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On the norms and Hadamard product of Toeplitz matrices involving Leonardo numbers

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Abstract. In this study, we consider the Toeplitz matrices with entries being Leonardo numbers. We have found upper and lower bounds for the spectral norms of these matrices, considering also the Hadamard product of this type of matrix.

1. Introduction and background

The topic of number sequences has been of great interest to many researchers in recent years. With this topic as the target of their study, identities, generating functions, generating matrices and their applications not only in matrix theory but also in other science subjects have been developed (see, for instance, the works [2, 9, 10, 13]). In particular, the well-known Fibonacci sequence and the Leonardo sequence are two examples of number sequences that have received a great deal of attention from researchers (see the works [1], [5], [7], [3], and many others).

This work will also be dedicated to another study related to these two sequences, which involves them in the construction of Toeplitz matrices, studying some of their norms, and obtaining interesting identities. Note that both the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ and the Leonardo sequence $\{Le_n\}_{n=0}^{\infty}$ are sequences defined by second order recurrence relations. So, for the Fibonacci sequence, we have the following.

$$F_{n+1} = F_n + F_{n-1}, \ (n \geq 1) \tag{1}$$

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with initial conditions $F_0 = 0$ and $F_1 = 1$, and for the Leonardo sequence

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \ (n \geq 2), \tag{2}$$

with initial conditions $Le_0 = Le_1 = 1$.

Equation (2) can be written as

$$Le_{n+1} = 2Le_n - Le_{n-2}, \ (n \geq 2)$$

which is Lemma 2.1 stated in [4]. An important result that we will use later is the relationship between the n-th Leonardo number and the $(n+1)$-th Fibonacci number. Such a result is as follows:

Lemma 1. *(Proposition 2.2, [6]) For $n \geq 0$,*

$$Le_n = 2F_{n+1} - 1. \tag{3}$$

The rules (1) and (2) can be used to extend the Fibonacci and Leonardo sequences backwards, thus

$$F_{-n} = (-1)^{n+1} F_n, \ (n \geq 1) \tag{4}$$

and

$$Le_{-n} = (-1)^n (Le_{n-2} + 1) - 1, \ (n \geq 2) \tag{5}$$

which is the identity (2.28) in [4] with $Le_{-1} = -1$.

Now, considering the identities (3), (4) and (5), we easily get the following result.

Lemma 2. *For $n \geq 1$,*

$$Le_{-n} = 2F_{-(n-1)} - 1. \tag{6}$$

Proof. We have

$$Le_{-n} = (-1)^n (Le_{n-2} + 1) - 1 = (-1)^n (2F_{n-1}) - 1 = (-1)^n (2F_{n-1}) - 1,$$

and the result follows. ☐

In [16] are presented the terms of the Leonardo sequence with a negative integer index.

The following formulae for the Fibonacci number are well known (see [8] and [15]),

$$\sum_{k=0}^{n} F_k = F_{n+2} - 1,$$

$$\sum_{k=1}^{n-1} F_k^2 = F_n F_{n-1}, \tag{7}$$

$$\sum_{k=1}^{n} F_k F_{k-1} = \begin{cases} F_n^2, & n \text{ even;} \\ F_n^2 - 1, & n\text{odd,} \end{cases} \tag{8}$$

$$\sum_{k=0}^{n}(-1)^{k+1}F_k = F_{n-1} + (-1)^{n+1}F_{n-2}, \tag{9}$$

and concerning Leonardo numbers (see Propositions 2.2 and 3.1 in [4] and also we can consult some other results in [6]),

$$\sum_{k=0}^{n} Le_k = Le_{n+2} - (n+2), \tag{10}$$

$$\begin{aligned}\sum_{k=0}^{n} Le_k^2 &= 4(F_{n+1}-1)(F_{n+2}-1) + (n+1)\\ &= (Le_n - 1)(Le_{n+1}-1) + (n+1),\end{aligned} \tag{11}$$

$$\sum_{k=0}^{n} Le_k F_{k+1} = F_{n+1}Le_{n+1} - F_{n+2} + 1, \tag{12}$$

$$\sum_{k=0}^{n} Le_{2k+1} = Le_{2n+2} - (n+2),$$

$$\sum_{k=0}^{n} Le_{2k} = Le_{2n+1} - n.$$

In addition to the equalities (5) and (6), we can also use the following recurrence relation for these Leonardo numbers with negative index given by:

$$Le_{-n} = -Le_{-n+1} + Le_{-n+2} - 1, \quad n > 0. \tag{13}$$

We state some of terms of the Leonardo sequence with a negative index in the following table:

Table 1. Some terms of Leonardo sequence with negative index

Le_{-1}	Le_{-2}	Le_{-3}	Le_{-4}	Le_{-5}	Le_{-6}	Le_{-7}	Le_{-8}	Le_{-9}	Le_{-10}
-1	1	-3	3	-7	9	-17	25	-43	67

As with Leonardo's numbers, Leonardo's numbers with a negative index are always odd numbers, alternating positivity and negativity, as shown in the next result, the proof of which is omitted.

Lemma 3. *The Leonardo number Le_{-j}, $(1 \leq j \leq n)$ is an odd number which is positive if j is even and negative if j is odd.*

Lemma 4. *For all natural numbers n, we have $|Le_{-n}| \leq Le_n$.*

Proof. For $n = 1$, we have $1 = |Le_{-1}| = Le_1$.

Assume that for all $k \leq n$, we have

$$|Le_{-k}| \leq Le_k.$$

To demonstrate this for the successors of n. Let's see

$$\begin{aligned}
|Le_{-(n+1)}| &= |-Le_{-(n+1)+1} + Le_{-(n+1)+2} - 1| \\
&= |-Le_{-n} + Le_{-(n-1)} - 1| \\
&\overset{\text{Triangle inequality}}{\leq} |-Le_{-n}| + |Le_{-(n-1)}| + |-1| \\
&= |Le_{-n}| + |Le_{-(n-1)}| + 1 \\
&\overset{\text{induction hypothesis}}{\leq} Le_n + Le_{n-1} + 1\ .
\end{aligned}$$

Since $Le_{n+1} = Le_n + Le_{n-1} + 1$, we conclude that $|Le_{-(n+1)}| \leq Le_{n+1}$. □

Also, we have the following result that will be used later.

Lemma 5. *For $n \geq 0$, the following identity is true:*

$$\sum_{k=0}^{n} Le_k Le_{k-1} = \begin{cases} (n+2) - (2Le_n - Le_{n-1}Le_{n+1}), & n \text{ even}, \\ (n-2) - (2Le_n - Le_{n-1}Le_{n+1}), & n \text{ odd}. \end{cases}$$

Proof. By Lemma 1 we have

$$\begin{aligned}
\sum_{k=0}^{n} Le_k Le_{k-1} &= \sum_{k=0}^{n}(2F_{k+1} - 1)(2F_k - 1) = \sum_{k=0}^{n}(4F_{k+1}F_k - 2(F_{k+1} + F_k) + 1) \\
&= 4\sum_{k=0}^{n} F_{k+1}F_k - 2\sum_{k=0}^{n} F_{k+2} + (n+1) \\
&= 4\sum_{k=0}^{n} F_{k+1}F_k - 2\left(\sum_{k=0}^{n} F_k - F_0 - F_1 + F_{n+1} + F_{n+2}\right) + (n+1) \\
&= 4\sum_{k=0}^{n} F_{k+1}F_k - 2\left(\sum_{k=0}^{n} F_k\right) + 2 - 2(F_{n+1} + F_{n+2}) + (n+1) \\
&= 4\sum_{k=0}^{n} F_{k+1}F_k - 2(F_{n+2} - 1) + 2 - 2(F_{n+1} + F_{n+2}) + (n+1) \\
&= 4\sum_{k=0}^{n} F_{k+1}F_k - 2F_{n+4} + (n+5)
\end{aligned}$$

Now, suppose that n is even. Then, by Equations (2), (3) and (8),

$$\begin{aligned}
\sum_{k=0}^{n} Le_k Le_{k-1} &= 4\sum_{k=0}^{n} F_{k+1}F_k - 2F_{n+4} + (n+5) \\
&= 4\sum_{k=1}^{n+1} F_k F_{k-1} - 2F_{n+4} + (n+5) \\
&= 4(F_n^2 + F_{n+1}F_n) - 2F_{n+4} + (n+5) \\
&= 4(F_n(F_n + F_{n+1}) - 2F_{n+4} + (n+5) \\
&= 4F_nF_{n+2} - 2F_{n+4} + (n+5) \\
&= (Le_{n-1}+1)(Le_{n+1}+1) - (Le_{n+3}+1) + (n+5) \\
&= Le_{n-1}Le_{n+1} + Le_{n-1} + Le_{n+1} - Le_{n+3} + (n+5) \\
&= Le_{n-1}Le_{n+1} + Le_{n-1} + Le_{n+1} \\
&\quad -(Le_{n+2} + Le_{n+1} + 1) + (n+5) \\
&= Le_{n-1}Le_{n+1} + Le_{n-1} - Le_{n+2} + (n+4) \\
&= Le_{n-1}Le_{n+1} + Le_{n-1} - (2Le_n + Le_{n-1} + 2) + (n+4) \\
&= Le_{n-1}Le_{n+1} - 2Le_n + (n+2).
\end{aligned}$$

Otherwise, if n is odd, we get, using again, Equations (2), (3) and (8),

$$\begin{aligned}
\sum_{k=0}^{n} Le_k Le_{k-1} &= 4\sum_{k=0}^{n} F_{k+1}F_k - 2F_{n+4} + (n+5) \\
&= 4\sum_{k=1}^{n+1} F_k F_{k-1} - 2F_{n+4} + (n+5) \\
&= 4(\sum_{k=1}^{n} F_k F_{k-1} + F_{n+1}F_n) - 2F_{n+4} + (n+5) \\
&= 4(F_n^2 + F_{n+1}F_n - 1) - 2F_{n+4} + (n+5) \\
&= 4(F_n(F_n + F_{n+1}) - 1) - 2F_{n+4} + (n+5) \\
&= 4F_nF_{n+2} - 2F_{n+4} - 4 + (n+5) \\
&= (Le_{n-1}+1)(Le_{n+1}+1) - (Le_{n+3}+1) + (n+1) \\
&= Le_{n-1}Le_{n+1} + Le_{n-1} + Le_{n+1} - Le_{n+3} + (n+1) \\
&= Le_{n-1}Le_{n+1} + Le_{n-1} + Le_{n+1} \\
&\quad -(Le_{n+2} + Le_{n+1} + 1) + (n+1) \\
&= Le_{n-1}Le_{n+1} + Le_{n-1} - Le_{n+2} + n \\
&= Le_{n-1}Le_{n+1} + Le_{n-1} - (2Le_n + Le_{n-1} + 2) + n \\
&= Le_{n-1}Le_{n+1} - 2Le_n + (n-2).
\end{aligned}$$

□

In this paper, we are interested in studying the norms of Toeplitz matrices whose entries are Leonardo numbers. We will also look at the Hadamard product between some of these matrices. In the following section we present the background related to Toeplitz matrices, matrix norms, and the Hadamard product between matrices. Section 3 is devoted to the Toeplitz matrix and norms involving Leonardo numbers. Section 4 is dedicated to the Hadamard product involving Toeplitz matrices with Leonardo numbers and Fibonacci numbers. Finally, conclusions are stated, providing a brief highlights of the most significant aspects of this research.

2. Toeplitz matrix, Hadamard product and Norms

An $n \times m$ matrix $T = (t_{i,j})$ is a Toeplitz matrix if $t_{i,j} = t_{i+j,i+j} = t_{i-j}$, for all valid i, j. A general Toeplitz matrix has the form:

$$T = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-m+1} \\ t_1 & t_0 & t_{-1} & \cdots & t_{-m+2} \\ t_2 & t_1 & t_0 & \cdots & t_{-m+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_0 \end{bmatrix}.$$

A Toeplitz matrix could also be square. A square matrix of order n is a Toeplitz matrix if all the entries along each of the $2n-1$ diagonals are the same and the elements in the first row and the first column are successive terms of a sequence.

Consider the vectorial space of the complex (or real) matrices $n \times m$. A norm $||.||$ is a function that associates, for each $n \times m$ matrix A, a non-negative real number that satisfies the properties:

$$||A|| = 0 \text{ if and only if } A = 0, \text{ for all } n \times m \text{ matrix } A;$$

$$||\lambda A|| = |\lambda| ||A||, \text{ for all scalar } \lambda \text{ and } n \times m \text{ matrix } A;$$

$$||A + B|| \leq ||A|| + ||B||, \text{ for all } n \times m \text{ matrices } A \text{ and } B;$$

$$||AB|| \leq ||A||\ ||B||, \text{ whenever the product } AB \text{ is well defined;}$$

$$||I|| = 1, \text{ where } I \text{ is the identity matrix.}$$

Now, we give some preliminary matrix results related to our study. Let $A = (a_{ij})$ be a matrix $m \times n$. When we want to study the norms of the matrices, the following norms are taken into account (see [12] and [17]).

The p- norm of the matrix A is defined by

$$||A||_p = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^p \right)^{1/p}.$$

If $p = 2$, the well-known Frobenius (Euclidean) norm is denoted and defined by

$$||A||_E = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

and, when $p = \infty$, we have

$$||A||_\infty = \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|.$$

Note that $||A||_\infty$ is related to the largest absolute sum of the rows in matrix A.

The 1-norm of a $n \times m$ matrix A is defined by

$$||A||_1 = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}|.$$

Similarly to what happens with $||A||_\infty$, the 1-norm of A is related to the largest absolute sum of the columns in matrix A.

The maximum column length norm $c_1(A)$ is defined by the maximum Euclidean norm (2-norm) among all columns of A,

$$c_1(A) = \max_{1 \le j \le n} \sqrt{\sum_{i=1}^{m} |a_{ij}|^2} = \max_j \|[a_{ij}]_{i=1}^{m}\|_E.$$

The maximum row length norm $r_1(A)$ is defined by the maximum Euclidean norm (2-norm) among all rows of A,

$$r_1(A) = \max_{1 \le i \le m} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} = \max_i \|[a_{ij}]_{j=1}^{n}\|_E.$$

There are general relationships between matrix norms that apply to any matrix, including Toeplitz matrices. For any square matrix A of order n, the following inequalities (see, [17]) hold:

$$\frac{1}{\sqrt{n}} \|A\|_E \le \|A\|_2 \le \|A\|_E, \tag{14}$$

and

$$\|A\|_\infty \le \sqrt{n} \|A\|_E. \tag{15}$$

Also, considering the 1-norm, and 2-norm, there exist constants c_1 and c_2 such that

$$c_1 ||A||_1 \le ||A||_2 \le c_2 ||A||_1,$$

for all $n \times m$ matrix A.

Given two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of size $m \times n$, their Hadamard product is denoted by $A \circ B$ and is defined as

$$(A \circ B)_{ij} = a_{ij} \cdot b_{ij} \quad \text{for all } i = 1, \ldots, m \text{ and } j = 1, \ldots, n.$$

For more details on the Hadamard product of two matrices, see, for example, the work of Reams in [14].

According to [11], if $C = A \circ B$, then the following inequality holds:

$$\|C\|_2 \leq r_1(A) \cdot c_1(B).$$

3. Leonardo Toeplitz matrix

This section introduces the Leonardo Toeplitz square matrix and provides some properties arising from the study of the norms of this matrix.

Definition 1. The Toeplitz matrix with Leonardo numbers denoted by $\mathbf{T}_{Le}^n$ is the square $n \times n$ matrix defined by

$$\mathbf{T}_{Le}^n = \begin{pmatrix} Le_0 & Le_{-1} & Le_{-2} & \ldots & Le_{1-n} \\ Le_1 & Le_0 & Le_{-1} & \ldots & Le_{2-n} \\ Le_2 & Le_1 & Le_0 & \ldots & Le_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Le_{n-1} & Le_{n-2} & Le_{n-3} & \ldots & Le_0 \end{pmatrix}, \tag{16}$$

and we call this matrix the Leonardo-Toeplitz matrix of order n.

The following examples show the Leonardo Toeplitz matrices of orders 3 and 4.

Example 1. The Leonardo Toeplitz matrices of order $n = 3$ and $n = 4$ are

$$\mathbf{T}_{Le}^3 = \begin{pmatrix} Le_0 & Le_{-1} & Le_{-2} \\ Le_1 & Le_0 & Le_{-1} \\ Le_2 & Le_1 & Le_0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 3 & 1 & 1 \end{pmatrix},$$

and

$$\mathbf{T}_{Le}^4 = \begin{pmatrix} Le_0 & Le_{-1} & Le_{-2} & Le_{-3} \\ Le_1 & Le_0 & Le_{-1} & Le_{-2} \\ Le_2 & Le_1 & Le_0 & Le_{-1} \\ Le_3 & Le_2 & Le_1 & Le_0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -3 \\ 1 & 1 & -1 & 1 \\ 3 & 1 & 1 & -1 \\ 5 & 3 & 1 & 1 \end{pmatrix},$$

respectively.

Looking at the rows and columns of the Leonardo Toeplitz matrix of order n, we denote by $R_i(\mathbf{T}_{Le}^n)$ and $C_j(\mathbf{T}_{Le}^n)$ the sum of the i-th row and the sum of the j-th column of $\mathbf{T}_{Le}^n$, respectively, and we can write that

$$R_i(\mathbf{T}^n_{Le}) = \sum_{m=-(n-i)}^{i-1} Le_m, \quad (i \geq 1),$$

and

$$C_j(\mathbf{T}^n_{Le}) = \sum_{m=1-j}^{n-j} Le_m, \quad (j \geq 1).$$

The next results are related to the row and column of $\mathbf{T}^n_{Le}$ and will be used to determine some norms of this matrix. More precisely, we state the relation between the sum of the j-th column of $\mathbf{T}^n_{Le}$ and the sum of the previous column.

Lemma 6. *Let $\mathbf{T}^n_{Le}$ be the Leonardo Toeplitz matrix given by* (16) *and let $C_j(\mathbf{T}^n_{Le})$ denote the sum of the j-th column of $\mathbf{T}^n_{Le}$,. Then, for all $j \geq 2$,*

$$C_j(\mathbf{T}^n_{Le}) = C_{j-1}(\mathbf{T}^n_{Le}) - Le_{n-(j-1)} + Le_{-(j-1)}.$$

Proof. The proof will be reduced by induction on j and we use equations (2), (10) and (13). Hence, for $j \geq 2$, we have

$$\begin{aligned}
C_2(\mathbf{T}^n_{Le}) &= \sum_{m=-1}^{n-2} Le_m = Le_{-1} + \sum_{m=0}^{n-2} Le_m \\
&= Le_{-1} + \sum_{m=0}^{n} Le_m - Le_{n-1} - Le_n \\
&= Le_{-1} + (Le_{n+2} - (n+2)) - Le_{n-1} - Le_n \\
&= Le_{-1} + (Le_{n+1} + Le_n + 1 - n - 2) - Le_{n-1} - Le_n \\
&= (-Le_0 + Le_1 - 1) + Le_{n+1} + Le_n - n - 1 - Le_{n-1} - Le_n \\
&= -2 + 1 + Le_{n+1} - n - 1 - Le_{n-1} = Le_{n+1} - Le_{n-1} - n - 2
\end{aligned}$$

On the other hand

$$\begin{aligned}
& C_1(\mathbf{T}^n_{Le}) - Le_{n-(2-1)} + Le_{-(2-1)} \\
&= \sum_{m=0}^{n-1} Le_m - Le_{n-1} + Le_{-1} \\
&= Le_{n+2} - n - 2 - Le_{n-1} + (-Le_0 + Le_1 - 1) - Le_n \\
&= Le_{n+2} - n - 3 - Le_{n-1} - Le_n \\
&= Le_{n+1} + Le_n + 1 - n - 3 - Le_{n-1} - Le_n \\
&= Le_{n+1} - Le_{n-1} - n - 2,
\end{aligned}$$

showing that the identity is true for $j = 2$.

Suppose that the statement is true for all values of $m \leq j$ and we claim that

$$C_{j+1}(\mathbf{T}^n_{Le}) = C_j(\mathbf{T}^n_{Le}) - Le_{n-j} + Le_{-j}.$$

We have

$$\begin{aligned} C_j(\mathbf{T}^n_{Le}) - Le_{n-j} + Le_{-j} &= \sum_{m=1-j}^{n-j} Le_m - Le_{n-j} + Le_{-j} \\ = \sum_{m=1-j}^{n-j-1} Le_m + Le_{n-j} - Le_{n-j} + Le_{-j} &= \sum_{m=1-j}^{n-(j+1)} Le_m + Le_{-j} \\ = \sum_{m=-j+1}^{n-(j+1)} Le_m + Le_{-j} &= \sum_{m=-j}^{n-(j+1)} Le_m - Le_{-j} + Le_{-j} \\ &= \sum_{m=-j}^{n-(j+1)} Le_m = C_{j+1}(\mathbf{T}^n_{Le}), \end{aligned}$$

as required. □

In the next result, we state the relation between the sum of the i-th row of $\mathbf{T}^n_{Le}$ and the sum of the previous row.

Lemma 7. *Let $\mathbf{T}^n_{Le}$ be the Leonardo Toeplitz matrix given by* (16) *and let $R_i(\mathbf{T}^n_{Le})$ denote the sum of the i-th row of $\mathbf{T}^n_{Le}$,. Then, for all $i \geq 2$,*

$$R_i(\mathbf{T}^n_{Le}) = R_{i-1}(\mathbf{T}^n_{Le}) + Le_{i-1} + Le_{i-n}.$$

Proof. Similarly to what we did in the proof of the previous lemma, we will proceed by induction on i. Hence, for $i = 2$, we have

$$\begin{aligned} R_2(\mathbf{T}^n_{Le}) &= \sum_{m=-(n-2)}^{1} Le_m = Le_1 + Le_{-n+2} + \sum_{m=-n+1}^{0} Le_m \\ &= Le_1 + Le_{-(n-2)} + \sum_{m=-(n-1)}^{0} Le_m \\ &= R_1(\mathbf{T}^n_{Le}) + Le_1 + Le_{2-n}, \end{aligned}$$

that is verified for $i = 2$.

Suppose that the statement is true for all values of $m \leq i$ and we claim that

$$R_{i+1}(\mathbf{T}^n_{Le}) = R_i(\mathbf{T}^n_{Le}) + Le_i + Le_{i+1-n}.$$

We have

$$\begin{aligned} R_i(\mathbf{T}^n_{Le}) + Le_i + Le_{i+1-n} &= \sum_{m=-(n-i)}^{i-1} Le_m + Le_i + Le_{i+1-n} \\ = \sum_{m=-n+i}^{i-1} Le_m + Le_i + Le_{i+1-n} &= \sum_{m=-n+i}^{i} Le_m + Le_{i+1-n} \\ = \sum_{m=-j+1}^{n-(j+1)} Le_m + Le_{-j} &= \sum_{m=-n+i+1}^{i} Le_m \\ &= \sum_{m=-(n-(i+1))}^{i} Le_m = R_{i+1}(\mathbf{T}^n_{Le}), \end{aligned}$$

as required. □

The next result relates the row to column of $\mathbf{T}^n_{Le}$.

Lemma 8. *Let* $\mathbf{T}^n_{Le}$ *be the Leonardo Toeplitz matrix given by* (16) *and let* $R_i(\mathbf{T}^n_{Le})$ *and* $C_j(\mathbf{T}^n_{Le})$ *denote the sum of the i-th row and the sum of the j-th column of* $\mathbf{T}^n_{Le}$*, respectively. Then, for all* $j \geq 1$,

$$C_j(\mathbf{T}^n_{Le}) = R_{n-(j-1)}(\mathbf{T}^n_{Le}).$$

Proof. We have

$$\begin{aligned} C_j(\mathbf{T}^n_{Le}) &= \sum_{m=1-j}^{n-j} Le_m = \sum_{m=-(j-1)-n+n}^{n-j+1-1} Le_m \\ &= \sum_{m=-(n-(n-(j-1)))}^{n-(j-1)-1} Le_m = R_{n-(j-1)}(\mathbf{T}^n_{Le}), \end{aligned}$$

and the result follows. □

Note that

$$C_1(\mathbf{T}^n_{Le}) = R_n(\mathbf{T}^n_{Le}) = \sum_{m=0}^{n-1} Le_m = \sum_{m=0}^{n} Le_m - Le_n = Le_{n+1} - (n+1).$$

The next result gives us the general expression of the sum of the j-th column, $(2 \leq j \leq n)$, of $\mathbf{T}^n_{Le}$.

Proposition 1. *Let* $\mathbf{T}^n_{Le}$ *be the Leonardo Toeplitz matrix* (16) *and let* $C_j(\mathbf{T}^n_{Le})$ *denote the sum of its j-th column. Then*

$$C_j(\mathbf{T}^n_{Le}) = \sum_{m=1-j}^{-1} Le_m + Le_{n-j+2} - (n-j+2),\ (2 \leq j \leq n).$$

Proof. Let j, $(2 \leq j \leq n)$ be a fixed column. Then we can write $j = n - p$ for some non-negative integer p. Therefore,

$$\begin{aligned}
C_j(\mathbf{T}^n_{Le}) &= Le_{1-j} + Le_{2-j} + \cdots + Le_{(j-1)-j} + \sum_{m=0}^{n-j} Le_m \\
&= Le_{1-(n-p)} + Le_{2-(n-p)} + \cdots + Le_{(n-p-1)-(n-p)} + \sum_{m=0}^{n-(n-p)} Le_m \\
&= \sum_{m=1-(n-p)}^{-1} Le_m + \sum_{m=0}^{p} Le_m \\
&= \sum_{m=1-(n-p)}^{-1} Le_m + Le_{p+2} - (p+2) \\
&= \sum_{m=1-j}^{-1} Le_m - Le_{n-j+2} - (n-j+2),
\end{aligned}$$

as required. □

Using the previous result and Lemma 8, we can easily find the general expression of the sum of the i-th row, $(1 \leq i \leq n-1)$, of $\mathbf{T}^n_{Le}$, so we have omitted the proof.

Proposition 2. *Let* $\mathbf{T}^n_{Le}$ *be the Leonardo Toeplitz matrix* (16) *and let* $R_i(\mathbf{T}^n_{Le})$ *denote the sum of its* i*-th row. Then*

$$R_i(\mathbf{T}^n_{Le}) = \sum_{m=i-n}^{-1} Le_m + Le_{i+1} - (i+1), \ (1 \leq i \leq n-1).$$

4. NORMS AND HADAMARD PRODUCTS WITH $\mathbf{T}^n_{Le}$

This section is dedicated to the norms of Leonardo Toeplitz matrix $\mathbf{T}^n_{Le}$ and some Hadamard products that involve this matrix.

Using Lemmas 1 and 2, more precisely Equations (3) and (6), we get

$$\mathbf{T}^n_{Le} = \begin{pmatrix} 2F_1 - 1 & 2F_0 - 1 & 2F_{-1} - 1 & \ldots & 2F_{-(n-2)} - 1 \\ 2F_2 - 1 & 2F_1 - 1 & 2F_0 - 1 & \ldots & 2F_{-(n-3)} - 1 \\ 2F_3 - 1 & 2F_2 - 1 & 2F_1 - 1 & \ldots & 2F_{-(n-4)} - 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2F_n - 1 & 2F_{n-1} - 1 & 2F_{n-2} - 1 & \ldots & 2F_1 - 1 \end{pmatrix}$$

$$= 2\begin{pmatrix} F_1 & F_0 & F_{-1} & \dots & F_{-(n-2)} \\ F_2 & F_1 & F_0 & \dots & F_{-(n-3)} \\ F_3 & F_2 & F_1 & \dots & F_{-(n-4)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_n & F_{n-1} & F_{n-2} & \dots & F_1 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1. \end{pmatrix}$$

$$= 2\mathbf{T}_F^n - X,$$

where $\mathbf{T}_F^n$ is the Fibonacci Toeplitz matrix and X is a matrix $n \times n$ with all entries equal to 1.

In general, using the properties of a norm of a matrix, we get

$$||\mathbf{T}_{Le}^n|| \leq 2||\mathbf{T}_F^n|| - ||X||.$$

Next, we have results involving the calculus of $||.||_\infty$ and $||.||_1$ norms of the matrix $\mathbf{T}_{Le}^n$.

Theorem 1. *Consider* $\mathbf{T}_{Le}^n$ *which is given in* (16)*, then the* $||.||_\infty$ *norm of the matrix* $\mathbf{T}_{Le}^n$ *is* $||\mathbf{T}_{Le}^n||_\infty = Le_{n+1} - (n+1)$*.*

Proof. For each $1 \leq i \leq n$, we denote by t_{ij} the entry of $\mathbf{T}_{Le}^n$ which is in the i-th row and in j-th column. Then we have

$$\begin{aligned} ||\mathbf{T}_{Le}^n||_\infty &= \max_{1\leq i\leq n} \sum_{j=1}^{n} |t_{ij}| \\ &= \max_{1\leq i\leq n} \{|t_{i1}| + |t_{i2}| + \cdots + |t_{in}|\} \end{aligned}$$

By Lemma 4, we conclude that

$$||\mathbf{T}_{Le}^n||_\infty = |Le_{n-1}| + |Le_{n-2}| + \cdots + |Le_0| = \sum_{m=0}^{n-1} |Le_m|.$$

Since $Le_m, (m \geq 0)$ is a positive integer, then using equations (2) and (10), we obtain that

$$\begin{aligned} ||\mathbf{T}_{Le}^n||_\infty &= \sum_{m=0}^{n-1} Le_m = \sum_{m=0}^{n} Le_m - Le_n \\ &= Le_{n+2} - Le_n - n - 2 \\ &= Le_{n+1} - (n+1), \end{aligned}$$

as required. □

Note that $||\mathbf{T}_{Le}^n||_\infty$ is the sum of absolute values of the entries of the last row of $\mathbf{T}_{Le}^n$, that is,

$$||\mathbf{T}_{Le}^n||_\infty = R_n(\mathbf{T}_{Le}^n).$$

Taking into account that $R_n(\mathbf{T}_{Le}^n) = C_1(\mathbf{T}_{Le}^n)$, the next result immediately follows, and then the proof is omitted.

Theorem 2. *Consider* $\mathbf{T}_{Le}^n$ *which is given in* (16)*, then the* $||.||_1$ *norm of the matrix* $\mathbf{T}_{Le}^n$ *is*

$$||\mathbf{T}_{Le}^n||_1 = ||\mathbf{T}_{Le}^n||_\infty.$$

The next result gives us the Frobenius norm of the matrix $\mathbf{T}_{Le}^n$.

Theorem 3. *Consider* $\mathbf{T}_{Le}^n$ *the matrix given in* (16)*. Then*

$$||\mathbf{T}_{Le}^n||_E^2 = \begin{cases} 2[(n-3) - Le_{n+1}(2 - Le_{n-1}) - 3Le_n] + (n+4)^2, & n \text{ even}, \\ -2[Le_{n+1}(2 - Le_{n-1}) + 3Le_n] + (n+3)^2 - 19, & n \text{ odd}. \end{cases}$$

Proof. We deduce from (11) that

$$\begin{aligned} ||\mathbf{T}_{Le}^n||_E^2 &= nLe_0^2 + 2\sum_{i=1}^{n-1}\sum_{k=1}^{i} Le_k^2 \\ &= n + 2\sum_{i=1}^{n-1}\left[(Le_i - 1)(Le_{i+1} - 1) + (i+1)\right] \\ &= n + 2\sum_{i=1}^{n-1}(Le_i - 1)(Le_{i+1} - 1) + 2\sum_{i=1}^{n-1}(i+1) \\ &= n + 2\sum_{i=1}^{n-1}(Le_iLe_{i+1} - Le_i - Le_{i+1} + 1) + 2\sum_{i=1}^{n-1}(i+1) \\ &= n + 2\sum_{i=1}^{n-1}Le_iLe_{i+1} - 4\sum_{i=0}^{n}Le_i + 6 + 2Le_n + 4\sum_{i=1}^{n-1}1 + 2\sum_{i=1}^{n-1}i, \end{aligned}$$

or equivalently,

$$||\mathbf{T}_{Le}^n||_E^2 = n^2 + 8n + 10 - 2Le_{n+3} + 2\sum_{i=0}^{n}Le_iLe_{i-1}.$$

Now we just need to consider the cases of even n and odd n, use Lemma 5 in each of these cases, simplify some calculations, and we will obtain the desired result. □

If we consider the spectral norm of $\mathbf{T}_{Le}^n$, and taking into account the equations (14) and (15), then we can conclude that $\frac{1}{\sqrt{n}}\,||\mathbf{T}_{Le}^n||_E \leq ||\mathbf{T}_{Le}^n||_2 \leq ||\mathbf{T}_{Le}^n||_E$, and $||\mathbf{T}_{Le}^n||_\infty \leq \sqrt{n}\,||\mathbf{T}_{Le}^n||_E$.

Therefore, using Theorems 1 and 3, we have the following two results.

Theorem 4. *Consider* $\mathbf{T}^n_{Le}$ *which is given in* (16). *Then*

$$||\mathbf{T}^n_{Le}||_2 \geq \begin{cases} \frac{1}{\sqrt{n}}\left\{2[(n-3) - Le_{n+1}(2 - Le_{n-1}) - 3Le_n] + (n+4)^2\right\}, & n \text{ even;} \\ \frac{1}{\sqrt{n}}\left\{-2[Le_{n+1}(2 - Le_{n-1}) + 3Le_n] + (n+3)^2 - 19\right\}, & n \text{ odd.} \end{cases}$$

and

$$||\mathbf{T}^n_{Le}||_2 \leq \begin{cases} \sqrt{n}\sqrt{(Le_{n-2}+1)(Le_{n-1}+5)+n}, & n \text{ even;} \\ \sqrt{n}\sqrt{(Le_{n-2}+1)(Le_{n-1}+1)+n}, & n \text{ odd.} \end{cases}$$

where $||.||_2$ *denotes the spectral norm and* Le_j *the* j*-th Leonardo number.*

Proof. The proof of the first two inequalities is omitted as it is a consequence of Theorem 3. For the proof of the other inequalities, let the $n \times n$ square matrices

$$A = [a_{ij}] = \begin{cases} a_{ij} = 1, & i = j; \\ a_{ij} = Le_{i-j}, & i \neq j; \end{cases}$$

and

$$B = [b_{ij}] = \begin{cases} b_{ij} = 1, & i \neq j; \\ b_{ij} = Le_{i-j}, & j = i; \end{cases}$$

that satisfy $C = A \circ B$. Then by Lemma 2, and equalities (4), (7), (9), we obtain

$$\begin{aligned} r_1(A) &= \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |a_i j|^2} = \sqrt{1 + \sum_{k=1}^{n-1} Le^2_{-k}} \\ &= \sqrt{1 + \sum_{k=1}^{n-1} \left(2F_{-(k-1)} - 1\right)^2} \\ &= \sqrt{1 + 4\sum_{k=1}^{n-1} F^2_{-(k-1)} - 4\sum_{k=1}^{n-1} F_{-(k-1)} + \sum_{k=1}^{n-1} 1} \\ &= \sqrt{1 + 4\sum_{k=1}^{n-1} \left((-1)^k F_k\right)^2 - 4\sum_{k=1}^{n-1} (-1)^k F_k + \sum_{k=1}^{n-1} 1} \\ &= \sqrt{1 + 4\left(F_n F_{n-1}\right) - 4\sum_{k=1}^{n-1} (-1)^k F_k + (n-1)} \\ &= \sqrt{4\left(F_n F_{n-1}\right) + 4\sum_{k=0}^{n} (-1)^{k+1} F_k - 4(-1)^{n+1} F_n + n} \end{aligned}$$

$$\begin{aligned}
&= \sqrt{4\left(F_nF_{n-1}\right)+4\left(F_{n-1}+(-1)^{n+1}F_{n-2}\right)-4(-1)^{n+1}F_n+n} \\
&= \sqrt{4F_{n-1}\left(F_n+1+(-1)^n\right)+n} \\
&= \sqrt{\left(2Le_{n-2}+2\right)\left(F_n+1+(-1)^n\right)+n}.
\end{aligned}$$

Suppose that n is even. Hence

$$\begin{aligned}
r_1(A) &= \sqrt{\left(2Le_{n-2}+2\right)\left(F_n+2\right)+n} \\
&= \sqrt{\left(Le_{n-2}+1\right)\left(2F_n+4\right)+n} \\
&= \sqrt{\left(Le_{n-2}+1\right)\left(Le_{n-1}+5\right)+n},
\end{aligned}$$

whereas when n is odd,

$$\begin{aligned}
r_1(A) &= \sqrt{\left(2Le_{n-2}+2\right)F_n+n} \\
&= \sqrt{\left(Le_{n-2}+1\right)2F_n+n} \\
&= \sqrt{\left(Le_{n-2}+1\right)\left(Le_{n-1}+1\right)+n},
\end{aligned}$$

as required. Now it is easy to show that $c_1(B)=\sqrt{n}$ and therefore the result follows. □

Yet another result which is a consequence of Theorem 2, which is why we have omitted its proof.

Theorem 5. *Consider* $\mathbf{T}_{Le}^n$ *which is given in* (16)*. Then*

$$\sqrt{n}\ ||\mathbf{T}_{Le}^n||_E \geq Le_{n+1}-(n+1).$$

5. Conclusion

In this study, we investigated Toeplitz matrices whose entries are Leonardo numbers, a sequence with intriguing mathematical properties. By establishing upper and lower bounds for the spectral norms of these matrices, we provided a deeper understanding of their structural behavior and potential applications. Furthermore, our analysis extended to the Hadamard product of such matrices, revealing additional insights into their norm characteristics and interactions.

These findings contribute to the broader study of structured matrices and special number sequences, offering a foundation for future research in numerical analysis, signal processing, and matrix theory. Further exploration could involve generalizing these results to other recursive sequences or examining the impact of different matrix operations on spectral properties.

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