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A note on infinitely divisible distribution on function fields

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ABSTRACT. In this note, we define a function associated with the zeta function on function fields of genus g over a finite field $\mathbb{F}_q$. We shown that the exponential of this function is the characteristic function of an infinitely divisible distribution on the real line, which is equivalent to the Riemann hypothesis on function fields. Furthermore, we give some special values of this characteristic function and derive several interesting summation formulas.

1. INTRODUCTION

The Riemann zeta function, a central topic in classical analytic number theory, is a function of a complex variable $s = \sigma + it$ with t, $\sigma \in \mathbb{R}$, defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s},$$

which converges for $\Re(s) = \sigma > 1$. This function admits a meromorphic continuation to the entire complex plane $\mathbb{C}$, where it is holomorphic everywhere except for a simple pole at $s = 1$. The Riemann ξ-function, introduced as

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{s/2}\Gamma(s/2)\zeta(s),$$

satisfies two functional equations $\xi(s) = \xi(1-s)$ and $\xi(s) = \overline{\xi(\overline{s})}$, where $\Gamma(s)$ is the gamma function and the bar denotes the complex conjugate. The Riemann Hypothesis (RH), one of the most famous unsolved problems in mathematics, asserts that all non-trivial (non-real) zeros of the Riemann zeta function ζ lie on the critical line $\Re(s) = 1/2$ which is equivalent to all the zeros of $\xi(1/2 - iz)$ being real. One of the particular charms of the study of the Riemann hypothesis is the great diversity of its equivalent formulations,

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which can be extended to the zeta function associated with the function field K of an arbitrary genus over a finite field of constants [3,4,6]. The context of function fields with a finite field of constants is particularly interesting due to the fact that the analog of the Riemann hypothesis holds true in this case [12]. Therefore, one is able to deduce many interesting results by using formulas equivalent to the Riemann hypothesis.

A distribution (or probability measure) μ on $\mathbb{R}$ is infinitely divisible if there exists a distribution μ_n on $\mathbb{R}$ such that $\mu = \mu_n * \cdots * \mu_n$ (n-fold) for every positive integer n. For any infinitely divisible distribution μ, there exists a triplet (a, b, υ) consisting of $a \in \mathbb{R}_{\geq 0}$, $b \in \mathbb{R}$ and a measure υ on $\mathbb{R}$ such that the characteristic function has the Lévy-Khintchine formula (see [10, Theorem 8.1 and Remark 8.4])

$$\hat{\mu}(t) = \exp\left[-\frac{1}{2}at^2 + ibt + \int_{-\infty}^{+\infty}\left(e^{it\lambda} - 1 - \frac{it\lambda}{1+\lambda^2}\right)\upsilon(d\lambda)\right], \tag{1}$$

$$\upsilon(\{0\}) = 0, \quad \int_{-\infty}^{+\infty} \min(1, \lambda^2)\upsilon(d\lambda) < \infty.$$

The measure υ is referred to as the Lévy measure for μ. If the Lévy measure υ satisfies $\int_{|\lambda|\leq 1} |\lambda|\upsilon(d\lambda) < \infty$, then (1) can be rewritten as (see [10, (8.7)])

$$\hat{\mu}(t) = \exp\left[-\frac{1}{2}at^2 + ib_0t + \int_{-\infty}^{+\infty}\left(e^{it\lambda} - 1\right)\upsilon(d\lambda)\right], \tag{2}$$

where $b_0 \in \mathbb{R}$ is called the drift of μ.

In [7], T. Nakamura and M. Suzuki introduced a function such that it is a characteristic function of an infinitely divisible distribution on the real line if and only if the Riemann Hypothesis is true.

The purpose of this paper is to get analogous Nakamura and Suzuki's results on function fields. Furthermore, we study some special values of this characteristic function. In addition, we derive some interesting summation formulas.

Consider a function field K with a finite field of constants $\mathbb{F}_q$ and let X be its smooth projective algebraic curve of genus g defined over $\mathbb{F}_q$. For more details we refer to [9]. The zeta function of K is defined as follows

$$Z_K(T) = \sum_{n=0}^{+\infty} C_n T^n = \prod_{D \text{ prime}} \left(1 - T^{\deg(D)}\right)^{-1},$$

where $C_n = \#\{D \in Div(K);\ D \geq 0,\ \deg(D) = n\}$; $Z_K(T)$ is actually a rational function

$$Z_K(T) = \frac{L(T)}{(1-T)(1-qT)}, \tag{3}$$

where $L(T)$ factors in $\mathbb{C}[T]$ as

$$L(T) = \prod_{j=1}^{2g} (1 - \alpha_j T) \in \mathbb{Z}[T]. \tag{4}$$

The special value $L(1) = \prod_{i=1}^{2g} (1 - \alpha_i)$ is the class number of K, denoted by h_K. The complex numbers $\alpha_1, \ldots, \alpha_{2g}$ are algebraic integers and can be arranged so that $\alpha_j \alpha_{g+j} = q$ holds for $j = 1, \ldots, g$. Since the Riemann hypothesis for function fields (abbreviated to RH) proved by Weil [12] states that α_i, $i = 1, \ldots, 2g$ have absolute value $q^{1/2}$, we may order the indices $j \in \{1, \ldots, g\}$ so that $\alpha_{g+j} = \overline{\alpha_j}$, and we then can write $\alpha_j = q^{1/2} \exp(i\theta_j)$ with $\theta_j \in [0, \pi]$.

Now, we define the (classical) zeta function ζ_K of K as follows: for $s \in \mathbb{C}$, we substitute T with q^{-s} in $Z_K(T)$ to get the function

$$\zeta_K(s) = Z_K(q^{-s}) = \sum_{n=0}^{+\infty} C_n q^{-ns},$$

which converges for $\Re(s) > 1$. We define the following completed zeta function

$$\xi_K(s) := q^s (1 - q^{-s})(1 - q^{1-s}) q^{(g-1)s} \zeta_K(s) = q^{gs} L(q^{-s}), \tag{5}$$

which is an entire function of order one, whose zeros coincide with the zeros of $\zeta_K(s)$. Moreover, $\xi_K(s)$ satisfies the functional equation

$$\xi_K(s) = \xi_K(1 - s). \tag{6}$$

By taking the logarithm and subsequently differentiating both sides of (5), we obtain

$$(\log \xi_K(s))' = g \log q + \sum_{\widetilde{q}} \frac{\alpha_K(\widetilde{q}) \log \widetilde{q}}{\widetilde{q}^s}, \qquad \Re(s) > 1, \tag{7}$$

where $\alpha_K(\widetilde{q}) = \sum_{j=1}^{2g} \frac{\alpha_j^n}{n}$, and $\widetilde{q} = q^n$.

Let us recall that all zeros of the zeta function ζ_K lie in the critical strip $0 \leq \Re(s) \leq 1$, and they are symmetric with respect to the real axis and the line $\Re(s) = 1/2$. Note that the RH in this context is equivalent to saying that the zeros of ζ_K lie on the line $\Re(s) = 1/2$. Let $\mathcal{Z}(K)$ be the set of the zeros ρ of ζ_K. Using (3) and (4), we obtain

$$\mathcal{Z}(K) = \left\{ \frac{1}{2} \pm i \frac{\theta_j}{\log q} + i \frac{2k\pi}{\log q},\ j \in \{1, \ldots, g\},\ k \in \mathbb{Z} \right\}.$$

2. Main results

In this section, we give the main results of the paper.

Let us define a function $g_{\zeta_K}(t)$ to be the even real-valued function on the real line by

$$g_{\zeta_K}(t) = -tg\log q - \sum_{\tilde{q}\le e^t} \frac{\alpha_K(\tilde{q})\log\tilde{q}}{\tilde{q}^{1/2}}(t-\log\tilde{q}), \tag{8}$$

for nonnegative t and $g_{\zeta_K}(t) = g_{\zeta_K}(-t)$ for negative t, where $\alpha_K(\tilde{q})$ is defined in (7).

In section 3, we provide relation between the RH and an infinitely divisible distribution is formulated using the function $g_{\zeta_K}(t)$ associated to the zeta function on function fields following Nakamura and Suzuki arguments given in [7, Theorems 1.1 and 1.2] for the classical Riemann zeta function.

Let us recall that the RH holds on function fields. In the following theorem we define a characteristic function of an infinitely divisible distribution.

Theorem 1. *Let $g_{\zeta_K}(t)$ be the function defined on the real line by equation* (8)*. Then,* $\exp(g_{\zeta_K}(t))$ *represents the characteristic function of an infinitely divisible distribution on* $\mathbb{R}$*.*

Under the RH, the Lévy measure associated with the corresponding infinitely divisible distribution can be explicitly described as follows.

Theorem 2. *Let $g_{\zeta_K}(t)$ be as in Theorem* (1)*. We introduce the measure*

$$\nu_{\zeta_K}(d\lambda) = \sum_{\gamma}\frac{ord(\gamma)}{\gamma^2}\delta_{-\gamma}(d\lambda), \tag{9}$$

where the sum runs over all zeros γ of $\xi_K(1/2-iz)$, counting multiplicity. Here, $ord(\gamma)$ denotes the multiplicity of the zero γ, and δ_x represents the delta measure at x. Then, the Lévy-Khintchine formula (2) *of* $\exp(g_{\zeta_K}(t))$ *holds for the triplet $(a, b_0, \nu) = (0, 0, \nu_{\zeta_K})$, that is,* $\exp(g_K(t))$ *is a characteristic function of a compound Poisson distribution.*

In section 4, another expression of the characteristic function of a compound Poisson distribution $\exp(g_K(t))$ is given in the following theorem.

Theorem 3. *Assume that $\xi_K(1/2)\neq 0$. Let $t\ge 0$ and $\theta_j\in[0,\pi]$, we have*

$$\exp(g_K(t)) = q^{-tg}\prod_{n=1}^{[t/\log q]}\prod_{j=1}^{2g} q^{-\cos(n\theta_j)(t-n\log q)}.$$

As an application of Theorem 3 we study special values of $\exp(g_K(t))$ at some points (see Corollary 1). Furthermore, we derive some interesting summation formulas (see Theorem 4 and Corollary 2).

3. Proofs of Theorems 1 and 2

In this section, we prove Theorems 1 and 2.

Let us recall that the function $g_{\zeta_K}(t)$ is defined by

$$g_{\zeta_K}(t) = -tg\log q - \sum_{\tilde{q}\leq e^t} \frac{\alpha_K(\tilde{q})\log\tilde{q}}{\tilde{q}^{1/2}}(t - \log\tilde{q}),$$

for nonnegative t, where $\alpha_K(\tilde{q})$ is defined in (7).

In the following proposition, we give some results on $g_{\zeta_K}(t)$ which will be useful for the proofs of Theorems 1 and 2.

Proposition 1. (i) *If $\Im(z) > 1/2$, we have*

$$\int_0^{+\infty} g_{\zeta_K}(t)e^{izt}dt = \frac{1}{z^2}\frac{\xi_K'}{\xi_K}\left(\frac{1}{2} - iz\right). \tag{10}$$

(ii) *Let $t \geq 0$, we have*

$$g_{\zeta_K}(t) = \sum_{\gamma\in\Gamma}\frac{\cos(\gamma t) - 1}{\gamma^2} = \sum_{\gamma\in\Gamma}\frac{e^{-i\gamma t} - 1}{\gamma^2}, \tag{11}$$

where Γ is the set of all zeros of $\xi_K(1/2 - iz)$ counting with multiplicity.

(iii) *The right-hand side of* (11) *converges absolutely and uniformly on any compact subsets of $\mathbb{R}$.*

Proof. (i) The proof closely follows the same approach as done in [5, Proposition 2.1 (i)] (see also [8, Theorem 1.1 (1)]), we replace $g_K(t)$ by $g_{\zeta_K}(t)$ such that $g_{\zeta_K}(t) = -g_K(t)$.

(ii) We only sketch the proof since it follows the lines of that [5, Proposition 2.1 (ii)] (see also [8, Theorem 1.1 (2)]), we replace $g_K(t)$ by $g_{\zeta_K}(t)$ such that $g_{\zeta_K}(t) = -g_K(t)$.

(iii) The proof follows very closely the lines of the proof of the corresponding results in [7, Lemma 2.1]
Let K be a compact subset of $\mathbb{R}$. The expression $|e^{-i\gamma t} - 1|$ is uniformly bounded for all γ and $t \in K$, as all zeros of $\xi_K(1/2 - iz)$ lie in the horizontal strip $|\Im(z)| \leq 1/2$. Additionally, the sum $\sum_\gamma ord(\gamma)|\gamma|^{-\alpha}$ converges for any $\alpha > 1$, because $\xi_K(1/2 - iz)$ is an entire function of order one. Consequently, the sum on the right-hand side of (11) converges absolutely and uniformly for $t \in K$. □

The right-hand side of (11) defines a continuous function on the real line. By the functional equation given in (6), if γ is a zero of $\xi_K(1/2 - iz)$, then $-\gamma$ is also a zero with the same multiplicity. As a result, the right-hand side of (11) is an even real-valued function. This is consistent with the definition of $g_{\zeta_K}(t)$.

Proofs of Theorems 1 and 2. The proof closely follows the same approach as done in [7, Section 2]. We will adapt that proof and keep some of it's notations.

Proof of RH $\Rightarrow$ Theorem 1 and Theorem 2: Since the Riemann Hypothesis (RH) holds for function fields, then the zeros γ of $\xi_K(1/2-iz)$ are real. Additionally, assuming $\xi_K(\frac{1}{2}) \neq 0$ the formula (9) defines a measure on $\mathbb{R}$ such that the identity

$$\exp(g_{\zeta_K}(t)) = \exp\left[\int_{-\infty}^{+\infty}(e^{it\lambda}-1)\upsilon_{\zeta_K}(d\lambda)\right],$$

holds for all $t \in \mathbb{R}$. Here, $\upsilon_{\zeta_K}(\{0\}) = 0$, and $\int_{|\lambda|\leq 1}|\lambda|\upsilon_{\zeta_K}(d\lambda) < \infty$ by equation (11). Furthermore,

$$\int_{-\infty}^{+\infty}\min(1,\lambda^2)\upsilon_{\zeta_K}(d\lambda) < \infty,$$

which follows from the convergence of $\sum_\gamma ord(\gamma)|\gamma|^{-2}$ in the proof of Proposition 1 (iii). Consequently, there exists an infinitely divisible distribution μ whose characteristic function is given by $\exp(g_{\zeta_K}(t))$, with the characteristic triplet $(0,0,\upsilon_{\zeta_K})$, as established in [10, Theorem 8.1 (iii)].

Proof of Theorem 1 $\Rightarrow$ RH: By Theorem 1, we have $\exp(g_{\zeta_K}(t)) = \hat{\mu}(t)$ for some infinitely divisible distribution μ on the real line. Since $\exp(g_{\zeta_K}(t))$ is an even function (a property independent of the Riemann Hypothesis, as noted before Theorem 1), it follows that $\exp(g_{\zeta_K}(t)) = \hat{\mu}(t) = \hat{\mu}(-t)$. The equality $\hat{\mu}(t) = \hat{\mu}(-t)$ implies

$$\hat{\mu}(t) = \int_{-\infty}^{+\infty}\cos(tx)\mu(dx).$$

Thus, $g_{\zeta_K}(t) = \log\hat{\mu}(t)$ is real-valued. Furthermore, $g_{\zeta_K}(t)$ is non-positive on $\mathbb{R}$, since $|\hat{\mu}(t)| \leq 1$ holds because μ is a distribution.

Since $g_{\zeta_K}(t)$ is non-positive, the pure imaginary point iy on the horizontal axis $\Im(z) = y$ of convergence for the function $f(z) = \int_0^{+\infty} g_{\zeta_K}(t)e^{izt}dt$ is a singularity of $f(z)$ as established in [13, Theorem 5b in Chap. II]. On the other hand, $(\xi_K'/\xi_K)(1/2-iz)$ has no singularity on the positive imaginary axis $i\mathbb{R}_{>0}$, because $\xi_K(s)$ is holomorphic and has no zeros on the positive real line $\mathbb{R}_{>0}$ ([11, Section 2.12]). Consequently, $f(z)$ converges in the upper half-plane $\{z \setminus \Im(z) > 0\}$ and defines a holomorphic function there. By (10), this implies that $\xi_K(1/2-iz)$ has no zeros in the upper half-plane. Furthermore, by (6) it also has no zeros in the lower half-plane $\{z \setminus \Im(z) < 0\}$. Therefore, all zeros of $\xi_K(1/2-iz)$ must lie on the real line, which is equivalent to the Riemann Hypothesis (RH). $\square$

4. Further results on $\exp(g_{\zeta_K}(t))$

In this section, we give another expression of the characteristic function of a compound Poisson distribution $\exp(g_{\zeta_K}(t))$. Furthermore, we study some special values of this characteristic function. In addition, we derive some interesting summation formulas.

Let us recall that the zeros ρ of the function ζ_K are denoted by

$$\rho = \frac{1}{2} + i\tau^{\pm}_{k,j}, \quad \textit{where} \;\; \tau^{\pm}_{k,j} = (\pm\theta_j + 2k\pi)/\log q, \quad j = 1, \dots, g; \;\; k \in \mathbb{Z},$$

and let Γ be the set of all zeros of $\xi_K(1/2 - iz)$ with counting multiplicity and the multiplicity of $\gamma \in \Gamma$ by $ord(\gamma)$. Throughout this section, we replace γ by $\tau^{\pm}_{k,j}$ where $j = 1, \dots, g$ and $k \in \mathbb{Z}$.

Remark. We have $\xi_K(1/2) = 0$ if and only if for some $j = 1, 2, \dots, g$, $\theta_j = 0$; in this case, instead of ξ_K, we may take the function $F_K(s) = \xi_K(s)/(s - 1/2)^m$, where m is the multiplicity of the eventual zero of ξ_K at $s = 1/2$. Functions F_K and ξ_K have the same zeros with $\Im(\rho) > 0$. For this reason, we assume in this section $\xi_K(1/2) \neq 0$.

From (11) for $t \geq 0$, we obtain

$$\exp(g_{\zeta_K}(t)) = \prod_{j=1}^{g} \prod_{k\in\mathbb{Z}} \exp\left(\frac{\cos(\tau^{\pm}_{k,j} t) - 1}{(\tau^{\pm}_{k,j})^2}\right). \tag{12}$$

Proof of Theorem 3. By (11) and $\widetilde{q} = q^n$, we have

$$\begin{aligned} g_{\zeta_K}(t) &= -tg\log q - \sum_{\tilde{q}\leq e^t} \frac{\alpha_K(\tilde{q})\log\tilde{q}}{\tilde{q}^{1/2}}(t - \log\tilde{q}) \\ &= -tg\log q - \sum_{q^n\leq e^t} \frac{\alpha_K(q^n)\log q^n}{q^{n/2}}(t - \log q^n) \\ &= -tg\log q - \sum_{n=1}^{[t/\log q]} \frac{\alpha_K(q^n)\log q^n}{q^{n/2}}(t - \log q^n). \end{aligned}$$

Let us recall that $\alpha_j = q^{1/2}\exp(i\theta_j)$ with $\theta_j \in [0, \pi]$ and $\alpha_K(q^n) = \sum_{j=1}^{2g} \frac{\alpha_j^n}{n}$. Since the function $g_{\zeta_K}(t)$ is real, we obtain

$$g_{\zeta_K}(t) = -tg\log q - \log q \sum_{n=1}^{[t/\log q]} \sum_{j=1}^{2g} \cos(n\theta_j)(t - n\log q).$$

Then

$$\begin{aligned} &\exp(g_{\zeta_K}(t)) = \\ &= \exp\left(-tg\log q - \log q \sum_{n=1}^{[t/\log q]} \sum_{j=1}^{2g} \cos(n\theta_j)(t - n\log q)\right) \\ &= q^{-tg} \prod_{n=1}^{[t/\log q]} \prod_{j=1}^{2g} q^{-\cos(n\theta_j)(t - n\log q)}. \end{aligned} \tag{13}$$

This complete the proof of Theorem 3. $\square$

In the following corollary, we give some special values of $\exp(g_{\zeta_K}(t))$ at some points.

Corollary 1. *Assume that $\xi_K(1/2) \neq 0$ and let $\theta_j \in [0, \pi]$. We have*

(i) $$\exp(g_{\zeta_K}(0)) = 1.$$

(ii) $$\exp(g_{\zeta_K}(\log q)) = \exp\left(-g \log^2 q\right).$$

(iii) $$\exp(g_{\zeta_K}(2\log q)) = \exp\left[-\log^2 q\left(2g + \sum_{j=1}^{2g} \cos(\theta_j)\right)\right].$$

Proof. The proof of Corollary 1 yields from equation (13) by replacing t by $0, \log q$ and $2\log q$. □

In the following theorem, we derive some interesting summation formulas.

Theorem 4. *Assume that $\xi_K(1/2) \neq 0$ and let $\theta_j \in [0, \pi]$. We have*

(i) $$\sum_{j=1}^{g} \sum_{k \in \mathbb{Z}} \frac{1 - \cos(\pm\theta_j)}{(\pm\theta_j + 2k\pi)^2} = g.$$

(ii) $$\sum_{j=1}^{g} \sum_{k \in \mathbb{Z}} \left(\frac{1 - \cos(\pm 2\theta_j)}{(\pm\theta_j + 2k\pi)^2}\right) = \sum_{j=1}^{2g} \left(\cos(\theta_j) + 1)\right).$$

Proof. By Corollary 1 (ii) we have

$$\exp(g_{\zeta_K}(\log q)) = \exp\left(-g \log^2 q\right).$$

On the other hand, (12) with $t = \log q$ and $\tau^{\pm}_{k,j} = (\pm\theta_j + 2k\pi)/\log q$, where $j = 1, \dots, g$ and $k \in \mathbb{Z}$ yields

$$\exp(g_{\zeta_K}(\log q)) = \prod_{j=1}^{g} \prod_{k \in \mathbb{Z}} \exp\left(\frac{\cos(\pm\theta_j + 2k\pi) - 1}{\left(\frac{\pm\theta_j + 2k\pi}{\log q}\right)^2}\right).$$

Then

$$\prod_{j=1}^{g} \prod_{k \in \mathbb{Z}} \exp\left(\frac{\cos(\pm\theta_j + 2k\pi) - 1}{\left(\frac{\pm\theta_j + 2k\pi}{\log q}\right)^2}\right) = \exp\left(-g \log^2 q\right).$$

By applying logarithm to two sides, we get

$$\sum_{j=1}^{g} \sum_{k \in \mathbb{Z}} \frac{1 - \cos(\pm\theta_j)}{(\pm\theta_j + 2k\pi)^2} = g.$$

Therefore, we get the first assertion (i).

Similarly, from Corollary 1 (iii) we have

$$\exp(g_{\zeta_K}(2\log q)) = \exp\left[-\log^2 q\left(2g + \sum_{j=1}^{2g} \cos(\theta_j)\right)\right].$$

On the other hand, (12) with $t = 2\log q$ and $\tau_{k,j}^{\pm} = (\pm\theta_j + 2k\pi)/\log q$, where $j = 1, \ldots, g$ and $k \in \mathbb{Z}$ yields

$$\exp(g_{\zeta_K}(2\log q)) = \prod_{j=1}^{g}\prod_{k\in\mathbb{Z}} \exp\left(\frac{\cos(\pm 2\theta_j + 4k\pi) - 1}{\left(\frac{\pm\theta_j + 2k\pi}{\log q}\right)^2}\right).$$

Then

$$\prod_{j=1}^{g}\prod_{k\in\mathbb{Z}} \exp\left(\frac{\cos(\pm 2\theta_j + 4k\pi) - 1}{\left(\frac{\pm\theta_j + 2k\pi}{\log q}\right)^2}\right) = \exp\left[-\log^2 q\left(2g + \sum_{j=1}^{2g}\cos(\theta_j)\right)\right].$$

By applying the logarithm to the two sides, we get

$$\sum_{j=1}^{g}\sum_{k\in\mathbb{Z}} \left(\frac{\cos(\pm 2\theta_j + 4k\pi) - 1}{\left(\frac{\pm\theta_j + 2k\pi}{\log q}\right)^2}\right) = -\log^2 q\left(2g + \sum_{j=1}^{2g}\cos(\theta_j)\right).$$

Simple computation yields

$$\sum_{j=1}^{g}\sum_{k\in\mathbb{Z}} \left(\frac{\cos(\pm 2\theta_j) - 1}{(\pm\theta_j + 2k\pi)^2}\right) = -\left(2g + \sum_{j=1}^{2g}\cos(\theta_j)\right).$$

The proof of Theorem 4 is complete. □

Now, we give an interesting summation formula in terms of special values of the completed zeta function ξ_K and its derivatives.

Corollary 2. *Assume that $\xi_K(1/2) \neq 0$ and let $\theta_j \in [0, \pi]$. We have*

$$\sum_{j=1}^{g}\left(\sum_{k\in\mathbb{Z}} \frac{\cos(\pm\theta_j)}{(\pm\theta_j + 2k\pi)^2} + 1\right) = \frac{2}{\log^2 q}\frac{\xi_K''(\frac{1}{2})}{\xi_K(\frac{1}{2})}. \tag{14}$$

Proof. Assume that $\xi_K(1/2) \neq 0$. From [1, Theorem 2.2], we have

$$\frac{\xi_K''\left(\frac{1}{2}\right)}{\xi_K\left(\frac{1}{2}\right)} = \log^2 q \sum_{j=1}^{g}\sum_{k\in\mathbb{Z}} \frac{1}{(\theta_j + 2k\pi)^2}. \tag{15}$$

On the other hand, by Theorem 4 (i) one has

$$\sum_{j=1}^{g}\sum_{k\in\mathbb{Z}} \frac{1 - \cos(\pm\theta_j)}{(\pm\theta_j + 2k\pi)^2} = g.$$

Easy computation yields

$$(16)\qquad \begin{aligned} \sum_{j=1}^{g}\left(\sum_{k\in\mathbb{Z}}\frac{\cos(\pm\theta_j)}{(\pm\theta_j+2k\pi)^2}+1\right) &= \sum_{j=1}^{g}\sum_{k\in\mathbb{Z}}\frac{1}{(\pm\theta_j+2k\pi)^2} \\ &= 2\sum_{j=1}^{g}\sum_{k\in\mathbb{Z}}\frac{1}{(\theta_j+2k\pi)^2}. \end{aligned}$$

The proof of Corollary 2 follows from equations (15) and (16). □

Remark. Assume that $\xi_K(1/2)\neq 0$ and let $\theta_j\in[0,\pi]$. Formula (14) can be written as follows

$$\begin{aligned} &\sum_{j=1}^{g}\left(\sum_{k\in\mathbb{Z}}\frac{\cos(\pm\theta_j)}{(\pm\theta_j+2k\pi)^2}+1\right)= \\ &=\frac{1}{q^{\frac{g}{2}+1}L(q^{-1/2})}\left[\frac{L''}{L}(q^{-1/2})-\left(\frac{L'}{L}(q^{-1/2})\right)^2+q^{1/2}\frac{L'}{L}(q^{-1/2})\right]. \end{aligned}$$

In fact, by (5) with $s=1/2$ we get

$$\xi_K\left(\frac{1}{2}\right)=q^{\frac{g}{2}}L(q^{-1/2})$$

and

$$\xi_K''\left(\frac{1}{2}\right)=\frac{\log^2 q}{q}\left[\frac{L''}{L}(q^{-1/2})-\left(\frac{L'}{L}(q^{-1/2})\right)^2+q^{1/2}\frac{L'}{L}(q^{-1/2})\right].$$

Then, we obtain

$$\begin{aligned} \frac{1}{\log^2 q}\frac{\xi_K''(\frac12)}{\xi_K(\frac12)} &= \frac{1}{q^{\frac{g}{2}+1}L(q^{-1/2})}\left[\frac{L''}{L}(q^{-1/2})-\left(\frac{L'}{L}(q^{-1/2})\right)^2+q^{1/2}\frac{L'}{L}(q^{-1/2})\right] \\ &= \sum_{j=1}^{g}\left(\sum_{k\in\mathbb{Z}}\frac{\cos(\pm\theta_j)}{(\pm\theta_j+2k\pi)^2}+1\right). \end{aligned}$$

□

Now, let us recall that the superzeta functions on function fields of the second kind is defined by (see [2, section 5])

$$(17)\qquad \mathcal{Z}_K(s,t)=\sum_{j=1}^{g}\sum_{k\in\mathbb{Z}}\frac{1}{((\tau_{k,j}^{\pm})^2+t^2)^s},\quad \Re(s)>1/2,$$

where $t\in\mathbb{C}$ such that $t^2+(\tau_{k,j}^{\pm})^2\notin\mathbb{R}_-$ for all k.

In the following proposition, we express $\exp(g_{\zeta_K}(t))$ in terms of special values of $\mathcal{Z}_K(s,t)$.

Proposition 2. *Assume that $\xi_K(1/2) \neq 0$ and let $t \geq 0$. We have*

$$\exp(g_{\zeta_K}(t)) = \exp\left(-\mathcal{Z}_K(1,0)\right) \prod_{j=1}^{g} \prod_{k \in \mathbb{Z}} \exp\left(\frac{\cos(\tau_{k,j}^{\pm} t)}{(\tau_{k,j}^{\pm})^2}\right).$$

Proof. By (12), we obtain

$$\begin{aligned}\exp(g_{\zeta_K}(t)) &= \prod_{j=1}^{g} \prod_{k \in \mathbb{Z}} \exp\left(\frac{\cos(\tau_{k,j}^{\pm} t) - 1}{(\tau_{k,j}^{\pm})^2}\right) \\ &= \prod_{j=1}^{g} \prod_{k \in \mathbb{Z}} \exp\left(\frac{\cos(\tau_{k,j}^{\pm} t)}{(\tau_{k,j}^{\pm})^2}\right) \prod_{j=1}^{g} \prod_{k \in \mathbb{Z}} \exp\left(\frac{-1}{(\tau_{k,j}^{\pm})^2}\right).\end{aligned}$$

From (17) for $s = 1$ and $t = 0$, we have

$$-\mathcal{Z}_K(1,0) = \sum_{j=1}^{g} \sum_{k \in \mathbb{Z}} \left(\frac{-1}{(\tau_{k,j}^{\pm})^2}\right). \tag{18}$$

By applying the exponential, (18) gives

$$\exp\left(-\mathcal{Z}_K(1,0)\right) = \prod_{j=1}^{g} \prod_{k \in \mathbb{Z}} \exp\left(\frac{-1}{(\tau_{k,j}^{\pm})^2}\right).$$

Hence, the proof of Proposition 2 is complete. □

Corollary 3. *Assume that $\xi_K(1/2) \neq 0$ and let $\theta_j \in [0, \pi]$. We have*

$$\prod_{j=1}^{g} \prod_{k \in \mathbb{Z}} \exp\left(\frac{\cos(\pm\theta_j)}{\left(\frac{\pm\theta_j + 2k\pi}{\log^2 q}\right)^2}\right) = \frac{\exp\left(\mathcal{Z}_K(1,0)\right)}{q^{g \log q}}.$$

Proof. By Theorem 3 and Proposition 2, we obtain

$$\begin{aligned}&\prod_{j=1}^{g} \prod_{k \in \mathbb{Z}} \exp\left(\frac{\cos(\tau_{k,j}^{\pm} t)}{(\tau_{k,j}^{\pm})^2}\right) \\ &= \exp\left(\mathcal{Z}_K(1,0)\right) q^{-tg} \prod_{n=1}^{[t/\log q]} \prod_{j=1}^{2g} q^{-\cos(n\theta_j)(t - n\log q)}.\end{aligned} \tag{19}$$

From (19), assume that $\xi_K(1/2) \neq 0$ and for $t = \log q$ such that $\tau_{k,j}^{\pm} = (\pm\theta_j + 2k\pi)/\log q$, $j = 1, \ldots, g$ and $k \in \mathbb{Z}$. Then, we have

$$\prod_{j=1}^{g} \prod_{k \in \mathbb{Z}} \exp\left(\frac{\cos(\pm\theta_j)}{\left(\frac{\pm\theta_j + 2k\pi}{\log^2 q}\right)^2}\right) = \frac{\exp\left(\mathcal{Z}_K(1,0)\right)}{q^{g \log q}}.$$ □

5. Concluding remarks

In this work, we introduce and analyze a new function linked to the zeta function of function fields of genus g over a finite field $\mathbb{F}_q$. We establish that the exponential of this function coincides with the characteristic function of an infinitely divisible distribution on the real line.

This finding provides a probabilistic interpretation equivalent to the Riemann hypothesis for function fields, thereby uncovering a novel connection between analytical and probabilistic approaches. Moreover, the evaluation of special values and the derivation of summation formulas emphasize the rich structural features of this function.

We believe that these results pave the way for further investigations into the profound interaction between zeta functions, probability theory, and the Riemann hypothesis in the framework of function fields.

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