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G-approximate best proximity pairs in metric space with a directed graph

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Abstract. Let (X, d) be a metric space endowed with a directed graph G where $V(G)$ and $E(G)$ represent the sets of vertices and edges corresponding to X, respectively. We establish sufficient conditions for the existence of a G-approximate best proximity pair for a mapping T in the metric space X equipped with the graph G such that the set $V(G)$ of vertices of G coincides with X.

1. Introduction

Fixed point theory is a powerful tool for addressing existence problems in various branches of mathematical analysis and its applications. In physics and engineering, this technique has been employed in areas such as image retrieval, signal processing, and the study of nonlinear integral equations. Graphs serve as models for relations and processes in diverse fields, including physical systems, biochemistry, electrical engineering, computer science, operations research, and biological systems. Their applications are well-documented in the literature [5, 12].

Let X be a metric space with nonempty subsets A and B. The distance between A and B is denoted by $d(A, B)$. A pair (x_0, y_0) satisfying $d(x_0, y_0) = d(A, B)$ is called a best proximity pair for A and B. The set of all such pairs is defined as:

$$prox(A, B) := \{(x, y) \in A \times B : \ d(x, y) = d(A, B)\}.$$

This concept generalizes the idea of best approximation, with key results found in [6, 9, 10].

Following [13] (see also[1, 2, 18–21]), best proximity points for sets A and B can be identified by considering a mapping $T : A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$. Notably, if $A \cap B \neq \emptyset$, every best proximity point becomes a fixed point of T.

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In 2012, Mohsenalhosseini et al.[11] introduced fixed point theory for completely normed spaces and the map T_α. Recent developments provide sufficient conditions for a mapping to be Picard when (X, d) is endowed with a graph. Jachymski [8] pioneered this direction, applying it to the Kelisky-Rivlin theorem on iterates of Bernstein maps in $C[0, 1]$.

Reich [16], Ćirić [17] and Rus [15] demonstrated that in a complete metric space (X, d), every Ćirić -Reich-Rus mapping has a unique fixed point. Bojor [3] explored fixed points for φ-contractions in metric spaces with graphs, while [4] extended this to Reich-type contractions.

The concept of G-approximation best proximity points arises in nonlinear analysis, optimization, and fixed-point theory. These points generalize classical best proximity points by incorporating a graph structure G that constrains the allowable pairs (x, Tx), making them useful in structured or constrained settings. This advancement makes them valuable in applications such as:

- Proximal algorithms for nonsmooth optimization,
- Structured machine learning (e.g., constrained regression),
- Modeling Nash equilibria where strategies must lie in disjoint sets,
- Variational inequalities and optimization (e.g., split feasibility problems in medical imaging and signal processing),
- Their ability to provide approximate solutions when exact fixed points do not exist ensures broad applicability in both theoretical and applied mathematics.

This paper investigates the existence of approximate best proximity pairs for cyclic mappings $T : A \cup B \cup C \to A \cup B \cup C$, where $T(A) \subseteq B$, $T(B) \subseteq C$, and $T(C) \subseteq A$, in metric spaces endowed with a graph G. We define G-approximate best proximity pairs and provide illustrative examples to support our results.

2. Preliminaries

We adopt the following notations and refer to [7] for graph distance theory and [11] for approximate best proximity pairs.
Graphs, as metric spaces with intrinsic path metrics, exhibit relationships between distance, diameter, and radius. A path in a graph is a sequence of distinct vertices where adjacent vertices in the sequence are connected by edges. For unweighted graphs, path length is the edge count; for weighted graphs, it is the sum of edge weights (assumed nonnegative). We focus on connected, undirected graphs.

Definition 1. Let u and v be two vertices of graph G. The *distance* between two vertices u and v, denoted $d_G(u, v)$, is the length of a shortest $u - v$ path, also called a $u - v$ geodesic.

Remark 1. If there is no path between two vertices u and v, the distance between them is infinite ($d_G(u, v) := \infty$).

The distance function is a metric on the vertex set of a (weighted) graph G. In particular, it satisfies the triangle inequality:

$$d_G(a, b) \leq d_G(a, c) + d_G(c, b)$$

for all vertices a, b, c of G.

Three of the most commonly observed parameters of a graph are its *eccentricity*, *radius* and *diameter*.

Definition 2. Let u and v, two vertices of graph G.

(i) The *eccentricity* of a vertex is the maximum distance from it to any other vertex

$$e(u) = \max\{d_G(u, v) : v \in V(G)\}.$$

(ii) The *diameter* of a connected graph G, denoted $diam(G)$, is the maximum *eccentricity*

$$diam(G) = \max\{e(u) : u \in V(G)\}.$$

(iii) The *radius*, denoted $rad(G)$, is the minimum eccentricity among all vertices of G

$$r(G) = \min\{e(u) : u \in V(G)\}.$$

Remark 2. The best way to calculate the diameter and radius of the graph is to use the $n \times n$ square matrix, where n is the order of the graph and the (i, j)-th entry of this matrix is the distance of vertex v_i from v_j.

Example 1. Find the diameter and radius of the graph in Figure 1.

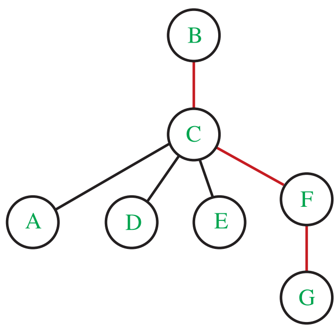

Figure 1.

Considering the graph G in Figure 1 with $B = v_1$, $C = v_2$, $A = v_3$, $D = v_4$, $E = v_5$, $F = v_6$, $G = v_7$, we have:

$$(d_G(v_i, v_j))_{i,j} = \begin{pmatrix} 0 & 1 & 2 & 2 & 2 & 2 & 3 \\ 1 & 0 & 1 & 1 & 1 & 1 & 2 \\ 2 & 1 & 0 & 2 & 2 & 2 & 1 \\ 2 & 1 & 2 & 0 & 2 & 2 & 3 \\ 2 & 1 & 2 & 2 & 0 & 2 & 3 \\ 2 & 1 & 2 & 2 & 2 & 0 & 3 \\ 3 & 2 & 1 & 3 & 3 & 3 & 0 \end{pmatrix}$$

Therefore, $e(v_1) = 3$, $e(v_2) = 2$, $e(v_3) = 2$, $e(v_4) = 3$, $e(v_5) = 3$, $e(v_6) = 3$, $e(v_7) = 3$.

Diameter is 3 because

$$diam(G) = \max\{e(u) : u \in V(G)\} = 3$$

and radius is 2 because

$$r(G) = \min\{e(u) : u \in V(G)\} = 2.$$

Definition 3 ([11])**.** Let $T : X \to X$, $\epsilon > 0$, $x_0 \in X$. Then $x_0 \in X$ is an $\epsilon-$fixed point for T if $d(x_0, Tx_0) < \epsilon$.

Definition 4 ([11])**.** Let $T : X \to X$. Then T has the approximate fixed point property (a.f.p.p) if

$$\forall \epsilon > 0, \ F_\epsilon(T) \neq \varnothing.$$

Theorem 1 ([11])**.** *Let $(X, \|.\|)$ be a complete norm space, $T : X \to X$, $x_0 \in X$ and $\epsilon > 0$. If $\|T^n(x_0) - T^{n+k}(x_0)\| \to 0$ as $n \to \infty$ for some $k > 0$, then T^k has an $\epsilon-$ fixed point.*

Definition 5 ([8])**.** We say that a mapping $T : X \to X$ is a G-contraction or simply G-contraction if T preserves edges of G, i.e.,

$$\forall x, y \in X, \quad \big((x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)\big), \tag{1}$$

and T decreases weights of edges of G in the following way:

$$\exists \alpha \in (0, 1)\ \forall x, y \in X, \quad \big((x, y) \in E(G) \Rightarrow d(Tx, Ty) \leq \alpha d(x, y)\big). \tag{2}$$

3. Main result

Throughout this section, G is a directed graph with $V(G) = X$, $E(G) \supseteq \Delta$, where Δ is the diagonal of $X \times X$, and no parallel edges. Building on Jachymski's framework [8], we examine the existence of G-approximate best proximity points for mappings $T : A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$, along with their diameter properties.

Definition 6. Let A and B be nonempty subsets of a metric space endowed with a directed graph $G = (V(G), E(G))$ such that $V(G) = A \cup B$. The map $T : A \cup B \to A \cup B$ be a map such that $T(A) \subseteq B$, $T(B) \subseteq A$. The point $x \in A \cup B$ is said to be a G-approximate best proximity point of the pair (A, B), if:

(i) $\big((x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)\big), \quad \forall x, y \in A \cup B$;
(ii) $\exists \epsilon > 0 \ \forall x \in A \cup B, \big((x, Tx) \in E(G) \Rightarrow d(x, Tx) \leq d(A, B) + \epsilon\big)$.

Remark 3. In this paper we will denote the set of all G-approximate best proximity point of the pair (A, B) of T, for a given ϵ, by:

$$P_T^{Ga}(A, B) = \big\{x \in A \cup B : ((x, Tx) \in E(G) \Rightarrow d(x, Tx) \leq d(A, B) + \epsilon)\big\}.$$

Example 2. Suppose Let $X = \mathbf{R}^2$ and

$$A = \{(x, y) \in X : (x - y)^2 + y^2 \leq 1\},$$
$$B = \{(x, y) \in X : (x + y)^2 + y^2 \leq 1\},$$

with $T(x, y) = (-x, y)$ for $(x, y) \in X$.
Define the graph G by $V(G) = X$. Then, there exists $((x, y), T(x, y)) \in E(G)$ such that $d((x, y), T(x, y)) \leq d(A, B) + \epsilon$ for some $\epsilon > 0$. Hence, $P_T^{Ga}(A, B) \neq \emptyset$.

We say that the pair (A, B) is a G-approximate best proximity pair if $P_T^{Ga}(A, B) \neq \emptyset$.

Proposition 1. *Let A and B be nonempty subsets of a metric space endowed with a directed graph $G = (V(G), E(G))$ such that $V(G) = A \cup B$. Suppose that the mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$. If $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = d(A, B)$, for some $x \in A \cup B$ satisfies the condition $(x, Tx) \in E(G)$ then the pair (A, B) is a G-approximate best proximity pair.*

Proof. Let $\epsilon > 0$ be given and $x \in A \cup B$ with $(x, Tx) \in E(G)$ such that $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = d(A, B)$; then there exists $N_0 > 0$ such that for all $n \geq N_0$,

$$d(T^n x, T^{n+1}x) < d(A, B) + \epsilon.$$

If $n = N_0$, then $d(T^{N_0}(x), T(T^{N_0}(x))) < d(A, B) + \epsilon$, then $T^{N_0}(x) \in P_T^{Ga}(A, B)$ and $P_T^{Ga}(A, B) \neq \emptyset$. $\square$

Definition 7. Let A and B be nonempty subsets of a metric space endowed with a graph G. The map $T : A \cup B \to A \cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ is said to be a G-Ćirić-Rich-Rus-Moh map if:

(i) $((x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)), \forall x, y \in A \cup B$;
(ii) There exists nonnegative numbers α, β, γ with $\alpha + 2\beta + \gamma < 1$, such that, for each $(x, y) \in E(G)$, we have:

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma d(A, B).$$

Theorem 2. *Let A and B be nonempty subsets of a metric space endowed with a graph G and $T : A \cup B \to A \cup B$ be a G-Ćirić-Rich-Rus-Moh map. If $x \in A \cup B$ satisfies the condition $(x, Tx) \in E(G)$, then the pair (A, B) is a G-approximate best proximity pair.*

Proof. Let $x \in A \cup B$ with $(x, Tx) \in E(G)$, then

$$d(Tx, T^2x) \leq \alpha d(x, Tx) + \beta[d(x, Tx) + d(Tx, T^2x)] + \gamma d(A, B).$$

Therefore

$$d(Tx, T^2x) \leq \frac{\alpha + \beta}{1 - \beta} d(x, Tx) + \frac{\gamma}{1 - \beta} d(A, B).$$

Now, if $k = \frac{\alpha+\beta}{1-\beta}$, then

$$d(Tx, T^2x) \leq kd(x, Tx) + (1 - k)d(A, B),$$

also

$$d(T^2x, T^3x) \leq k^2 d(x, Tx) + (1 - k^2)d(A, B).$$

Therefore

$$d(T^n x, T^{n+1}x) \leq k^n d(x, Tx) + (1 - k^n)d(A, B),$$

and so

$$d(T^n x, T^{n+1}x) \to d(A, B), \text{ as } n \to \infty.$$

Therefore, by Proposition 1, $P_T^{Ga}(A, B) \neq \emptyset$, then pair (A, B) is a G-approximate best proximity pair. □

Definition 8. Let A and B be nonempty subsets of a metric space endowed with a graph G. Suppose that the mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$. We say that the sequence $\{z_n\} \subseteq A \cup B$ with $(z_n, Tz_n) \in E(G)$ is GT-minimizing if

$$\lim_{n\to\infty} d(z_n, Tz_n) = d(A, B).$$

Theorem 3. *Let A and B be nonempty subsets of a metric space endowed with a graph G, suppose that the mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$. If $\{T^n x\}$ is a GT-minimizing for some $x \in A \cup B$ satisfies the condition $(x, Tx) \in E(G)$, then (A, B) is a G-approximate best pair proximity.*

Proof. Since

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = d(A, B)$$

for some $x \in A \cup B$ satisfies the condition $(x, Tx) \in E(G)$, then by Proposition 1 $P_T^{Ga}(A, B) \neq \emptyset$. Therefore, pair (A, B) is a G-approximate best proximity pair. □

Theorem 4. *Let A and B be nonempty subsets of a metric space endowed with a graph G such that $E(G)$ is compact. Suppose that the mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$, T is continuous and $\|Tx - Ty\| \leq \|x - y\|$, where $(x, y) \in E(G)$. Then $P_T^{Ga}(A, B)$ is nonempty and compact.*

Proof. Since $E(G)$ compact, there exists a $z_0 \in E(G)$ such that

$$\|z_0 - Tz_0\| = \inf_{z \in E(G)} \|z - Tz\|. \tag{3}$$

If $\|z_0 - Tz_0\| > d(A, B)$, then $\|Tz_0 - T^2z_0\| < \|z_0 - Tz_0\|$, which is contradiction to the definition of z_0 ($Tz_0 \in E(G)$) and (3). Therefore $\|z_0 - Tz_0\| = d(A, B) \leq d(A, B) + \epsilon$ for some $\epsilon > 0$ and $z_0 \in P_T^{Ga}(A, B)$. Therefore $P_T^{Ga}(A, B)$ is nonempty.

Also, if $\{z_n\} \subseteq P_T^{Ga}(A, B)$, then $\|z_n - Tz_n\| < d(A, B) + \epsilon$ for some $\epsilon > 0$, and by compactness of $E(G)$, there exists a subsequence z_{n_k} and a $z_0 \in E(G)$ such that $z_{n_k} \to z_0$ and so

$$\|z_0 - Tz_0\| = \lim_{k \to \infty} \|z_{n_k} - Tz_{n_k}\| < d(A, B) + \epsilon$$

for some $\epsilon > 0$ hence $P_T^{Ga}(A, B)$ is compact. □

Example 3. If $A = [-3, -1], B = [1, 3]$ and $T : A \cup B \to A \cup B$, such that

$$T(x) = \begin{cases} \frac{1-x}{2} & x \in A \\ \frac{-1-x}{2} & x \in B \end{cases}$$

Then $P_T^{Ga}(A, B)$ is compact, we have

$$\begin{aligned} P_T^{Ga}(A, B) &= \{x \in A \cup B : \ d(x, Tx) < d(A, B) + \epsilon \quad \text{for some } \epsilon > 0\} \\ &= \{x \in A \cup B : \ d(x, Tx) < 2 + \epsilon \quad \text{for some } \epsilon > 0\} \\ &= \{1, -1\}, \end{aligned}$$

that is compact.

In the following, by $diam(P_T^{Ga}(A, B))$ for a set $P_T^{Ga}(A, B) \neq \emptyset$, we will understand the diameter of the set $P_T^{Ga}(A, B)$.

Definition 9. Let A and B be nonempty subsets of a metric space endowed with a graph G such that $T(A) \subseteq B$, $T(B) \subseteq A$ and $\epsilon > 0$. We define diameter of $P_T^{Ga}(A, B)$ by

$$diam(P_T^{Ga}(A, B)) = \sup\{d(x, y) : \ x, y \in P_T^{Ga}(A, B)\}. \tag{4}$$

Theorem 5. *Let A and B be nonempty subsets of a metric space endowed with a graph G. Suppose that the mapping $T : A \cup B \to A \cup B$, such that $T(A) \subseteq B$, $T(B) \subseteq A$ and $\epsilon > 0$. If T be a G-contraction then*

$$diam(P_T^{Ga}(A, B)) \leq \frac{2\epsilon}{1 - \alpha} + \frac{2d(A, B)}{1 - \alpha}.$$

Proof. If $x, y \in P_T^{Ga}(A, B)$, then

$$\begin{aligned} d(x, y) &\leq d(x, Tx) + d(Tx, Ty) + d(Ty, y) \\ &\leq \epsilon_1 + \alpha d(x, y) + 2d(A, B) + \epsilon_2. \end{aligned}$$

Put $\epsilon = \max\{\epsilon_1, \epsilon_2\}$, therefore

$$d(x, y) \leq \frac{2\epsilon}{1-\alpha} + \frac{2d(A, B)}{1-\alpha}.$$

Hence,

$$diam(P_T^{Ga}(A, B)) \leq \frac{2\epsilon}{1-\alpha} + \frac{2d(A, B)}{1-\alpha}. \qquad \square$$

4. G-APPROXIMATE BEST PROXIMITY FOR TWO MAPS

In this section we will consider the existence of G-approximate best proximity points for two maps $T : A \cup B \to A \cup B$, $S : A \cup B \to A \cup B$, and its diameter.

Definition 10. Let A and B be nonempty subsets of a metric space endowed with a graph G and $T : A \cup B \to A \cup B$, $S : A \cup B \to A \cup B$ be two maps such that $T(A) \subseteq B$, $S(B) \subseteq A$. A point (x, y) in $A \times B$ is said to be a G-approximate-pair fixed point for (T, S), if:

(i) $\forall (x, y) \in A \times B, \quad \big((x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G), (Sx, Sy) \in E(G)\big)$;

(ii) $\exists \alpha \in (0, 1)\ \forall (x, y) \in A \times B, \quad \big((x, y) \in E(G) \Rightarrow d(Tx, Sy) \leq d(A, B) + \epsilon\big)$.

We say that the pair (T, S) has the G-approximate-pair fixed property in X if $P_{(T,S)}^{Ga}(A, B) \neq \emptyset$, where

$$P_{(T,S)}^{Ga}(A, B) = \{(x, y) \in E(G) : \ d(Tx, Sy) \leq d(A, B) + \epsilon \quad \text{for some } \epsilon > 0\}.$$

Theorem 6. *Let A and B be nonempty subsets of a metric space endowed with a graph G and $T : A \cup B \to A \cup B$, $S : A \cup B \to A \cup B$ be two maps such that $T(A) \subseteq B$, $S(B) \subseteq A$. If, for every $(x, y) \in A \times B$,*

$$d(T^n(x), S^n(y)) \to d(A, B),$$

then (T, S) has the G-approximate-pair fixed property.

Proof. Let $\epsilon > 0$ be given and $(x, y) \in A \times B$ with $((x, y), (Tx, Ty)) \in E(G)$. Since $d(T^n(x), S^n(y)) \to d(A, B)$, there exist $n_0 > 0$ such that for all $n \geq n_0$,

$$d(T^n(x), S^n(y)) < d(A, B) + \epsilon.$$

Then $d(T(T^{n-1}(x), S(S^{n-1}(y)) < d(A, B) + \epsilon$ for every $n \geq n_0$. Put $x_0 = T^{n_0-1}(x)$ and $y_0 = S^{n_0-1}(y))$. Hence, $d(T(x_0), S(y_0)) \leq d(A, B) + \epsilon$ and $P_{(T,S)}^{Ga}(A, B) \neq \emptyset$. $\square$

Definition 11. Let A and B be nonempty subsets of a metric space endowed with a graph G. The map $T : A \cup B \to A \cup B$, $S : A \cup B \to A \cup B$ satisfying $T(A) \subseteq B$, $S(B) \subseteq A$ is said to be a G-Ćirić-Rich-Rus map if:

(i) $\forall (x,y) \in A \times B \ \big((x,y) \in E(G) \Rightarrow (Tx,Ty) \in E(G), (Sx,Sy) \in E(G)\big)$;

(ii) There exist nonnegative numbers α, β, γ with $\alpha + 2\beta + \gamma < 1$, such that for each $(x,y) \in E(G)$ we have:

$$d(Tx,Sy) \leq \alpha d(x,y) + \beta[d(x,Tx) + d(y,Sy)] + \gamma d(A,B).$$

Theorem 7. *Let A and B be nonempty subsets of a metric space endowed with a graph G and $T : A \cup B \to A \cup B$, $S : A \cup B \to A \cup B$ are a G-Ćirić-Rich-Rus map. If $x, y \in A \cup B$ satisfies the condition $(x,Tx),(x,Sx) \in E(G)$, $(y,Ty),(y,Sy) \in E(G)$ and if x is a G-approximate fixed point for T, or y is a G-approximate fixed point for S, then $P^{Ga}_{(T,S)}(A,B) \neq \emptyset$.*

Proof. Let $x, y \in A \cup B$ with $(x,Tx),(x,Sx) \in E(G)$, $(y,Ty),(y,Sy) \in E(G)$, then

$$d(Tx,S(Tx)) \leq \alpha d(x,Tx) + \beta[d(x,Tx) + d(Tx,S(Tx)] + \gamma d(A,B).$$

Therefore

$$d(Tx,S(Tx)) \leq \frac{\alpha+\beta}{1-\beta} d(x,Tx) + \frac{\gamma}{1-\beta} d(A,B).$$

Now, if $k = \frac{\alpha+\beta}{1-\beta}$, then

$$d(Tx,S(Tx)) \leq kd(x,Tx) + (1-k)d(A,B), \tag{5}$$

$$d(Sy,T(Sy)) \leq kd(y,Sy) + (1-k)d(A,B). \tag{6}$$

If x is a G-approximate fixed point for T, then there exists a $\epsilon > 0$ and by 5

$$\begin{aligned} d(Tx,S(Tx)) &\leq kd(x,Tx) + (1-k)d(A,B) \\ &\leq k(d(A,B)+\epsilon) + (1-k)d(A,B) \\ &= d(A,B) + k\epsilon \\ &< d(A,B) + \epsilon \end{aligned}$$

and $(x,Tx) \in P^{Ga}_{(T,S)}(A,B)$, also if y is a G-approximate fixed point for S, then there exists a $\epsilon > 0$ and by 6

$$\begin{aligned} d(Sy,T(Sy)) &\leq kd(y,Sy) + (1-k)d(A,B) \\ &\leq k(d(A,B)+\epsilon) + (1-k)d(A,B) \\ &= d(A,B) + k\epsilon \\ &< d(A,B) + \epsilon \end{aligned}$$

and $(y,Sy) \in P^{Ga}_{(T,S)}(A,B)$.

Therefore, $P^{Ga}_{(T,S)}(A,B) \neq \emptyset$. □

Theorem 8. *Let A and B be nonempty subsets of a metric space endowed with a graph G and $T : A\cup B \to A\cup B$, $S : A\cup B \to A\cup B$ be two continuous maps such that $T(A) \subseteq B$, $S(B) \subseteq A$. We suppose that:*

(i) *For every $(x, y) \in E(G)$, $d(Tx, Sy) \leq \alpha d(x, y) + \gamma d(A, B)$, where $\alpha, \gamma \geq 0$ and $\alpha + \gamma = 1$;*

(ii) *For any $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ in $A \cup B$ as flowing:*

$$x_{n+1} = Sy_n,\ y_{n+1} = Tx_n \quad \textit{for some } (x_1, y_1) \in E(G) \textit{ for } n \in N.$$

(iii) *For any $\{x_n\}_{n\in N}$ in $A \cup B$, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in N$ then there is a subsequence convergent $\{x_{n_k}\}_{k\geq 1}$ with $(x_{n_k}, x) \in E(G)$ for $n \in N$.*

Then there exists $x \in A$ such that $d(x, Tx) = d(A, B)$.

Proof. We have

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &= d(Tx_n, Sy_n) \\ &\leq \alpha d(x_n, y_n) + \gamma(d(A, B) \\ &\leq \cdots \\ &\leq \alpha^{n+1} d(x_0, y_0) + (1 + \alpha + \cdots + \alpha^n)\gamma d(A, B). \end{aligned}$$

If $\{x_{n_k}\}_{k\geq 1}$ is converge to $x_1 \in A$, that is $x_{n_k} \to x_1$ with $(x_{n_k}, x_1) \in E(G)$ for all $n \in N$. Then

$$d(x_{n_{K+1}}, y_{n_{k+1}}) \leq \alpha^{n_k+1} d(x_0, y_0) + (1 + \alpha + \cdots + \alpha^{n_k})\gamma d(A, B).$$

Since T is continuous, then

$$d(x_{n_{k+1}}, Tx_{n_k}) \to \frac{\gamma}{1-\alpha} d(A, B) = d(A, B).$$

Therefore, $d(x_1, Tx_1) = d(A, B)$. □

Definition 12. Let A and B be nonempty subsets of a metric space endowed with a graph G and let $T : A\cup B \to A\cup B$, $S : A\cup B \to A\cup B$ be continues maps such that $T(A) \subseteq B$, $S(B) \subseteq A$. We define diameter $P^{Ga}_{(T,S)}(A, B)$ by

$$diam(P^{Ga}_{(T,S)}(A, B)) = \sup\{d(x, y) :\ d(Tx, Ty) \leq \epsilon + d(A, B) \ \text{ for some } \epsilon > 0\}.$$

Example 4. Suppose $A = \{(x, 0) :\ 0 \leq x \leq 1\}$, $B = \{(x, 1) :\ 0 \leq x \leq 1\}$, $T(x, 0) = T(x, 1) = (\frac{1}{2}, 1)$ and $S(x, 1) = S(x, 0) = (\frac{1}{2}, 0)$. Then $d(T(x, 0), S(y, 1)) = 1$ and $diam(P^{Ga}_{(T,S)}(A, B)) < diam(A \times B) = \sqrt{2}$.

(i) $d(T(x, 0), S(y, 1)) = 1$ for all $x, y \in [0, 1]$, since T maps all points to $\left(\frac{1}{2}, 1\right)$ and S maps all points to $\left(\frac{1}{2}, 0\right)$, and the distance between these two points is 1.

(ii) $\text{diam}(A\times B) = \sqrt{2}$, since the maximum distance between two points in $A \times B$ under the product metric is

$$d((0,0),(1,1)) = \sqrt{(0-1)^2 + (0-1)^2} = \sqrt{2}.$$

(iii) $\text{diam}(P^{Ga}_{(T,S)}(A,B)) = 0$ because $P^{Ga}_{(T,S)}(A,B)$ is a singleton set.

Now by (ii) and (iii) it follows that $\text{diam}(P^{Ga}_{(T,S)}(A,B)) = 0$ is less than $\sqrt{2}$.

Theorem 9. *Let A and B be nonempty subsets of a metric space endowed with a graph G and let $T : A\cup B \to A\cup B$, $S : A\cup B \to A\cup B$ be continues maps such that $T(A) \subseteq B$, $S(B) \subseteq A$. If there exists a $k \in [0,1]$,*

$$d(x,Tx) + d(Sy,y) \leq kd(x,y).$$

Then

$$diam(P^{Ga}_{(T,S)}(A,B)) \leq \frac{\epsilon}{1-k} + \frac{d(A,B)}{1-k} \quad \textit{for some } \epsilon > 0.$$

Proof. If $(x,y) \in P^{Ga}_{(T,S)}(A,B)$, then

$$\begin{aligned} d(x,y) &\leq d(x,Tx) + d(Tx,Sy) + d(Sy,y) \\ &\leq \epsilon + kd(x,y) + d(A,B). \end{aligned}$$

Therefore

$$d(x,y) \leq \frac{\epsilon}{1-k} + \frac{d(A,B)}{1-k}.$$

Then

$$diam(P^{Ga}_{(T,S)}(A,B)) \leq \frac{\epsilon}{1-k} + \frac{d(A,B)}{1-k}.$$ □

5. Conclusions

Fixed points, approximate fixed points, and graph-based metric spaces play pivotal roles in mathematics and its applications, including physics, biochemistry, electrical engineering, and computer science. This work introduces new classes of mappings and contractions, presenting results on G-approximate best proximity points and their diameters in metric spaces endowed with a graph G where $V(G) = X$. Using general theorems for cyclic mappings, we establish several G-approximate best proximity point theorems and provide applied examples to illustrate our findings.

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