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Multivariate trigonometric Korovkin theorem within a fuzzy framework

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Abstract. In this paper, the trigonometric fuzzy Korovkin theorem, originally established by G. A. Anastassiou and S. G. Gal (Nonlinear Functional Analysis and Applications, 11 (2006), 385–395), is extended to the k-dimensional setting. The proof is based on a new approach that differs from the original one and does not rely on the fuzzy modulus of continuity. An illustrative example is included to confirm the applicability and validity of the proposed generalization.

1. Motivation and preliminaries

In classical setting, there are two possibilities for an element, i.e., a set includes an element or does not include because of its well-defined definition. Zadeh has introduced another possibility to this case by defining fuzzy sets. Fuzzy set theory was introduced by Zadeh [13] in 1965. Fuzzy set is defined by values in the range of $[0, 1]$, which shows the degree of belonging of each possible element in fuzzy set. The larger this value, the more the element belongs the fuzzy set. After Zadeh's work, many researchers applied the known results of classical set theory to this theory.

In traditional mathematical analysis, approximation functions plays a crucial role in various fields such as numerical analysis, functional analysis and approximation theory. The classical Korovkin theory provides a framework for understanding convergence properties of sequences of positive linear operators as they approximate functions. However, this theory relies on crisp(non-fuzzy) sets and does not effectively capture the inherit uncertainty and imprecision found in many real-world applications. Fuzzy Korovkin theory is a branch of mathematics that deals with the approximation of functions using fuzzy sets [1–4, 7, 11, 12]. It is an extension of the classical Korovkin theory which focuses on the approximation of functions

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by linear positive operators. At the same time, fuzzy Korovkin theory introduces the concept of fuzzy sets to enhance the approximation capabilities and handle uncertainties inherit in real-world problems. Fuzzy Korovkin theory represents an important development in the field of approximation theory enabling the use of fuzzy sets to tackle the challenges posed by uncertainty.

In this study, the trigonometric fuzzy Korovkin theorem, previously introduced in [2], is generalized to the k-dimensional case. The proof of this theorem employs an alternative method distinct from that utilized in [2]. When $k = 1$, the proposed theorem reduces to Theorem 5 in [2]. Furthermore, an example is provided to demonstrate the applicability and validity of the generalized theorem.

First of all, we give fundamental definitions and concepts that will be used during this paper.

Definition 1. The following conditions

1. μ is normal i.e. there exists $u_0 \in \mathbb{R}$ with $\mu(u_0) = 1$;
2. for every $u, v \in \mathbb{R}$ and $r \in [0,1]$, $\mu(ru + (1-r)v) \geq min\{\mu(u), \mu(v)\}$;
3. for every $u_0 \in \mathbb{R}$, $\varepsilon > 0$, there exists a neighborhood $V(u_0)$ with for every $u \in V(u_0)$
$$\mu(u) \leq \mu(u_0) + \varepsilon,$$
i.e., μ is upper semicontinuous;
4. the closure of supp $(\mu) := \{u \in \mathbb{R}; \mu(u) > 0\}$ is compact in $\mathbb{R}$;

hold for $\mu : \mathbb{R} \to [0,1]$, then it is said that μ is a fuzzy real number and the set of all fuzzy real numbers is denoted by $\mathbb{R}_{\mathcal{F}}$ [10].

One can easily see that for every $u_0 \in \mathbb{R}$, $\chi_{\{u_0\}}$ is a fuzzy real number, where $\chi_{\{u_0\}}$ is the characteristic function at u_0.

For $0 < p \leq 1$, $\mu \in \mathbb{R}_{\mathcal{F}}$, consider the following sets

$$[\mu]^p := \{u \in \mathbb{R} : \mu(u) \geq p\}$$

and

$$[\mu]^0 := \overline{\{u \in \mathbb{R} : \mu(u) > 0\}}.$$

$[\mu]^p$ is a closed and bounded interval of $\mathbb{R}$ for each $p \in [0,1]$. For $\mu, \beta \in \mathbb{R}_{\mathcal{F}}$ and $t \in \mathbb{R}$, the sum $\mu \oplus \beta$ and the product $t \otimes \mu$ are defined as follows

$$[\mu \oplus \beta]^p = [\mu]^p + [\beta]^p, \quad [t \otimes \mu]^p = t[\mu]^p, \quad \text{for every } p \in [0,1],$$

where the expression $[\mu]^p + [\beta]^p$ denotes the standard addition of two intervals considered as subsets of $\mathbb{R}$ and $t[\mu]^p$ means the usual product between a scalar and a subset of $\mathbb{R}$. Notice that $1 \otimes \mu = \mu$ and $\mu \oplus \beta = \beta \oplus \mu$. If $0 \leq p_1 \leq p_2 \leq 1$ then $[\mu]^{p_2} \subseteq [\mu]^{p_1}$. Actually $[\mu]^p = [\mu_-^p, \mu_+^p]$, where $\mu_-^p \leq \mu_+^p$, $\mu_-^p, \mu_+^p \in \mathbb{R}$, for every $p \in [0,1]$ and also $\mu \precsim \beta$ if $\mu_-^p \leq \beta_-^p$ and $\mu_+^p \leq \beta_+^p$, every $p \in [0,1]$. Define

$$D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+$$

by

$$D(\mu,\beta) := \sup_{p\in[0,1]} \max\left\{|\mu_-^p - \beta_-^p|, |\mu_+^p - \beta_+^p|\right\}.$$

Then it is easy to see that D defines a metric on $\mathbb{R}_{\mathcal{F}}$ and also $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, satisfying the properties

$$\begin{aligned}
&D(\mu\oplus\omega,\beta\oplus\omega) = D(\mu,\beta), \quad \forall\mu,\beta,\omega\in\mathbb{R}_{\mathcal{F}},\\
&D(\lambda\otimes\mu,\lambda\otimes\beta) = |\lambda| D(\mu,\beta), \quad \forall\mu,\beta\in\mathbb{R}_{\mathcal{F}}, \text{ for all } \lambda\in\mathbb{R},\\
&D(\mu\oplus\beta,\omega\oplus\eta) \le D(\mu,\omega) + D(\beta,\eta), \quad \forall\mu,\beta,\omega,\eta\in\mathbb{R}_{\mathcal{F}}.
\end{aligned}$$

Let $I=[a,b]\subset\mathbb{R}$ and $f,g: I\to\mathbb{R}_{\mathcal{F}}$ be fuzzy real number valued functions. The distance between f and g is given by

$$D^*(f,g) := \sup_{u\in I} D\big(f(u), g(u)\big).$$

2. Main results

In this section, the trigonometric fuzzy Korovkin theorem, previously introduced in [2], is generalized to the k-dimensional case. The proof of this theorem employs an alternative method distinct from that utilized in [2]. When $k=1$, the proposed theorem reduces to Theorem 5 in [2]. Furthermore, an example is provided to demonstrate the applicability and validity of the generalized theorem.

Lemma 1 ([8]). *Consider continuous separating trigonometric function $\gamma:\mathbb{R}^k\times\mathbb{R}^k\to[0,\infty)$ with 2π-periodic according to each variable. Let also $f\in C_{2\pi}(\mathbb{R}^k)$. Then:*

(1) $\forall\varepsilon>0$, $\exists\eta_\varepsilon>0$ *such that for every positive linear operator $G: C_{2\pi}(\mathbb{R}^k)\to C_{2\pi}(\mathbb{R}^k)$ and $\forall u\in\mathbb{R}^k$ the following inequality holds*

$$\big|G(f)(u)-f(u)\big| \le \varepsilon G(1)(u) + \eta_\varepsilon G\big(\gamma(.,u)\big)(u) + \big|f(u)\big|\big|G(1)(u)-1\big|.$$

(2) $\forall\varepsilon>0$, $\exists\eta_\varepsilon>0$ *such that for every positive linear operator $G: C_{2\pi}(\mathbb{R}^k)\to C_{2\pi}(\mathbb{R}^k)$ the following inequality holds*

$$\big\|G(f)-f\big\| \le \varepsilon\big\|G(1)\big\| + \eta_\varepsilon \sup_{u\in\mathbb{R}^k} G\big(\gamma(.,u)\big)(u) + \big\|f\big\|\big\|G(1)-1\big\|.$$

In the above lemma, $C_{2\pi}(\mathbb{R}^k)$ denotes the set of all continuous function $f:\mathbb{R}^k\to\mathbb{R}$ which is 2π-periodic according to each variable. Also $C_{2\pi}^{\mathcal{F}}(\mathbb{R}^k)$ denotes the set of all fuzzy continuous function $f:\mathbb{R}^k\to\mathbb{R}_{\mathcal{F}}$ which is 2π-periodic according to every variable. This result is similar to Shisha-Mond inequality [9]. In order to give the alternative proof of Theorem 4 in [1], we use this inequality.

In the next result we do not use the fuzzy modulus of continuity in contrast to Anastassiou and Gal [2].

Theorem 1. *Let $\{G_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy positive linear operators from $C_{2\pi}^{\mathcal{F}}(\mathbb{R}^k)$ to $C_{2\pi}^{\mathcal{F}}(\mathbb{R}^k)$. Let $\{\tilde{G}_n\}_{n\in\mathbb{N}}$ be a corresponding sequence of positive linear operators from $C_{2\pi}(\mathbb{R}^k)$ to $C_{2\pi}(\mathbb{R}^k)$ which satisfy the following equality for every $p\in[0,1]$, any $f\in C_{2\pi}^{\mathcal{F}}(\mathbb{R}^k)$*

$$\big\{G_n(f)\big\}_{\pm}^{(p)}=\tilde{G}_n(f_{\pm}^{(p)}).$$

Also assume that $\{\tilde{G}_n(1)\}_{n\in\mathbb{N}}$ is bounded in n over $[0,2\pi]^k\subseteq\mathbb{R}^k$. Then for $n\in\mathbb{N}$, we have for $\forall\varepsilon>0$, $\exists\eta_\varepsilon$, $\eta_\varepsilon^{'}>0$ such that

$$D^*\big(G_nf,f\big)\leq\varepsilon\big\|\tilde{G}_n(1)\big\|+\big\|\tilde{G}_n(1)-1\big\|D^*(f,\tilde{o})+\max\{\eta_\varepsilon,\eta_\varepsilon^{'}\}\sup_{u\in\mathbb{R}^k}\tilde{G}_n\big(\gamma(.,u)\big).$$

Proof. Let f be in $C_{2\pi}^{\mathcal{F}}(\mathbb{R}^k)$. By using Lemma 1, we get

$$\begin{aligned}
&D^*(G_nf,f)\\
&=\sup_{u\in\mathbb{R}^k}D\big((G_n)(u),f(u)\big)\\
&=\sup_{u\in\mathbb{R}^k}\sup_{p\in[0,1]}\max\Big\{\big|(G_nf)_-^p(u)-f_-^p(u)\big|,\big|(G_nf)_+^p(u)-f_+^p(u)\big|\Big\}\\
&=\sup_{u\in\mathbb{R}^k}\sup_{p\in[0,1]}\max\Big\{\big|\tilde{G}_n(f_-^{(p)})(u)-f_-^{(p)}\big|,\big|\tilde{G}_n(f_+^{(p)})(u)-f_+^{(p)}\big|\Big\}\\
&=\sup_{p\in[0,1]}\max\Big\{\big\|\tilde{G}_n(f_-^{(p)})-f_-^{(p)}\big\|,\big\|\tilde{G}_n(f_+^{(p)})-f_+^{(p)}\big\|\Big\}\\
&\leq\sup_{p\in[0,1]}\max\Big\{\varepsilon\big\|\tilde{G}_n(1)\big\|+\eta_\varepsilon\sup_{u\in\mathbb{R}^k}\tilde{G}_n(\gamma(.,u))(u)+\big\|f_-^{(p)}\big\|\big\|\tilde{G}_n(1)-1\big\|,\\
&\qquad\qquad\varepsilon\big\|\tilde{G}_n(1)\big\|+\eta_\varepsilon^{'}\sup_{u\in\mathbb{R}^k}\tilde{G}_n(\gamma(.,u))(u)+\big\|f_+^{(p)}\big\|\big\|\tilde{G}_n(1)-1\big\|\Big\}\\
&\leq\varepsilon\big\|\tilde{G}_n(1)\big\|+\big\|\tilde{G}_n(1)-1\big\|\sup_{p\in[0,1]}\max\Big\{\big\|f_-^{(p)}\big\|,\big\|f_+^{(p)}\big\|\Big\}\\
&\quad+\max\{\eta_\varepsilon,\eta_\varepsilon^{'}\}\sup_{u\in\mathbb{R}^k}\tilde{G}_n\big(\gamma(.,u)\big)(u)\\
&=\varepsilon\big\|\tilde{G}_n(1)\big\|+\big\|\tilde{G}_n(1)-1\big\|D^*(f,\tilde{o})\\
&\quad+\max\{\eta_\varepsilon,\eta_\varepsilon^{'}\}\sup_{u\in\mathbb{R}^k}\tilde{G}_n\big(\gamma(.,u)\big)(u).
\end{aligned}$$

□

Theorem 2. *Consider a sequence of fuzzy positive linear operators $\{G_n\}_{n\in\mathbb{N}}$ from $C_{2\pi}^{\mathcal{F}}(\mathbb{R}^k)$ to $C_{2\pi}^{\mathcal{F}}(\mathbb{R}^k)$. Let $\{\tilde{G}_n\}_{n\in\mathbb{N}}$ be a corresponding sequence of positive linear operators from $C_{2\pi}(\mathbb{R}^k)$ to $C_{2\pi}(\mathbb{R}^k)$ which satisfy the following equality for every $p\in[0,1]$ and any $f\in C_{2\pi}^{\mathcal{F}}(\mathbb{R}^k)$*

$$\big\{G_n(f)\big\}_{\pm}^{(p)}=\tilde{G}_n(f_{\pm}^{p}).$$

Consider the separating trigonometric function, $\gamma : \mathbb{R}^k \times \mathbb{R}^k \to [0,\infty)$ with 2π-periodic according to each variable such that $\gamma(u,u) = 0$ for all $u \in \mathbb{R}^k$.

If

$$\lim_{n\to\infty} \tilde{G}_n(\gamma(.,u))(u) = 0$$

uniformly in $u \in \mathbb{R}^k$ and

$$\lim_{n\to\infty} \tilde{G}_n(1) = 1$$

uniformly on $\mathbb{R}^k$. Then

$$\lim_{n\to\infty} G_n(f) = f$$

holds uniformly on $\mathbb{R}^k$ for any $f \in C_{2\pi}^{\mathcal{F}}(\mathbb{R}^k)$.

Proof. According to Theorem 1, it is well-known that for any $n \in \mathbb{N}$

$$\begin{aligned} D^*(G_n f, f) \le \varepsilon \big\|\tilde{G}_n(1)\big\| &+ \big\|\tilde{G}_n(1) - 1\big\|_\infty D^*(f, \tilde{o}) \\ &+ \max\big\{\eta_\varepsilon, \eta_\varepsilon^{'}\big\} \sup_{u\in\mathbb{R}^k} \tilde{G}_n\big(\gamma(.,u)\big)(u). \end{aligned}$$

Since $\lim_{n\to\infty} \tilde{G}_n\big(\gamma(.,u)\big)(u) = 0$ uniformly in $u \in \mathbb{R}^k$, we can write

$$\exists \eta_\varepsilon \in \mathbb{N} \ni \sup_{u\in\mathbb{R}^k} \tilde{G}\big(\gamma(.,u)\big)(u) < \frac{\varepsilon}{\max\big\{\eta_\varepsilon, \eta_\varepsilon^{'}\big\}}, \quad \forall n \ge \eta_\varepsilon.$$

On the other hand, since $\lim_{n\to\infty} \tilde{G}_n(1) = 1$ uniformly on $\mathbb{R}^k$, we have

$$\exists m_\varepsilon \in \mathbb{N} \ni \big\|\tilde{G}_n(1) - 1\big\| \le \varepsilon, \quad \forall n \ge m_\varepsilon$$

and for every $n \in \mathbb{N}$, $\exists M > 0 \ni \big\|\tilde{G}_n(1)\big\| \le M$.
Then for every $n \ge \max\{n_\varepsilon, m_\varepsilon\}$, we get

$$\begin{aligned} D^*(G_n f, f) &\le \varepsilon M + \max\big\{\eta_\varepsilon, \eta_\varepsilon^{'}\big\} \frac{\varepsilon}{\max\big\{\eta_\varepsilon, \eta_\varepsilon^{'}\big\}} + \varepsilon D^*(f, \tilde{o}) \\ &= \varepsilon\big(M + 1 + D^*(f, \tilde{o})\big), \end{aligned}$$

which completes the proof. □

Example 1. Consider the Fejer Kernel $K_n^{(i)} : \mathbb{R}^k \to [0,\infty)$ for each $i = 1, 2, \ldots, k$ and any $n \in \mathbb{N}$ defined by

$$K_n^{(i)}(x) = \frac{1}{n}\left(\frac{\sin\frac{nx_i}{2}}{\sin\frac{x_i}{2}}\right)^2.$$

By using this kernel, fuzzy Fejer operator can be defined as follows:

$$(H_n(f))(u) = \frac{1}{(2\pi)^k n^k}(FR)\int_{[-\pi,\pi]^k} \left\{\begin{array}{l} \left(\frac{\sin\frac{nx_1}{2}}{\sin\frac{x_1}{2}}\right)^2 \cdots \left(\frac{\sin\frac{nx_k}{2}}{\sin\frac{x_k}{2}}\right)^2 \otimes \\ f(u_1 - x_1, \ldots, u_k - x_k)\,dx_1 \cdots dx_k, \end{array}\right\}$$

where $(FR)\int$ denotes the fuzzy Riemann integral (for details, see [2,5,6]).

Consider the separating trigonometric function, $\gamma : \mathbb{R}^k \times \mathbb{R}^k \to [0, \infty)$ with 2π-periodic according to each variable, defined as

$$\gamma(t, u) = \sin^2\left(\frac{t_1 - u_1}{2}\right) + cdots + \sin^2\left(\frac{t_k - u_k}{2}\right).$$

Observe that $\gamma(., u) \in C_{2\pi}(\mathbb{R}^k)$. From Corollary 3-(i) in [8] we have

$$\begin{aligned}
&\lim_{n\to\infty} (\tilde{H}_n(\gamma(., u)))(u) \\
&= \lim_{n\to\infty} \frac{1}{(2\pi)^k n^k} \int_{[-\pi,\pi]^k} \left\{\begin{array}{l} \left(\frac{\sin\frac{nx_1}{2}}{\sin\frac{x_1}{2}}\right)^2 \cdots \left(\frac{\sin\frac{nx_k}{2}}{\sin\frac{x_k}{2}}\right)^2 \\ \gamma((t_1 - x_1, \ldots, t_k - x_k), (u_1, u_2, \ldots, u_k)) \\ dx_1 \cdots dx_k \end{array}\right\} \\
&= \gamma(u, u) = 0
\end{aligned}$$

uniformly in $u \in \mathbb{R}^k$. At the same time it is clear that

$$\lim_{n\to\infty} (\tilde{H}_n(1)) = 1$$

uniformly on $\mathbb{R}^k$. This operator satisfies the conditions of Theorem 2. Also, another example can be given by using Jackson kernel.

3. Conclusion

In this paper, we extend the trigonometric fuzzy Korovkin theorem, which was previously introduced in [2], to the k-dimensional setting. The proof of the generalized theorem is established through an alternative approach that differs from the method employed in [2]. It is worth noting that, for $k = 1$, the proposed result coincides with Theorem 5 in [2]. Moreover, an illustrative example is presented to confirm both the applicability and validity of the generalized theorem.

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