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$\nu-$Wedge FDK-spaces

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Abstract. The (weak) wedgeness for FK-spaces was first defined by Bennett in 1974. Then, some results of Bennett (1974) were improved by İnce (2002) and Dağadur (2004) for all (weak) wedge FK spaces. In this paper, the concept of wedgeness for an FDK-space X containing Φ is defined, and some fundamental characterizations related to this space and compactness of the inclusion mapping are studied. Also, some results for a summability domain $X_A^{(\nu)}$ to be (weak) $\nu-$wedge are obtained. Moreover, necessary and sufficient conditions for some double sequence spaces are given.

1. Introduction

An important class of spaces (X,τ) with interesting applications in Schauder basis theory and summability theory is the family of wedge spaces introduced by Bennett [3]. Accordingly, let (X,τ) be a K-space. X is said to be a wedge space provided $\delta^n \to 0$ in τ; a weak wedge space if $\delta^n \to 0$ in $\sigma(X, X^*)$, where X^* is topological dual of X and $\sigma(X, X^*)$ is the weak topology on X. As a generalization of Bennett's results, İnce [11] studied (weak) Cesàro wedge FK spaces, Dağadur [6] continued to work on C_λ-wedge FK spaces and to give some characterizations. Also, as a generalization of the concepts of [11] and [6], Sezgek and Dağadur [12] studied the deferred wedge FK-spaces. These studies motivated us to define the concept of $\nu-$wedge FDK space by using $\nu-$convergence for double sequences, where ν represents one of the notions Pringsheim, bounded and regular convergence.

The set of all complex valued double sequences is denoted by Ω. The set Ω forms a vector space with coordinatewise addition and scalar multiplication. Any vector subspace of Ω is called as a double sequence space.

A subspace X of the vector space Ω is called DK-space, if all the semi-norms $r_{kl} : X \to \mathbb{R}, x \mapsto |x_{kl}|$ $(k,l \in \mathbb{N})$ are continuous. An FDK space is a

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DK-space with a complete, metrizable, locally convex topology. A normable FDK-space is called BDK-space.

The set of all continuous linear functionals on a space X will be denoted by X' and called the dual spaces of X. Also the definitions of α and f-duals of any subspaces X is given as follows

$$X^{\alpha} := \Big\{x = (x_{kl}) \ : \ \sum_{k,l=1}^{\infty,\infty} |x_{kl}y_{kl}| < \infty, \quad \forall y = (y_{kl}) \in X\Big\},$$
$$X^{f} := \Big\{(f(\delta^{kl})) \ : \ \forall f \in E'\Big\}.$$

We denote by (δ^{ij}), $i,j = 1,2,\dots$ the double sequence whose $(i,j)-$term is 1 and all other terms are 0, and e the sequence with all terms 1. Also,

$$\Phi := span\big\{e^{kl} \ : \ k,l \in \mathbb{N}\big\} \quad \text{and} \quad \Phi_1 := \Phi \cup \big\{e\big\}.$$

All bounded double sequences is defined as

$$\mathcal{M}_u := \Big\{x \in \Omega \ : \ \|x\|_{\infty} := \sup_{k,l} |x_{kl}| < \infty\Big\},$$

and it is a Banach space with the norm $\|.\|_{\infty}$. A double sequence $x = (x_{kl})$ of real or complex numbers is said to converge to a in Pringsheim's sense (shortly, p-converge to a) if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ : \ k,l > N \quad \Rightarrow \quad |x_{kl} - a| < \varepsilon.$$

If, in addition, $\sup_{k,l} |x_{kl}| < \infty$, or the limits $\lim_k x_{kl}$ $(l \in \mathbb{N})$ and $\lim_l x_{kl}$ $(k \in \mathbb{N})$ exist, then x said to be boundedly convergent to a in Pringsheim's sense (shortly, bp-converge to a) and regularly convergent to a (shortly, r-converge to a). Throughout the paper, for any notion of convergence ν, the space of all $\nu-$convergent double sequences is be denoted by $\mathcal{C}_{\nu}$. The set of all null sequences contained in the space $\mathcal{C}_{\nu}$ is denoted by $\mathcal{C}_{\nu 0}$. In addition we consider the spaces

$$\mathcal{L}_u := \Big\{x \in \Omega \ : \ \sum_{k,l} |x_{kl}| < \infty\Big\},$$
$$\mathcal{L}_{\varphi} := \Big\{x \in \mathcal{L}_u \ : \ \forall k \ (x_{kl})_l \in \varphi \text{ and } \forall l \ (x_{kl})_k \in \varphi\Big\},$$
$$\mathcal{BV} := \Big\{x \in \Omega \ : \ \|x\|_{\mathcal{BV}} := \sum_{k,l} |x_{kl} - x_{k+1,l} - x_{k,l+1} + x_{k+1,l+1}| < \infty\Big\},$$
$$\mathcal{BV}_0 := \Big\{x \in \mathcal{BV} \ : \ \forall k, \ \lim_{l} x_{kl} = 0 \text{ and } \forall l, \ \lim_{k} x_{kl} = 0\Big\},$$
$$\mathcal{BS} := \Big\{x \in \Omega \ : \ \sup_{m,n} \Big|\sum_{k,l}^{m,n} x_{kl}\Big| < \infty\Big\}.$$

A double sequence $x = (x_{kl})$ of real numbers is called almost convergent to a limit L if

$$p\text{-}\lim_{p,q\to\infty} \sup_{s,t\in\mathbb{N}} \Big| \frac{1}{(p+1)(q+1)} \sum_{k,l=s,t} x_{kl} - L \Big| = 0,$$

that is the average value of $x = (x_{kl})$ taken over any rectangle

$$\big\{(k,l) : s \leq k \leq s+p,\ t \leq l \leq t+q\big\}$$

tends to L as both p and q tend to ∞, and this convergence is uniform in $s, t \in \mathbb{N}$.

Recall that a metric space X is said to be compact if every sequence in X has a convergent subsequence. Moreover, A subset M of X is said to be relatively compact if the closure of M is compact.

Let us consider the isomorphism T defined by Zeltser [18] as

$$T : \Omega \to w, \quad x \to (z_i) := (x_{\psi^{-1}(i)}),$$

where $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is the bijection defined by

$$\begin{aligned}
&\psi[(1,1)] = 1,\\
&\psi[(1,2)] = 2, \quad \psi[(2,2)] = 3, \quad \psi[(2,1)] = 4,\\
&\qquad\qquad \vdots\\
&\psi[(1,n)] = (n-1)^2 + 1, \quad \psi[(2,n)] = (n-1)^2 + 2, \ldots\\
&\psi[(n,n)] = (n-1)^2 + n, \quad \psi[(n,n-1)] = n^2 - n + 2, \ldots\\
&\psi[(n,1)] = n^2,\\
&\qquad\qquad \vdots
\end{aligned}$$

It is clear that a double sequence $x = (x_{kl})$ is convergent to a, if the sequence $T(x)$ is convergent to a. Also, a double sequence space X is a (separable) FDK space if and only if the sequence space $T(X)$ is a (separable) FDK space [18].

Let E be a sequence space, then $x \in X$ is said to have AK(ν) if $x = \nu - \sum_{k,l} x_{kl}\delta^{kl}$. If each $x \in X$ has AK(ν), then X is called AK(ν)-space [17].

Let $A = (a_{mnkl})$ be any 4-dimensional scalar matrix and ν be some convergence notion of double sequences. We define

$$\Omega_A^{(\nu)} := \Big\{ x \in \Omega \mid \forall m, n \in \mathbb{N} \ : \ [Ax]_{mn} := \nu - \sum_{k,l} a_{mnkl} x_{kl} \text{ exists} \Big\}.$$

The map

$$A : \Omega_A^{(\nu)} \to \Omega,\ x \mapsto Ax := \big([Ax]_{mn}\big)_{m,n}$$

is called a matrix map of type ν. Summability domain of a four dimensional matrix $A=(a_{mnkl})$ is defined as

$$X_A^{(\nu)}=\big\{x\in\Omega\ :\ Ax \text{ exists and } Ax\in X\big\}.$$

Let $A=(a_{mnkl})$ be a four dimensional infinitive matrix. $\{a_{mnkl}\}_{k,l=1}^{\infty,\infty}$ is called the (k,l)-th row of the matrix A, say $\xi_A^{(mn)}$ and $\{a_{mnkl}\}_{m,n=1}^{\infty,\infty}$ is called the (k,l)-th column of the matrix A, say $\zeta_A^{(kl)}$.

Theorem 1 ([18])**.** *Let ν be some notion of convergence for double sequences such that $\mathcal{C}_\nu$ is an FDK space and let $\{t_k : k\in\mathbb{N}\}$ be a system of seminorms, defining the FDK topology of $\mathcal{C}_\nu$. Let $A=(a_{mnkl})$ be a four dimensional matrix and E be an FDK space with the FDK topology generated by a system of seminorms $\{\varrho_k : k\in\mathbb{N}\}$.*

(i) *The space $E_A^{(\nu)}$ is an FDK space and the FDK topology is generated by the system of seminorms*

$$\big\{r_{mn} : m,n\in\mathbb{N}\big\}\cup\big\{t_r\circ A_{mn} : r,m,n\in\mathbb{N}\big\}\cup\big\{\varrho_r\circ A : r\in\mathbb{N}\big\},$$

where

$$A_{mn}(x):=\left(\sum_{k=1}^{s}\sum_{l=1}^{t}a_{mnkl}x_{kl}\right)_{s,t},\quad x\in E_A^{(\nu)}.$$

(ii) *The topological dual $(E_A^{(\nu)})'$ consist of all linear functionals f of the form*

$$f(x)=g(x)+h(Ax),\quad x\in E_A^{(\nu)}$$

with certain $g\in(\Omega_A^{(\nu)})'$ and $h\in E'$.

(iii) *If $\mathcal{C}_\nu$ and E are separable, then $E_A^{(\nu)}$ is separable.*

2. Main results

In this section, the (weak) $\nu-$wedge FDK spaces are introduced and some characterizations related to those spaces and compactness of inclusion mapping are examined.

Definition 1. Let $(X,\tau)\supset\Phi$ be a DK-space. (X,τ) is called a $\nu-$wedge FDK space, if the sequence (δ^{ij}) is $\nu-$convergent to 0 in τ.

Definition 2. Let $(X,\tau)\supset\Phi$ be a DK-space. (X,τ) is called a weak $\nu-$wedge FDK space, if the sequence (δ^{ij}) is weak $\nu-$convergent to 0 in τ.

Definition 3. Let $s = (s_m)$, $t = (t_n)$ denote throughout strictly increasing sequences of nonnegative integers with $s_1 = 0$, $t_1 = 0$. Let $m|(s,t)|$ designate the space define by

$$m|(s,t)| = \left\{ x \in \Omega \quad : \quad \sup_{m,n} \sum_{\substack{k=s_m+1 \\ l=t_n+1}}^{s_{m+1},t_{n+1}} |x_{kl}| < \infty \right\}.$$

Then, $m|(s,t)|$ is a BDK-space under the follwing norm

$$x \to \sup_{m,n} \sum_{\substack{k=s_m+1 \\ l=t_n+1}}^{s_{m+1},t_{n+1}} |x_{kl}|.$$

Recall that, A sequence space X is called normal (or solid) if $y \in X$ whenever $|y_n| \leq |x_n|$, $n \geq 1$, for some $x \in X$. It is obvious that $m|(s,t)|$ is always solid.

Now we give some lemmas which help to prove the following theorems.

Lemma 1. *If $z^{(mn)} \in \mathcal{C}_{\nu 0}$, $(m,n = 1,2,\ldots)$, then we can choose an $z \in \mathcal{C}_{\nu 0}$ such that*

$$\lim_{i,j\to\infty} \frac{z_{ij}^{(mn)}}{z_{ij}} = 0, \quad m,n = 1,2,ldots$$

Furthermore, $z^\alpha \subseteq \bigcap_{m,n=1}^{\infty,\infty} \{z^{(mn)}\}^\alpha$ for any such z.

Proof. Let $z^{(mn)} \in \mathcal{C}_{\nu 0}$. It is easy to see that we can find two sequences (i_k), (j_l) of positive integers such that

$$1 = i_0 < i_1 < i_2 < \cdots, \quad 1 = j_0 < j_1 < j_2 < \cdots$$

and for $i \geq i_k$, $j \geq j_l$, $k,l = 1,2,\ldots$

$$\max_{\substack{1\leq m\leq k \\ 1\leq n\leq l}} |z_{ij}^{(mn)}| < \frac{1}{4^{kl}}.$$

For $i_k \leq i < i_{k+1}$, $j_l \leq j < j_{l+1}$, $k,l = 0,1,2,\ldots$, we can define $z \in \Omega$ as follows

$$z_{ij} = \frac{1}{2^{kl}}.$$

Clearly, $z \in \mathcal{C}_{\nu 0}$, $i \geq i_k$, $j \geq j_l$, $k \geq m$, $l \geq n$ and any fixed m,n

$$\left| \frac{z_{ij}^{(mn)}}{z_{ij}} \right| < \frac{1}{2^{kl}}.$$

Thus, $\lim_{i,j\to\infty} \frac{z_{ij}^{(mn)}}{z_{ij}} = 0$, for all m,n.

Now let $x \in z^{\alpha}$, then

$$\sum_{i,j=1}^{\infty,\infty} |x_{ij} z_{ij}| < \infty.$$

Moreover, for $m, n = 1, 2, \ldots$, we have

$$\sum_{i,j=1}^{\infty,\infty} |x_{ij} z_{ij}^{(mn)}| < \sum_{i,j=1}^{\infty,\infty} |x_{ij} z_{ij}| \frac{1}{2^{kl}} < \frac{1}{2^{kl}} \sum_{i,j=1}^{\infty,\infty} |x_{ij} z_{ij}| < \infty.$$

Hence, $x \in \bigcap_{m,n=1}^{\infty,\infty} \{z^{(mn)}\}^{\alpha}$. □

Lemma 2. *Let M denote a subset of an FDK-AK(ν)-space $X \supset \Phi$ such that M is coordinatewise bounded, and $x^{(mn)} \to x$ uniformly in M. Then any sequence $(x^{kl}) \subset M$ has a Cauchy subsequence.*

Proof. Let $\varepsilon > 0$ and consider any continuous seminorm q on X. If $x^{(mn)} \to x$ uniformly on M, there exist positive integers m_0, n_0 such that

$$\sup_{k,l} \, q\left(x^{(kl)} - x_{kl}^{(m_0 n_0)}\right) < \frac{\varepsilon}{3}.$$

On the other hand, since $\left(x_{kl}^{(m_0 n_0)}\right)$ is a bounded sequence in an $m_0 n_0$-dimensional subspace of X, it has a convergent subsequence, say $\left(y_{kl}^{(m_0 n_0)}\right)$. So, there exist k_0, l_0 such that

$$q\left(\left(y_{ij}^{(m_0 n_0)}\right) - \left(y_{kl}^{(m_0 n_0)}\right)\right) < \frac{\varepsilon}{3},$$

are obtained whenever $i, k \geq k_0$, $j, l \geq l_0$. From the triangle inequality, we have

$$\begin{aligned} q\left(y^{(ij)} - y^{(kl)}\right) &= q\left(y_{ij}^{(m_0 n_0)} - y^{(ij)} - y_{ij}^{(m_0 n_0)} + y_{kl}^{(m_0 n_0)} - y^{(kl)} - y_{kl}^{(m_0 n_0)}\right) \\ &< q\left(y_{ij}^{(m_0 n_0)} - y^{(ij)}\right) + q\left(y_{ij}^{(m_0 n_0)} + y_{kl}^{(m_0 n_0)}\right) \\ &\quad + q\left(y^{(kl)} - y_{kl}^{(m_0 n_0)}\right) \\ &< \varepsilon. \end{aligned}$$

Therefore, (y^{kl}) is a Cauchy subsequence of (x^{kl}).

This complete the proof. □

Theorem 2. *Let (X, τ) be an FDK-space. The following statements are equivalent*

(i) *X is a $\nu-$wedge FDK space,*
(ii) *$X \supset z^{\alpha}$ for some $z \in \mathcal{C}_{\nu 0}$,*
(iii) *$X \supset m|(s,t)|$ for some s, t and the inclusion mapping $I : m|(s,t)| \to X$ is compact,*
(iv) *$X \supset \mathcal{L}_{\varphi}$ and the inclusion mapping $I : \mathcal{L}_{\varphi} \to X$ is compact.*

Proof. ($i \Rightarrow ii$) Let the topology τ be generated by seminorms $\{q_{mn}\}$ and let $z^{(mn)} \in \mathcal{C}_{\nu 0}$ be defined by

$$z_{ij}^{(mn)} = q_{mn}(\delta^{ij}), \qquad m, n, i, j = 1, 2, \ldots.$$

Suppose $y \in \bigcap\limits_{m,n=1}^{\infty,\infty} \{z^{(mn)}\}^{\alpha}$. Then

$$\sum_{i,j} |y_{ij} z_{ij}^{(mn)}| < \infty,$$

for all m, n. Therefore,

$$\sum_{i,j} |y_{ij} q_{mn}(\delta^{ij})| = \sum_{i,j} q_{mn}(y_{ij}\delta^{ij}) < \infty$$

is obtained. Since the space X is complete $\sum\limits_{i,j} y_{ij}\delta^{ij}$ converges in (X, τ) to, say x, or $y^{(mn)} \to x$. Thus $y_{ij}^{(mn)} \to x_{ij}$ for every i, j, also we always have $y_{ij}^{(mn)} \to y_{ij}$ for every i, j. Consequently $y = x$. That gives us

$$\bigcap_{m,n=1}^{\infty,\infty} \left\{z^{(mn)}\right\}^{\alpha} \subseteq X.$$

Choosing z as in Lemma 1, (ii) follows.

($ii \Rightarrow iii$) Let us choose strictly increasing sequences (s_m), (t_n) of positive integers such that $s_1 = 0$, $t_1 = 0$ and

$$|z_{ij}| \leq \frac{1}{2^{mn}}$$

whenever $i \geq s_m$, $j \geq t_n$, $m, n \geq 2$.

For $x \in m|(s,t)|$ and any positive integers k, l, u, v such that $l \geq k$ and $v \geq u$ we have

$$\sum_{\substack{i=s_k+1 \\ j=t_p+1}}^{s_{l+1}, t_{v+1}} |x_{ij} z_{ij}| = \sum_{m=k,n=u}^{l,v} \sum_{\substack{i=s_m+1 \\ j=t_n+1}}^{s_{m+1}, t_{n+1}} |x_{ij} z_{ij}| \leq \|x\| \sum_{m=k,n=u}^{l,v} \frac{1}{2^{mn}}.$$

Hence $x \in z^{\alpha}$. That is, $m|(s,t)| \subseteq z^{\alpha} \subseteq X$. Also, the mapping

$$i : (m|(s,t)|, \|.\|) \to (X, \tau)$$

is compact.

($iii \Rightarrow iv$) $\mathcal{L}_{\varphi} \subset m|(s,t)|$ and so, $i : \mathcal{L}_{\varphi} \to m|(s,t)|$ is continuous. Consequently, $I : \mathcal{L}_{\varphi} \to X$ is compact, since $\mathcal{L}_{\varphi} \subset m|(s,t)| \subset X$ and $I : (m|(s,t)|, \|.\|) \to (X, \tau)$ is compact.

($iv \Rightarrow i$) From (iv), the set $A = \{\delta^{kl} : k, l = 1, 2, \ldots\}$ is a bounded subset of $\mathcal{L}_{\varphi}$. Since $I : \mathcal{L}_{\varphi} \to X$ is compact, the set $I(A) = A$ is relatively compact. So the coordinatwise convergence topology and the topolog τ is coincide.

Hence, according to the coordinatwise convergence topology generated by the seminorms $r_{mn}(x) = |x_{mn}|$, $m, n = 1, 2, \dots$, we get

$$r_{mn}(\delta^{kl}) = \begin{cases} 1, & (k,l) = (m,n); \\ 0, & \text{otherwise.} \end{cases}$$

Since $r_{mn}(\delta^{kl}) \to 0$, $\delta^{kl} \to 0$ $(k, l \to \infty)$ in (X, τ), which means that X is a $\nu-$wedge FDK space. □

Theorem 3. *A $\nu-$wedge FDK space X contains a bounded double sequence that is not almost convergent.*

Proof. Since X is a $\nu-$wedge FDK space, $z^\alpha \subseteq X$ for $z \in \mathcal{C}_{\nu 0}$ by Theorem 2. If we insert sufficiently many zeros in the following double sequence

$$\begin{pmatrix}
1 & 0\cdots0 & 1 & 1 & 0\cdots0 & 1 & 1 & 1 & 0 & \cdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \\
\vdots & \ddots & \vdots & & & & & & & \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \\
1 & 0\cdots0 & 1 & 1 & 0\cdots0 & 1 & 1 & 1 & 0 & \cdots \\
1 & 0\cdots0 & 1 & 1 & 0\cdots0 & 1 & 1 & 1 & 0 & \cdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0\cdots0 & 1 & 1 & 0\cdots0 & 1 & 1 & 1 & 0 & \cdots \\
1 & 0\cdots0 & 1 & 1 & 0\cdots0 & 1 & 1 & 1 & 0 & \cdots \\
1 & 0\cdots0 & 1 & 1 & 0\cdots0 & 1 & 1 & 1 & 0 & \cdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},$$

we can obtain a bounded double sequence in z^α that is not almost convergent. □

The following result demonstrates that the space obtained in the intersection can vary depending on the chosen notion of convergence.

Corollary 1. (i) *Let X_n be $p-$wedge FDK-spaces. Then $\bigcap X_n = \mathcal{L}_\varphi$.*
(ii) *Let X_n be $\nu-$wedge FDK-spaces for $\nu \in \{bp, r\}$. Then $\bigcap X_n = \mathcal{L}_u$.*

Proof. Let X be a $\nu-$wedge FDK-space. So we get

$$\bigcap X = \bigcap \{z^\alpha \ : \ z \in \mathcal{C}_{\nu 0}\} = \mathcal{C}_{\nu 0}^\alpha = \begin{cases} \mathcal{L}_\varphi, & \nu = p; \\ \mathcal{L}_u, & \nu \in \{bp, r\}. \end{cases}$$ □

Theorem 4. *If $X = m|(s,t)|$, then $(X, \tau(X, X^\alpha))$ is a $\nu-$wedge FDK space.*

Proof. Let $X = m|(s,t)|$ and assume that the topological space $(X, \tau(X, X^\alpha))$ is not a $\nu-$wedge FDK space. Then for some $\tau(X, X^\alpha)$-neighborhood $\mathcal{U}$ of zero and increasing sequences $\{i_k\}$, $\{j_l\} \subset \mathbb{N}$

$$\delta^{i_k j_l} \notin \mathcal{U}, \quad k, l \geq 1.$$

We can choose a subset of $\{i_k\}$ as

$$s_1 < i_1 \leq s_2 < i_2 \leq s_3 < \cdots \quad (\text{that is } i_k \in (s_k, s_{k+1}],\ k \geq 1)$$

and a subset of $\{j_l\}$ as

$$t_1 < j_1 \leq t_2 < j_2 \leq t_3 < \cdots \quad (\text{that is } j_l \in (t_l, t_{l+1}],\ l \geq 1).$$

Now, if we take

$$x_{ij} = \begin{cases} 1, & i = i_k,\ j = j_l,\ k,l \geq 1; \\ 0, & \text{otherwise}; \end{cases}$$

then $x \in X$, $||x|| = 1$. Since X is solid

$$\sum_{k,l \geq 1} \delta^{i_k j_l} = x.$$

The convergence is related to $\tau(X, X^\alpha)$. Hence, $\delta^{i_k j_l} \in \mathcal{U}$ $(\forall k, l)$. This contradicts the assumption.

Thus, the proof is completed. □

Theorem 5. *$(\mathcal{BS}, \tau(\mathcal{BS}, \mathcal{BV}_0))$ is a $\nu-$wedge FDK space.*

Proof. If we show that $(\mathcal{M}_u, \tau(\mathcal{M}_u, \mathcal{L}_u))$ is topologically isomorphic to $(\mathcal{BS}, \tau(\mathcal{BS}, \mathcal{BV}_0))$, we have the desired result. Firstly, let us consider the surjection mapping $S^{(2)}$ of Ω to itself

$$S^{(2)}x = \begin{pmatrix} x_{11} & x_{11}+x_{12} & \cdots & \cdots & \cdots \\ x_{11}+x_{21} & \begin{Bmatrix} x_{11}+x_{12}+ \\ x_{21}+x_{22} \end{Bmatrix} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ \cdots & \cdots & \cdots & \sum_{k,l=1}^{m,n} x_{kl} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$(S^{(2)})^{-1}x = \begin{pmatrix} x_{11} & x_{12}-x_{11} & \vdots & \cdots & \cdots \\ x_{21}-x_{11} & \begin{Bmatrix} x_{22}-x_{12}- \\ x_{21}+x_{11} \end{Bmatrix} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ \cdots & \cdots & \cdots & \begin{Bmatrix} x_{mn}-x_{m,n-1}- \\ x_{m-1,n}+x_{m-1,n-1} \end{Bmatrix} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Clearly, $(S^{(2)})^{-1} : \mathcal{M}_u \to \mathcal{BS}$.

Take an arbitrary $y \in \mathcal{BV}_0$ and define $u \in \mathcal{L}_u$ as

$$u_{ij} = y_{ij} - y_{i,j+1} - y_{i+1,j} + y_{i+1,j+1}, \qquad i,j \geq 1.$$

Since $x \in \mathcal{M}_u$,

$$\begin{aligned}\langle (S^{(2)})^{-1}x, y\rangle &= \sum_{i,j\geq 1}(x_{ij} - x_{i-1,j} - x_{i,j-1} + x_{i-1,j-1})y_{ij}\\ &= \sum_{i,j\geq 1}(y_{ij} - y_{i,j+1} - y_{i+1,j} + y_{i+1,j+1})x_{ij}\\ &= \sum_{i,j\geq 1} u_{ij}x_{ij} = \langle x, u\rangle.\end{aligned}$$

Thus the mapping $(S^{(2)})^{-1} : (\mathcal{M}_u, \sigma(\mathcal{M}_u, \mathcal{L}_u)) \to (\mathcal{BS}, \sigma(\mathcal{BS}, \mathcal{BV}_0))$ is continuous. Consequently, $(S^{(2)})^{-1}$ is $\tau(\mathcal{M}_u, \mathcal{L}_u)-\tau(\mathcal{BS}, \mathcal{BV}_0)$ continuous. Similarly, we can show that $S^{(2)} : (\mathcal{BS}, \tau(\mathcal{BS}, \mathcal{BV}_0)) \to (\mathcal{M}_u, \tau(\mathcal{M}_u, \mathcal{L}_u))$ is continuous.

Now

$$(S^{(2)})^{-1}(e - e^{(mn)}) = \delta^{m+1,n+1},$$

and since $e^{(mn)} \to e$ in $\tau(\mathcal{M}_u, \mathcal{L}_u)$, then $\delta^{m+1,n+1} \to 0$ in $\tau(\mathcal{BS}, \mathcal{BV}_0)$.

The proof is completed. □

Theorem 6. *Let (X, τ) be a separable FDK-space and $\mathcal{BS} \subset X$. Then (X, τ) is a $\nu-$wedge FDK space.*

Proof. Let the surjection map $F^{(2)} : \mathcal{L}_u \to \mathcal{BV}_0$ be defined as $F^{(2)}(x) = \left\{\sum_{(i,j)\geq(m,n)} x_{ij}\right\}$. Choose an arbitrary $x \in \mathcal{L}_u$ and for $y \in \mathcal{BS}$ define $u \in \mathcal{M}_u$ such that $\sum_{(i,j)=(1,1)}^{(m,n)} y_{ij}$. Then

$$\begin{aligned}\langle F^{(2)}(x), y\rangle = y_{11} &\sum_{(i,j)\geq(1,1)} x_{ij} + y_{12} \sum_{(i,j)\geq(1,2)} x_{ij} + \cdots\\ &+ \ y_{21} \sum_{(i,j)\geq(2,1)} x_{ij} + y_{22} \sum_{(i,j)\geq(2,2)} x_{ij} + \cdots\\ &\vdots\\ &+ \ y_{m1} \sum_{(i,j)\geq(m,1)} x_{ij} + \ldots + y_{mn} \sum_{(i,j)\geq(m,n)} x_{ij} + \cdots\\ &+ \ \cdots\\ &= y_{11}x_{11} + (y_{11} + y_{12})x_{12} + \ldots + (\sum_{(i,j)=(1,1)}^{(m,n)} y_{ij})x_{mn} \cdots\\ &= \langle x, u\rangle.\end{aligned}$$

Thus, $F^{(2)}$ is $\sigma(\mathcal{L}_u, \mathcal{M}_u) - \sigma(\mathcal{BV}_0, \mathcal{BS})$ continuous.

Similarly, the inclusion mapping

$$(F^{(2)})^{-1} : \mathcal{BV}_0 \to \mathcal{L}_u,$$
$$(F^{(2)})^{-1}(x) = x_{mn} - x_{m+1,n} - x_{m,n+1} + x_{m+1,n+1}$$

is $\sigma(\mathcal{BV}_0, \mathcal{BS}) - \sigma(\mathcal{L}_u, \mathcal{M}_u)$ continuous. Since $\mathcal{M}_u$ is a solid sequence space, the space $\mathcal{BV}_0$ is $\sigma(\mathcal{BV}_0, \mathcal{BS})$-sequentially complete.
The graph of the identity mapping $I : (\mathcal{BS}, \tau(\mathcal{BS}, \mathcal{BV}_0)) \to (X, \tau)$ is closed. Hence I is $\tau(\mathcal{BS}, \mathcal{BV}_0) - \tau$ continuos.

From Theorem 5, the proof is completed. □

Theorem 7. *Let X be an FDK space such that $X^f \subset \mathcal{BS}$. In that case, X is a weak $\nu-$wedge FDK space if and only if $\mathcal{L}_\varphi \subset X$ and the inclusion mapping is weakly compact.*

Proof. Necessary. Let (X, τ) be a weak $\nu-$wedge FDK space. Then, $\delta^{kl} \to 0$ $(k, l \to \infty)$ in τ. That is, $f(\delta^{kl}) \to 0$ $(k, l \to \infty)$ for each $f \in E^f$. From hypothesis, $(f(\delta^{kl})) \in \mathcal{M}_u$ is obtained. Hence $\{\delta^{kl}\}$ is weakly bounded. Since weakly boundedness is equivalent boundedness in weak topology, $\{\delta^{kl}\}$ is bounded. Then $X^f \subset \mathcal{M}_u = \mathcal{L}_u \subset \mathcal{L}_\varphi$ is obtained. So $\mathcal{L}_\varphi \subset X$ and the inclusion mapping $I : \mathcal{L}_\varphi \to X$ is continuous. Thus, for $f \in X'$ and $x \in \mathcal{L}_\varphi$, we have

$$f(x) = f\left(\sum_{k,l=1}^{\infty} x_{kl}\delta^{kl}\right) = \sum_{k,l=1}^{\infty} x_{kl} f(\delta^{kl}).$$

From hypothesis, we get $\{f(\delta^{kl})\} \in \mathcal{C}_{\nu 0}$. So $I : \sigma(\mathcal{L}_\varphi, \mathcal{C}_{\nu 0}) \to \sigma(X, X')$ is continuous. Since the unit ball $\{x \ : \ x \in \mathcal{L}_\varphi, ||x|| \leq 1\}$ is $\sigma(\mathcal{L}_\varphi, \mathcal{C}_{\nu 0})$-compact, it is $\sigma(X, X')$-compact.

Sufficient. By hypothesis, the unit ball in $\mathcal{L}_\varphi$ is $\sigma(X, X')$-compact (relatively) in X. Thus $\delta^{kl} \to 0$ $(k, l \to \infty)$ in $\sigma(X, X')$.

This proves the theorem. □

Corollary 2. (i) *Let X_n be weak $p-$wedge FDK-spaces. Then $\bigcap X_n = \mathcal{L}_\varphi$.*

(ii) *Let X_n be weak $\nu-$wedge FDK-spaces for $\nu \in \{bp, r\}$. Then $\bigcap X_n = \mathcal{L}_u$.*

Proof. Let Z be the intersection of all weak $\nu-$wedge FDK-space. From Theorem 7, $\mathcal{L}_\varphi \subset Z$. From Theorem 2, for $z \in \mathcal{C}_{\nu 0}$, z^α is a $\nu-$wedge FDK-space and is also a weak $\nu-$wedge FDK-space. Thus we obtain the following

$$\bigcap X = \bigcap \{z^\alpha \ : \ z \in \mathcal{C}_{\nu 0}\} = \mathcal{C}^\alpha_{\nu 0} = \begin{cases} \mathcal{L}_\varphi, & \nu = p; \\ \mathcal{L}_u, & \nu \in \{bp, r\}. \end{cases}$$ □

Theorem 8. *Let X be FDK-space.*

(i) *If X contains a $\nu-$wedge FDK-space, then it is a $\nu-$wedge FDK space.*

(ii) *If X is a $\nu-$wedge space, then every closed subspace, containing Φ, of X is a $\nu-$wedge FDK space.*

(iii) *A countable intersection of $\nu-$wedge FDK-space is a $\nu-$wedge FDK space.*

Proof. The proof can obained from fundamental properties of FDK spaces (e.g. [15]). So, we omit this proof. □

3. APPLICATIONS OF $\nu-$WEDGE FDK SPACES TO SUMMABILITY DOMAINS

In this section, we shall apply some of the theorems of the main results section of the paper to summability domains.

Theorem 9. *Let $\mathcal{C}_\nu$ and X be FDK-spaces and $A = (a_{mnkl})$ be a four dimensional matrix. Then the following statements are equivalent*

(i) $X_A^{(\nu)}$ *is a $\nu-$wedge FDK-space;*

(ii) $\mathcal{L}_\varphi \subset X_A^{(\nu)}$, $\xi_A^{(mn)} \in \mathcal{C}_{\nu 0}$ $(m,n = 1,2\ldots)$ *and* $A : \mathcal{L}_\varphi \to X$ *is compact;*

(iii) $\zeta_A^{(kl)} \in X$ $(k,l = 1,2,\ldots)$ *and* $\zeta_A^{(kl)} \to 0$ *in* X.

Proof. $(i \Rightarrow ii)$ Let $X_A^{(\nu)}$ is a $\nu-$wedge FDK-space. By Theorem 2, we get $\mathcal{L}_\varphi \subset X_A^{(\nu)}$ and $I : \mathcal{L}_\varphi \to X_A^{(\nu)}$ is compact. Since the matrix transformations between FDK-spaces is continuous, $A : X_A^{(\nu)} \to X$ is continuous. Thus, the mapping $A = A \circ I : \mathcal{L}_\varphi \to X_A^{(\nu)} \to X$ is compact. Because $X_A^{(\nu)}$ is $\nu-$wedge and $A : X_A^{(\nu)} \to X$ is continuous, we obtain $A(\delta^{kl}) \to 0$ $(k,l \to \infty)$ in X. Since coordinate functionals of X are continuous, we have

$$P_{mn}(A(\delta^{kl})) = (a_{mnkl})_{kl} \to 0, \quad k,l \to \infty.$$

That is, $\xi_A^{(mn)} \in \mathcal{C}_{\nu 0}$ $(m,n = 1,2,\ldots)$.

$(ii \Rightarrow iii)$ Suppose that $\mathcal{L}_\varphi \subset X_A^{(\nu)}$. Since $\delta^{kl} \in \mathcal{L}_\varphi \subset X_A^{(\nu)}$, we get

$$\zeta_A^{(kl)} = A(\delta^{kl}) \in X, \quad k,l = 1,2,\ldots$$

The set $\{\delta^{kl} : k,l = 1,2,\ldots\}$ is bounded in $\mathcal{L}_\varphi$, it is also bounded in $X_A^{(\nu)}$. $\mathcal{L}_\varphi \subset X_A^{(\nu)}$ if and only if $A \in (\mathcal{L}_\varphi, X)$. This is equivalent that the columns of A is a bounded set in X ([8]). Also, since $A : \mathcal{L}_\varphi \to X$ is compact, the set $K = \{A(\delta^{kl}) : k,l = 1,2,\ldots\}$ is relatively compact in X. Hence, the coordinatwise convergence topology in K and the topology of X are coincident. On the other hand, since $\xi_A^{(mn)} \in \mathcal{C}_{\nu 0}$, we have $A(\delta^{kl}) \to 0$ $(k,l = 1,2,\ldots)$ and so we get $A(\delta^{kl}) = (a_{mnkl})_{mn} = \zeta_A^{(kl)} \in X$. Thus, we obtain $K = \{A(\delta^{kl}) :\ k,l = 1,2,\ldots\} \to 0$ $(k,l = 1,2,\ldots)$.

($iii \Rightarrow i$) Let the columns of A in X be convergent to zero. So $\Phi \in X_A^{(\nu)}$. From Theorem 1, $X_A^{(\nu)}$ is an FDK-spaces with the seminorms $\{r_{mn} : m,n \in \mathbb{N}\} \cup \{t_r \circ A_{mn} : r,m,n \in \mathbb{N}\} \cup \{\varrho_r \circ A : r \in \mathbb{N}\}$. Since

$$r_{mn}(\delta^{kl}) \to 0, \quad k,l \to \infty;$$
$$(t_r \circ A_{mn})(\delta^{kl}) = t_r(A_{mn}(\delta^{kl}) = t_r(\zeta_A^{(kl)}) \to 0, \quad k,l \to \infty;$$
$$(\varrho_r \circ A)(\delta^{kl}) = \varrho_r(\zeta_A^{(kl)}) \to 0, \quad k,l \to \infty;$$

we obtain that $X_A^{(\nu)}$ is a $\nu-$wedge FDK space. □

Theorem 10. *Let $\mathcal{C}_\nu$ and X be FDK-spaces, $X^f \subset \mathcal{BS}$ and $A = (a_{mnkl})$ be a four dimensional matrix. The following statements are equivalent*

(i) $X_A^{(\nu)}$ *is a weak $\nu-$wedge FDK-space;*
(ii) $\mathcal{L}_\varphi \subset X_A^{(\nu)}$, $\xi_A^{(mn)} \in \mathcal{C}_{\nu 0}$ $(m,n = 1,2\ldots)$ *and* $A : \mathcal{L}_\varphi \to X$ *is compact;*
(iii) $\zeta_A^{(kl)} \in X$ $(k,l = 1,2,\ldots)$ *and* $\zeta_A^{(kl)} \to 0$ *(weakly) in* X.

Proof. ($i \Rightarrow ii$) Let X be a weak $\nu-$wedge FDK-spaces. Then, $\mathcal{L}_\varphi \subset X_A^{(\nu)}$ and $A : \mathcal{L}_\varphi \to X_A^{(\nu)}$ is weakly compact by $X^f \subset \mathcal{BS}$. Since the matrix mapping between FDK spaces is continuous, $A : X_A^{(\nu)} \to X$ is continuous. Thus, the mapping $A = A \circ I : \mathcal{L}_\varphi \to X_A^{(\nu)} \to X$ is weakly compact. So, $A(\delta^{kl}) \to 0$ (weakly) $(k,l \to \infty)$. Since the coordinate functionals of FDK space X are continuous, we get

$$P_{mn}(A(\delta^{kl})) = (a_{mnkl})_{kl} \to 0, \quad k,l \to \infty.$$

That means $\xi_A^{(mn)} \in \mathcal{C}_{\nu 0}$ $(m,n = 1,2,\ldots)$.

($ii \Rightarrow iii$) Assume that $\mathcal{L}_\varphi \subset X_A^{(\nu)}$. Since $\delta^{kl} \in \mathcal{L}_\varphi \subset X_A^{(\nu)}$, we get $\zeta_A^{(kl)} = A(\delta^{kl}) \in X$ $(k,l = 1,2,\ldots)$ and $\zeta_A^{(kl)}$ is bounded in $X_A^{(\nu)}$. Hence, the convergence topology in K and the weak topology of X are coincident. Consequently, since $\xi_A^{(mn)} \in \mathcal{C}_{\nu 0}$, we get $A(\delta^{kl}) \to 0$ $(k,l \to \infty)$ in X.

($iii \Rightarrow i$) Let $\zeta_A^{(kl)} \in X$ $(k,l = 1,2,\ldots)$ and $\zeta_A^{(kl)} \to 0$ (weakly) in X. Then, for all $f \in (X_A^{(\nu)})'$, $f(x) = g(x) + h(Ax)$, $\big(x \in X_A^{(\nu)}, g \in (\Omega_A^{(\nu)})', h \in X'\big)$ and for $M > 0$,

$$|\alpha x| \le M \sup_s \left| \sum_{l=1}^{s} \sum_k a_{mnkl} x_{kl} \right| = M\ell_{mn}(x), \quad m,n = 1,2,\ldots$$

So, we obtain the following inequality

$$|\alpha \delta^{ij}| \le M\ell_{mn}(\delta^{ij}), \quad m,n = 1,2,\ldots$$

Since $a_{mnij} \to 0$ (weakly) $(m, n = 1, 2, \ldots)$ in X, we get $P_{mn}(a_{mnij}) = (a_{mnij})_{mn} \to 0$ $(i, j \to \infty)$ $(m, n = 1, 2, \ldots)$. Thus, $\ell_{mn}(\delta^{ij}) \to 0$ $(i, j \to \infty)$ $(m, n = 1, 2, \ldots)$ and so we get $g(\delta^{ij}) \to 0$ $(i, j \to \infty)$.

Now $f(\delta^{ij}) = g(\delta^{ij}) + h(A(\delta^{ij}))$, for $x \in X_A^{(\nu)}, g \in (\Omega_A^{(\nu)})', h \in X'$. And by hypothesis, $h(A(\delta^{ij})) \to 0$ $(i, j \to \infty)$ for all $h \in X'$. So we get $f(\delta^{ij}) \to 0$ $(i, j \to \infty)$ for all $f \in (X_A^{(\nu)})'$, which completes the proof. □

Theorem 11. *Let A be a four dimensional matrix. If weak convergence and strong convergence coincide in an FDK space X, it is equivalent for the space $X_A^{(\nu)}$ to be $\nu-$wedge FDK space and weak $\nu-$wedge FDK space.*

Proof. Let $X_A^{(\nu)}$ be a $\nu-$wedge FDK space. Then the columns of A is in X and $\{A\left(e - \sum \delta^{kl}\right)\} \to 0$. By hypothesis, the columns of A is weakly convergent in X. That is, $X_A^{(\nu)}$ is a weak $\nu-$wedge FDK space. □

For example, if we choose $X = \mathcal{L}_u$, $\mathcal{M}_u$ and $\mathcal{BV}$, we obtain the following results by Theorem 11, since weak convergence and strong convergence coincide in these spaces.

Theorem 12. *Let A be a four dimensional matrix.*

(i) $(\mathcal{L}_u)_A$ *is a (weak) $\nu-$wedge FDK space if and only if*

$$\lim_{k,l} \sum_{m,n} |a_{mnkl}| = 0.$$

(ii) $(\mathcal{M}_u)_A$ *is a (weak) $\nu-$wedge FDK space if and only if*

$$\text{a)} \quad \sup_{m,n} |a_{mnkl}| < \infty \quad (k, l = 1, 2, \ldots),$$

$$\text{b)} \quad \lim_{k,l} \sup_{m,n} |a_{mnkl}| = 0.$$

(iii) $\mathcal{BV}_A$ *is a (weak) $\nu-$wedge FDK space if and only if*

$$\lim_{k,l} \sum_{m,n} |a_{mnkl} - a_{m+1,n,k,l} - a_{m,n+1,k,l} + a_{m+1,n+1,k,l}| = 0.$$

Proof.

(i) Let $X_A^{(\nu)} = (\mathcal{L}_u)_A$ in Theroem 9. Then $(\mathcal{L}_u)_A$ is a (weak) $\nu-$wedge FDK-space if and only if the columns of A in $\mathcal{L}_u$ and converge to zero. So we obtain $A(\delta^{kl}) = (a_{mnkl})_{mn} = \zeta_A^{(kl)} \in \mathcal{L}_u$ and

$$\lim_{k,l} \left\| (A(\delta^{kl}))_{mn} \right\|_{\mathcal{L}_u} = 0 \quad \Leftrightarrow \quad \lim_{k,l} \sum_{m,n} |a_{mnkl}| = 0.$$

(ii) Let $X_A^{(\nu)} = (\mathcal{M}_u)_A$ in Theroem 9. Then $(\mathcal{L}_u)_A$ is a (weak) $\nu-$wedge FDK-space if and only if the columns of A in $\mathcal{M}_u$ and converge to zero. So we obtain $A(\delta^{kl}) = (a_{mnkl})_{mn} = \zeta_A^{(kl)} \in \mathcal{M}_u$, i.e., $\sup_{m,n} |a_{mnkl}| < \infty$ and $\lim_{k,l} \sup_{m,n} |a_{mnkl}| = 0$.

(iii) Let $X_A^{(\nu)} = \mathcal{BV}_A$ in Theroem 9. Then $\mathcal{BV}_A$ is a (weak) $\nu-$wedge FDK-space if and only if the columns of A in $\mathcal{BV}$ and converge to zero. So we obtain $A(\delta^{kl}) = (a_{mnkl})_{mn} = \zeta_A^{(kl)} \in \mathcal{BV}$ and

$$\lim_{k,l} \sum_{m,n} |a_{mnkl} - a_{m+1,n,k,l} - a_{m,n+1,k,l} + a_{m+1,n+1,k,l}| = 0. \qquad \square$$

4. Conclusion

In conclusion, this paper introduces the concept of (weak) ν-wedge FDK-spaces by drawing inspiration from the concept of (weak) wedge in FK-spaces, using ν-convergence for double sequences. We have provided fundamental characterizations and examples of these spaces, contributing to a deeper understanding of their structure. Additionally, by applying our results to summability domains, we have derived necessary and sufficient conditions for the ν-wedgeness of specific spaces, including $\mathcal{BV}_A$, $(\mathcal{L}_u)_A$, and $(\mathcal{M}_u)_A$. The results established in this paper are not limited to the context of (weak) ν-wedge FDK-spaces. They can be extended to broader settings, including paranormed double sequence spaces and ratio sequence spaces. This work not only builds upon previous results in the field, but also paves the way for further exploration and application of ν-wedge FDK-spaces in various mathematical contexts.

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