(z-1)}(t,0) f(t)\Big|_{t=0}^{t\to\infty} - \int_0^\infty e^{\Delta}_{\ominus(z-1)}(t,0) f(t)\,\Delta t\right\}. \end{aligned} \tag{12}$$

Letting $t \to \infty$, and using equation (2) and property (vii) from Lemma 2, we obtain

$$\begin{aligned} &\mathscr{L}_{\mathbb{T}}\left\{f^{\Delta}(t)\right\}(z) \\ &= z\left\{-f(0) + \int_0^\infty \frac{z-1}{1+\mu(t)(z-1)} e_{\ominus(z-1)}(t,0) f(t)\,\Delta t\right\} \\ &= z\left\{-f(0) + (z-1)\int_0^\infty \frac{1}{1+\mu(t)(z-1)} e_{\ominus(z-1)}(t,0) f(t)\,\Delta t\right\}. \end{aligned} \tag{13}$$

We note that

$$e_{\ominus(z-1)}(\sigma(t),0) = \left[1+\mu(t)(\ominus(z-1))\right] e_{\ominus(z-1)}(t,0),$$

i.e.,

$$e_{\ominus(z-1)}(\sigma(t),0) = \frac{1}{1+\mu(t)(z-1)} e_{\ominus(z-1)}(t,0).$$

From this, (12) becomes

$$\begin{aligned} \mathscr{L}_{\mathbb{T}}\left\{f^{\Delta}(t)\right\}(z) &= z\left\{-f(0) + \frac{z-1}{z}\int_0^\infty z e_{\ominus(z-1)}(\sigma(t),0) f(t)\,\Delta t\right\} \\ &= z\left\{-f(0) + \frac{z-1}{z}\mathscr{L}_{\mathbb{T}}\{f(t)\}(z)\right\}, \end{aligned}$$

i.e.,

$$\mathscr{L}_{\mathbb{T}}\left\{f^{\Delta}(t)\right\}(z) = -zf(0) + (z-1)\mathscr{L}_{\mathbb{T}}\{f(t)\}(z),$$

which proves (10). □

Theorem 4. *Let $f : \mathbb{T}_0 \to \mathbb{C}$ be such that $f^{\Delta^{(n)}}$ is regulated for some $n \in \mathbb{N}$. Then*

$$\begin{aligned} &\mathscr{L}_{\mathbb{T}}\left\{f^{\Delta^n}(t)\right\}(z) \\ &= (z-1)^n \mathscr{L}_{\mathbb{T}}\{f(t)\}(z) - z\sum_{i=0}^{n-1}(z-1)^i f^{\Delta^{n-1-i}}(0) \end{aligned} \tag{14}$$

for those $z \in \mathbb{C}$ satisfying $\lim_{t\to\infty}\left\{f^{\Delta^{(i)}}(t)\, e_{\ominus(z-1)}(t,0)\right\} = 0$.

Proof. We will use mathematical induction.

(1) For $n = 1$ the assertion follows from Theorem 3.

(2) For $n = 2$, applying Definition 4 and integrating by parts (Theorem 1, (iv)) we find

$$\mathscr{L}_{\mathbb{T}}\left\{f^{\Delta\Delta}(t)\right\}(z) = \mathscr{L}_{\mathbb{T}}\left\{\left(f^{\Delta}(t)\right)^{\Delta}\right\}(z)$$

$$= \int_0^{\infty} z e_{\ominus(z-1)}(\sigma(t),0)\left(f^{\Delta}(t)\right)^{\Delta}\Delta t$$

$$= z\left\{ e_{\ominus(z-1)}(t,0) f^{\Delta}(t)\Big|_{t=0}^{t\to\infty} - \int_0^{\infty} e^{\Delta}_{\ominus(z-1)}(t,0) f^{\Delta}(t)\Delta t\right\}.$$

Letting t tend to infinity, taking into account that $e_{\ominus(z-1)}(0,0) = 1$ and assumption in Theorem 4, we have

$$\begin{aligned}
&\mathscr{L}_{\mathbb{T}}\left\{f^{\Delta\Delta}(t)\right\}(z)\\
&= \lim_{t\to\infty} e_{\ominus(z-1)}(t,0) f^{\Delta}(t) - f^{\Delta}(0)\\
&\quad + \int_0^{\infty} \frac{z-1}{1+\mu(t)(z-1)} e_{\ominus(z-1)}(t,0) f^{\Delta}(t)\Delta t\\
&= -z f^{\Delta}(0) + (z-1)\int_0^{\infty} z e_{\ominus(z-1)}(\sigma(t),0) f^{\Delta}(0)\Delta t\\
&= -z f^{\Delta}(0) + (z-1)\mathscr{L}\left\{f^{\Delta}(t)\right\}(z).
\end{aligned}$$

From Theorem 3, we obtain

$$\begin{aligned}
&\mathscr{L}_{\mathbb{T}}\left\{f^{\Delta\Delta}(t)\right\}(z)\\
&= -z f^{\Delta}(0) + (z-1)\left[-z f(0) + (z-1)\mathscr{L}_{\mathbb{T}}\{f(t)\}(z)\right]\\
&= -z f^{\Delta}(0) - z(z-1) f(0) + (z-1)^2\mathscr{L}\{f(t)\}(z).
\end{aligned}$$

(3) Assume that (14) is true for some $n \in \mathbb{N}$.

(4) We will prove that

$$\begin{aligned}
&\mathscr{L}_{\mathbb{T}}\left\{f^{\Delta^{n+1}}(t)\right\}(z)\\
&= (z-1)^{n+1}\mathscr{L}_{\mathbb{T}}\{f(t)\}(z) - z\sum_{i=0}^{n}(z-1)^i f^{\Delta^{n-i}}(0),
\end{aligned}$$

for those $z \in \mathbb{C}$ satisfying

$$\lim_{t\to\infty}\left\{f^{\Delta^{(l)}}(t)\, e_{\ominus(z-1)}(t,0)\right\} = 0, \quad 0 \le 1 \le n.$$

Applying (10), we obtain

$$\begin{aligned}
\mathscr{L}_{\mathbb{T}}\left\{f^{\Delta^{(n+1)}}(t)\right\}(z) &= \mathscr{L}_{\mathbb{T}}\left\{\left(f^{\Delta^{(n)}}(t)\right)^{\Delta}\right\}(z)\\
&= -z f^{\Delta^{(n)}}(0) + (z-1)\mathscr{L}_{\mathbb{T}}\left\{f^{\Delta^{(n)}}(t)\right\}(z).
\end{aligned}$$

Next, by the assumption (14), we get

$$\begin{aligned}
&\mathscr{L}_{\mathbb{T}}\left\{f^{\Delta^{(n+1)}}(t)\right\}(z)\\
&= -zf^{\Delta^{(n)}}(0) + (z-1)\Bigg[(z-1)^n \mathscr{L}_{\mathbb{T}}\{f(t)\}(z)\\
&\qquad\qquad - z\sum_{i=0}^{n-1}(z-1)^i f^{\Delta^{(n-1-i)}}(0)\Bigg].
\end{aligned}$$

Finally,

$$\begin{aligned}
&\mathscr{L}_{\mathbb{T}}\left\{f^{\Delta^{(n+1)}}(t)\right\}(z)\\
&= (z-1)^{n+1}\mathscr{L}_{\mathbb{T}}\{f(t)\}(z) - z\sum_{i=0}^{n}(z-1)^i f^{\Delta^{(n-i)}}(0),
\end{aligned}$$

which completes the proof. □

Theorem 5. *Let $f:\mathbb{T}_0\to\mathbb{C}$ be regulated. If $g(t)=\int_0^t f(\tau)\Delta\tau$ for $t\in\mathbb{T}_0$, then*

$$\mathscr{L}_{\mathbb{T}}\{g(t)\}(z) = \frac{z}{z-1}\mathscr{L}_{\mathbb{T}}\{f(t)\}(z) \tag{15}$$

for those regressive $z\in\mathbb{C}\setminus\{0\}$ satisfying $\lim_{t\to\infty}\left\{g(t)\,e_{\ominus(z-1)}(t,0)\right\}=0$.

Proof. First, using (ii) and (vii) from Lemma 2, we obtain

$$\begin{aligned}
\mathscr{L}_{\mathbb{T}}\{g(t)\}(z) &= \mathscr{L}_{\mathbb{T}}\left\{\int_0^t f(\tau)\Delta\tau\right\}(z) = \int_0^\infty zg(t)\,e_{\ominus(z-1)}(\sigma(t),0)\,\Delta t\\
&= -\frac{z}{(z-1)}\int_0^\infty g(t)\,\frac{-(z-1)}{1+\mu(t)(z-1)}e_{\ominus(z-1)}(t,0)\,\Delta t\\
&= -\frac{z}{(z-1)}\int_0^\infty g(t)\,e^{\Delta}_{\ominus(z-1)}(t,0)\,\Delta t.
\end{aligned}$$

Now, we use integration by parts (Theorem 1) and assumptions of Theorem 5, to obtain

$$\begin{aligned}
&\mathscr{L}_{\mathbb{T}}\left\{\int_0^t f(\tau)\Delta\tau\right\}(z)\\
&= -\frac{z}{z-1}\left[g(t)\,e_{\ominus(z-1)}(t,0)\Big|_{t=0}^{t\to\infty} - \int_0^\infty g^{\Delta}(t)\,e_{\ominus(z-1)}(\sigma(t),0)\,\Delta t\right]\\
&= -\frac{z}{z-1}\left[-g(0) - \int_0^\infty f(t)\,e_{\ominus(z-1)}(\sigma(t),0)\,\Delta t\right] = \frac{z}{z-1}\mathscr{L}_{\mathbb{T}}\{f(t)\}(z),
\end{aligned}$$

which completes the proof of equation (15). □

Theorem 6. *Assume $h_k(t,0)$, $k \in \mathbb{N}_0$, are defined as in* (3). *Then*

$$\mathscr{L}_{\mathbb{T}}\{h_k(t,0)\}(z) = \left(\frac{z}{z-1}\right)^{k+1} \tag{16}$$

for those regressive $z \in \mathbb{C}$ satisfying

$$\lim_{t\to\infty}\left\{h_k(t,0)\,e_{\ominus(z-1)}(t,0)\right\} = 0. \tag{17}$$

Proof. Fix $k \in \mathbb{N}$. We will prove by mathematical induction that

$$\mathscr{L}_{\mathbb{T}}\{h_i(t,0)\}(z) = \left(\frac{z}{z-1}\right)^{i+1} \tag{18}$$

for $0 \le i \le k$. First note that (17) implies $\lim_{t\to\infty}\left\{h_i(t,0)\,e_{\ominus(z-1)}(t,0)\right\} = 0$, for $0 \le i \le k$, and (18) holds for $i = 0$ by (9). Now assume that $1 \le i \le k$ and (18) holds while i replaced $i-1$. Then by (3) and by the principle of mathematical induction, we obtain

$$\begin{aligned}
\mathscr{L}_{\mathbb{T}}\{h_i(t,0)\}(z) &= \mathscr{L}_{\mathbb{T}}\left\{\int_0^t h_{i-1}(\tau,0)\,\Delta\tau\right\}(z)\\
&= \frac{z}{z-1}\mathscr{L}_{\mathbb{T}}\{h_{i-1}(t,0)\}(z)\\
&= \frac{z}{z-1}\mathscr{L}_{\mathbb{T}}\left\{\int_0^t h_{i-2}(\tau,0)\,\Delta\tau\right\}(z)\\
&= \left(\frac{z}{z-1}\right)^2\mathscr{L}_{\mathbb{T}}\{h_{i-2}(t,0)\}(z).
\end{aligned}$$

Continuing this process after i step, we get

$$\begin{aligned}
\mathscr{L}_{\mathbb{T}}\{h_i(t,0)\}(z) &= \left(\frac{z}{z-1}\right)^i\mathscr{L}_{\mathbb{T}}\{h_1(t,0)\}(z)\\
&= \left(\frac{z}{z-1}\right)^i\mathscr{L}_{\mathbb{T}}\left\{\int_0^t h_0(\tau,0)\,\Delta\tau\right\}(z)\\
&= \left(\frac{z}{z-1}\right)^i\mathscr{L}_{\mathbb{T}}\{h_0(t,0)\}(z)\\
&= \left(\frac{z}{z-1}\right)^i\mathscr{L}_{\mathbb{T}}\{1\}(z) = \left(\frac{z}{z-1}\right)^{i+1}.
\end{aligned}$$

Taking into account that $h_0(t,0) = 1$ and equation (9), the proof is completed. $\square$

3.2. Real-shifting and complex-shifting properties on time scales

In this subsection, we prove two theorems for real-shifting and complex-shifting properties on time scales.

Theorem 7 (Real-shifting property on time scales)**.** *Let $\alpha, \beta \in \mathbb{R}$, and $F_{\mathbb{T}} = \mathscr{L}_{\mathbb{T}} \{f_\beta (t)\} (z)$, whereby $f(t)$ is one of the functions $e(t,0)$, $\sin(t,0)$, $\cos(t,0)$, $\sinh(t,0)$, $\cosh(t,0)$. Assume that $1+\alpha\mu(t) \neq 0$, $1+\beta\mu(t) \neq 0$, for all $x \in \mathbb{T}_0$, then*

$$\mathscr{L}_{\mathbb{T}}\Big\{e_\alpha (t,0) f_{\frac{\beta}{1+\alpha\mu}} (t,0)\Big\} (z) = F_{\mathbb{T}} (z-\alpha), \tag{19}$$

provided

$$\lim_{t\to\infty} e_\alpha (t,a) \Big(f_{\frac{\beta}{1+\alpha\mu}} (t,0)\Big) = 0, \quad \lim_{t\to\infty} e_\alpha (t,a) \Big(f_{\frac{\beta}{1+\alpha\mu}} (t,0)\Big)^\Delta = 0.$$

Proof. Let $f(t) = e(t,0)$. The right side of equation (19) becomes

$$\begin{aligned}
F_{\mathbb{T}} (z-\alpha) &= \mathscr{L}_{\mathbb{T}} (e_\beta (t,0)) (z-\alpha) \\
&= \int_0^\infty z e_\beta (t,0)\, e_{\ominus(z-1-\alpha)} (\sigma(t),0) \,\Delta t \\
&= \frac{z}{\beta-(z-1-\alpha)} \int_0^\infty \frac{\beta-(z-1-\alpha)}{1+(z-1-\alpha)\mu(t)} e_{\beta\ominus(z-1-\alpha)} (t,0)\,\Delta t \\
&= \frac{z}{\beta-(z-1-\alpha)} \int_0^\infty e^\Delta_{\beta\ominus(z-1-\alpha)} (t,0)\,\Delta t \\
&= \frac{z}{\beta-(z-1-\alpha)} \Big[\lim_{t\to\infty} e_{\beta\ominus(z-1-\alpha)} (t,0) - e_{\beta\ominus(z-1-\alpha)} (0,0)\Big] \\
&= \frac{z}{(z-1-\alpha)-\beta}.
\end{aligned} \tag{20}$$

Now, let $p(t) = e_\alpha (t,0)\, e_{\frac{\beta}{1+\mu\alpha}} (t,0)$. Then we have

$$\begin{aligned}
p^\Delta (t) &= e^\Delta_\alpha (t,0)\, e_{\frac{\beta}{1+\mu\alpha}} (t,0) + e_\alpha (\sigma(t),0)\, e^\Delta_{\frac{\beta}{1+\mu\alpha}} (t,0) \\
&= (\alpha+\beta)\, e_\alpha (t,0)\, e_{\frac{\beta}{1+\mu\alpha}} (t,0) \\
&= (\alpha+\beta)\, p(t).
\end{aligned} \tag{21}$$

By determining the initial conditions we get

$$\begin{cases} p^\Delta (t) = (\alpha+\beta)\, p(t), \\ p(0) = 1. \end{cases} \tag{22}$$

By applying the Laplace transform to (22), we get

$$(z-1)\, \mathscr{L}_{\mathbb{T}} \{p\} (z) - z p(0) = (\alpha+\beta)\, \mathscr{L}_{\mathbb{T}} \{p\} (z).$$

Hence

$$\mathscr{L}_{\mathbb{T}} \{p\} (z) = \frac{z}{(z-1-\alpha)-\beta}. \tag{23}$$

From (20) and (23) we obtain equation (19).

To prove (19) for function $f(t) = \sin(t,0)$, we could use the relation (19) for function $e(t,0)$, but we will conduct the proof in a different way. Based on (23), the right hand side of (19) becomes

$$
\begin{aligned}
F_{\mathbb{T}}(z-\alpha) &= \mathscr{L}_{\mathbb{T}}\left\{\sin_{\beta}(t,0)\right\}(z) \\
&= \mathscr{L}_{\mathbb{T}}\left\{\frac{e_{i\beta}(t,0) - e_{-i\beta}(t,0)}{2i}\right\}(z) \\
&= \frac{1}{2i}\Big[\mathscr{L}_{\mathbb{T}}\left\{e_{i\beta}(t,0)\right\}(z) - \mathscr{L}_{\mathbb{T}}\left\{e_{-i\beta}(t,0)\right\}(z)\Big] \\
&= \frac{z\beta}{(z-1-\alpha)^2 + \beta^2}.
\end{aligned} \tag{24}
$$

Denote $p(t) = e_{\alpha}(t,0)\sin_{\frac{\beta}{1+\mu\alpha}}(t,0)$. We determine Δ derivative of a function $p(t)$

$$
\begin{aligned}
p^{\Delta}(t) &= e_{\alpha}^{\Delta}(t,0)\sin_{\frac{\beta}{1+\mu\alpha}}(t,0) + e_{\alpha}(\sigma(t),0)\sin^{\Delta}_{\frac{\beta}{1+\mu\alpha}}(t,0) \\
&= \alpha e_{\alpha}(t,0)\sin_{\frac{\beta}{1+\mu\alpha}}(t,0) + \beta e_{\alpha}(t,0)\cos_{\frac{\beta}{1+\mu\alpha}}(t,0).
\end{aligned}
$$

We determine $\Delta\Delta$ derivative of a function $p(t)$

$$
\begin{aligned}
p^{\Delta\Delta}(t) = {}& \alpha e_{\alpha}^{\Delta}(t,0)\sin_{\frac{\beta}{1+\mu\alpha}}(t,0) + \alpha e_{\alpha}(\sigma(t),0)\sin^{\Delta}_{\frac{\beta}{1+\mu\alpha}}(t,0) \\
&+ \beta e_{\alpha}^{\Delta}(t,0)\cos_{\frac{\beta}{1+\mu\alpha}}(t,0) + \beta e_{\alpha}(\sigma(t),0)\cos^{\Delta}_{\frac{\beta}{1+\mu\alpha}}(t,0).
\end{aligned}
$$

By elementary calculations in last equation, we get

$$
\begin{aligned}
p^{\Delta\Delta}(t) &= \alpha^2 e_{\alpha}(t,0)\sin_{\frac{\beta}{1+\mu\alpha}}(t,0) + 2\alpha\beta e_{\alpha}(t,0)\cos_{\frac{\beta}{1+\mu\alpha}}(t,0) \\
&\quad - \beta^2 e_{\alpha}(t,0)\sin_{\frac{\beta}{1+\mu\alpha}}(t,0) \\
&= \left(\alpha^2 - \beta^2\right)p(t) + 2\alpha\beta e_{\alpha}(t,0)\cos_{\frac{\beta}{1+\mu\alpha}}(t,0).
\end{aligned}
$$

We form a system of equations

$$
\begin{cases}
p^{\Delta}(t) = \alpha e_{\alpha}(t,0)\sin_{\frac{\beta}{1+\mu\alpha}}(t,0) + \beta e_{\alpha}(t,0)\cos_{\frac{\beta}{1+\mu\alpha}}(t,0), \\
p^{\Delta\Delta}(t) = \left(\alpha^2 - \beta^2\right)p(t) + 2\alpha\beta e_{\alpha}(t,0)\cos_{\frac{\beta}{1+\mu\alpha}}(t,0),
\end{cases} \tag{25}
$$

and determine the initial conditions

$$
\begin{aligned}
p(0) &= e_{\alpha}(0,0)\sin_{\frac{\beta}{1+\mu\alpha}}(0,0) = 0, \\
p^{\Delta}(0) &= \alpha e_{\alpha}(0,0)\sin_{\frac{\beta}{1+\mu\alpha}}(0,0) + \beta e_{\alpha}(0,0)\cos_{\frac{\beta}{1+\mu\alpha}}(0,0) = \beta.
\end{aligned}
$$

From the system of equations (25) and above initial conditions we get the following Boundary Value Problem (BVP)

$$
\begin{cases}
p^{\Delta\Delta}(t) - 2\alpha p^{\Delta}(t) + \left(\alpha^2 + \beta^2\right)p(t) = 0, \\
p(0) = 0, \quad p^{\Delta}(0) = \beta.
\end{cases} \tag{26}
$$

By applying the Laplace transform to (26), it follows equation

$$\begin{aligned}&- zp^{\Delta}(0) - z(z-1)p(0) + (z-1)^2 \mathscr{L}_{\mathbb{T}}\{p(t)\}(z) \\ &- 2\alpha\left[-zp(0) + (z-1)\mathscr{L}_{\mathbb{T}}\{p(t)\}(z)\right] + \left(\alpha^2+\beta^2\right)\mathscr{L}_{\mathbb{T}}\{p(t)\}(z) = 0,\end{aligned}$$

i.e.,

$$\left[(z-1)^2 - 2\alpha(z-1) + \alpha^2 + \beta^2\right]\mathscr{L}_{\mathbb{T}}\{p(t)\}(z) = z\beta.$$

We determine $\mathscr{L}_{\mathbb{T}}\{p(t)\}(z)$ from there

$$\mathscr{L}_{\mathbb{T}}\{p(t)\}(z) = \frac{z\beta}{(z-1-\alpha)^2+\beta^2}. \tag{27}$$

From (26) and (27) follows (19).

For functions $f(t) = \cos(t,0)$, $f(t) = \sinh(t,0)$ and $f(t) = \cosh(t,0)$ the proof is analoguous. □

Theorem 8 (Complex-shifting property on time scales)**.** *Let $a \in \mathbb{T}_0$, $a > 0$. Assume that $f_\alpha(t,0)$ is one of the functions $e(t,0)$, $\sin(t,0)$, $\cos(t,0)$, $\sinh(t,0)$, $\cosh(t,0)$. If $1 + z\mu(t) \neq 0$, $1 + \alpha\mu(t) \neq 0$, for all $x \in \mathbb{T}_0$, then*

$$\mathscr{L}_{\mathbb{T}}\{u_a(t)f(t,a)\}(z) = e_{\ominus(z-1)}(a,0)\mathscr{L}_{\mathbb{T}}\{f(t,0)\}(z), \tag{28}$$

provided

$$\lim_{t\to\infty} e_{\alpha\ominus(z-1)}(t,a) = \lim_{t\to\infty} e_{i\alpha\ominus(z-1)}(t,a) = \lim_{t\to\infty} e_{-i\alpha\ominus(z-1)}(t,a) = 0.$$

Proof. Let $f(t) = e(t,0)$. Then,

$$\begin{aligned}&e_{\ominus(z-1)}(a,0)\mathscr{L}_{\mathbb{T}}\{e_\alpha(t,0)\}(z) \\ &= e_{\ominus(z-1)}(a,0)\int_0^\infty z e_\alpha(t,0)\, e_{\ominus(z-1)}(\sigma(t),0)\,\Delta t \\ &= \frac{z}{\alpha-(z-1)} e_{\ominus(z-1)}(a,0)\int_0^\infty e^{\Delta}_{\alpha\ominus(z-1)}(t,0)\,\Delta t \\ &= \frac{z}{\alpha-(z-1)} e_{\ominus(z-1)}(a,0)\; e_{\alpha\ominus(z-1)}(t,0)\Big|_{t=0}^{t\to\infty} \\ &= \frac{z}{(z-1)-\alpha} e_{\ominus(z-1)}(a,0).\end{aligned}$$

From this follows (28).

We have the proof for functions

$$\begin{aligned}f(t) &= \sin(t,0), \quad f(t) = \sinh(t,0), \\ f(t) &= \cos(t,0), \quad f(t) = \cosh(t,0),\end{aligned}$$

by the same method as in Theorem 7. □

3. Conclusion

We define Laplace transform in time scales, in such a way that by taking $\mathbb{T} = \mathbb{Z}$ we get the classical $\mathscr{Z}$-transform, and by taking $\mathbb{T} = \mathbb{R}$ we get

$$F(z) = \int_0^\infty z e^{-(z-1)t} f(t)\, dt = z\mathscr{L}\{f(t)\}(z-1).$$

The main result of our paper is derivation of formulas for real and complex shifting properties of Laplace transform on time scales of elementary functions. The motivation for introducing alternative definition of Laplace transform is to obtain formulas for shift properties that are simpler and more convenient for applications.

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