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On a family of hypergeometric polynomials

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Abstract. We work with sequences of integrals that we call SCE integrals. We establish their expressions in terms of five families of polynomials. We demonstrate the relations between these integrals and we focus on one of the three integrals: the determination of the family of polynomials noted e_n ($n \in \mathbb{N}$). We show that these polynomials are hypergeometric. From this property, the NU method can be applied to this family. We determine the Rodrigues formula. These polynomials have properties that distinguish them from classical hypergeometric polynomials. We state and demonstrate the theorem adapted to the determination of the e_n generating function. Finally, the sequence of polynomials studied is expressed in terms of associated Laguerre polynomials with negative upper indices.

1. Introduction

The problems of mathematical physics often lead to the resolution of mathematical equations of any form. The literature presents analytical, approximate or numerical resolution methods. This is where our problem lies. We will pose the problem of determining three sequences of indefinite integrals. These integral problems are related to the determination of some polynomial families. These are solutions of the following differential equation:

$$y'' + P(x)y' + Q(x)y = R(x), \tag{1}$$

where P, Q, R are functions of the real variable x. In the case where these functions are polynomial or rational, a particular solution of (1) is a polynomial. The Frobenius method makes it possible to obtain the families of polynomials sought, solutions of differential equations of the type (1).

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The detailed study of the problems posed reveals that these families of polynomials are linked and the study of only one of the cases is sufficient to determine the others. The differential equation (1) has been the subject of many studies, for example H. Ciftci *et al.* [9] and Y-Z. Zhang [38], who were interested in the general conditions so that a homogeneous equation of the type (1) admits polynomial solutions as well as the underlying applications.

In some cases, the resolution of (1) leads to the study of special functions of Physics. This theory is an active branch of research in Theoretical and Mathematical Physics. Literature often presents each special function as an isolated case and requires a detailed study on its own. These studies often lead to the establishment of Rodrigues formulas [35], recurrence and other relationships for each type of "special" polynomials (See [1,6,7,17,26,27,33]).

The method of Nikiforov-Uvarov (NU Method) [26,27] presents all special functions as arising from a single theory: the theory of differential equations of the hypergeometric type[1]. This method proposes a construction of special functions from a simple idea: to treat them from a single point of view. In some cases, the solutions of (1) are hypergeometric functions (See [6, 7, 27, 37]). Note that the NU method has been used successfully to solve some theoretical and mathematical physics problems. One can, for example, cite: the resolution of the Schrödinger's equation by Abu-Shady [2], Antia *et al.* [4], B.I. Ita *et al.* [18, 19, 21], I. B. Okon *et al.* [28], C. A. Onate *et al.* [29]; the resolution of Klein-Gordon's equation by B. I. Ita *et al.* [23]; some applications in Mathematical Physics by B. Gönul *et al.* [15] and in High Energy Physics, by example, M. Abu-Shady *et al.* [3].

In the present paper, we generated a family of hypergeometric polynomials whose particular property will allow to present the adapted integral representation. The consequence is that, it leads to the correct expression of the generating function. Finally, we have found a link with the associated Laguerre polynomials with negative orders[2]. Similar cases of studies of Laguerre polynomials of particular orders have been carried out by K. N. Boyadzieh [8] and I. K. Khabibrakhmanov *et al.* [20].

The paper is structured as follows. In the second section, we begin with a brief overview of the NU method as presented in [26] and [27]. Section 3 is devoted to the resolution of the problematic (integral problems). Differential equations deduced from these problems, if solved by the Frobenius method, lead to the explicit expressions of the families of polynomials. The section closes by determining the generating functions for each family of polynomials. Literature gives various resolution method, we can cite, by example, operational methods [10–12, 14]. We have been able to relate our results with this method. In Section 4, we use the NU method to determine the

[1] It is (1) homogeneous, with P and Q polynomials or rational functions of some form.

[2] We call *order* of an associated Laguerre polynomials $L_n^{(\alpha)}$, the higher index α of these polynomials.

Rodrigues formula and the generating function. The particular and adapted integral representation of the polynomials is discussed. Section 5 is devoted to the complete resolution of the second-order differential equation to which a family of polynomials determined in the second section obeys. From this study follows a relationship between the family of associated Laguerre polynomials with negative orders and the polynomials of the study. The last section deals with a generalization of the E integral. Finally, the paper ends with conclusions.

2. Some results from the Nikiforov-Uvarov method (NU method)

In the following lines, we will recall some essential elements of the NU method [26] that will be useful in the study.

Consider the following second-order differential equation:

$$u'' + \frac{C(x)}{A(x)}\, u' + \frac{\tilde{C}(x)}{A^2(x)}\, u = 0, \tag{2}$$

where u is a function of the real or complex variable x, C is a polynomial of degree not greater than one, A et $\tilde{C}$, are polynomials of degrees no greater than two.

Indeed, any differential equation of the second order having at most three singular points can be written in the form (2) by using a suitable change of variable [10].

Consider the change of variable:

$$u = \varphi(x)\, y, \tag{3}$$

the starting equation takes the following form:

$$y'' + \left(2\,\frac{\varphi'}{\varphi} + \frac{C}{A}\right) y' + \left(\frac{\varphi''}{\varphi} + \frac{C}{A}\,\frac{\varphi'}{\varphi} + \frac{\tilde{C}}{A^2}\right) y = 0. \tag{4}$$

In order to simplify (4), the following substitutions are made:

$$\frac{\varphi'}{\varphi} = \frac{D}{A},\ D = \frac{1}{2}\left(B - A\right),$$

with D and B polynomials of degrees no greater than one.

The coefficient of y' in (4) becomes $\frac{B}{A}$ and the one of y; $\frac{F}{A^2}$ with

$$F = \tilde{C} + C\ D + D^2 + D'\ A - D\ A'.$$

The polynomials B and F have degrees, respectively, not greater than one and two. Previous results generate a class of transformations(See (3)) leaving (2) invariant. Indeed, the equation (2) becomes:

$$y'' + \frac{B(x)}{A(x)}\, y' + \frac{F(x)}{A^2(x)}\, y = 0. \tag{5}$$

The choice of the function D is arbitrary, so we can choose its coefficients such as

$$F(x) = \lambda\ A(x), \tag{6}$$

where λ is a constant.

The equation (5) takes, at this moment, the form below:

$$A(x)\ y'' + B(x)\ y' + \lambda\ y = 0. \tag{7}$$

The equation (7) is called *second-order differential equation of the hypergeometric type* and its solutions are *the hypergeometric functions.* If the solution of the equation(7) is a polynomial, we will speak of a *hypergeometric polynomial.*

The NU method proposes a procedure to turn a differential equation of the form (2) to the form (7) (See [26]).

By restricting oneself to the case of hypergeometric polynomials, it is possible to determine the generalized Rodrigues formula for special functions. To do this, the NU theory begins with this important property: *"the derivatives of any order of hypergeometric type functions are hypergeometric functions."* Thus, the derivative of the nth order of the (7) is the following hypergeometric differential equation:

$$A(x)\ y_n'' + B_n(x)\ y_n' + \mu_n\ y_n = 0, \tag{8}$$

with $B_n = B + nA'$; $\mu_n = \lambda + nB' + \frac{n(n-1)}{2}A''$ and $y_n(x) = \frac{d^n}{dx^n}y(x)$.

In the case of hypergeometric polynomials, we can have $\mu_n = 0$, so (See Bateman *et al.* [7] and Nikiforov *et al.* [26]):

$$\lambda = \lambda_n \equiv -nB' - \frac{n\,(n-1)}{2}A''. \tag{9}$$

Thus, the explicit expression of hypergeometric polynomials, of degree n, is:

$$y(x) = \frac{\beta_n}{\rho(x)}\ \frac{d^n}{dx^n}\Big[A^n(x)\ \rho(x)\Big],\ n = 0, 1, 2, \cdots \tag{10}$$

This is the *Rodrigues formula* sought, with β_n a normalization factor and the function ρ, a solution of the following differential equation:

$$(A\rho)' = B\rho. \tag{11}$$

The function ρ allows to write (7) in a self-adjoint form. For the differential equation (8), the function ρ_n allowing to write it in a self-adjoint form is such that:

$$(A\rho_n)' = B_n\rho_n. \tag{12}$$

From (11) and (12), we have:

$$\rho_n(x) = A^n(x)\rho(x). \tag{13}$$

The NU method provides a general expression of the generating functions for hypergeometric polynomials [27]. The starting point is the integral representation of these polynomials in the complex plane. By replacing this representation in the definition of the generating functions, we obtain a relation allowing to find the generating function of any family of hypergeometric polynomials.

Let ρ be a function solution of (11) and let us replace n by ν in (9). There is a function ρ_ν given by (13), such that the function u (solution of (2)) given by:

$$u(x) = \int_{(C)} \frac{\rho_\nu(s)}{(s-x)^{\nu+1}} ds. \tag{14}$$

That is true if:

(1) by calculating the derivatives u' and u'', derivation and integration operations can be swapped:

$$\begin{aligned} u'(x) &= (\nu+1) \int_{(C)} \frac{\rho_\nu(s)}{(s-x)^{\nu+2}} ds, \\ u''(x) &= (\nu+1)(\nu+2) \int_{(C)} \frac{\rho_\nu(s)}{(s-x)^{\nu+3}} ds; \end{aligned} \tag{15}$$

(2) the integration contour (C) is chosen such that

$$\left. \frac{A^{\nu+1}(s)\rho(s)}{(s-x)^{\nu+2}} \right|_{s_1}^{s_2} = 0, \tag{16}$$

where s_1 and s_2 are the ends of the (C) contour.

Then, the equation (7) admits particular solutions of the form:

$$\begin{aligned} y_\nu(x) &= \frac{\gamma_\nu}{\rho(x)} \int_{(C)} \frac{A^\nu(s)\rho(s)}{(s-x)^{\nu+1}} ds, \\ \gamma_\nu &= \frac{\nu!}{2\pi i} \beta_\nu. \end{aligned} \tag{17}$$

For ν an integer, y_ν is a polynomial. When ν is not an integer, (17) still gives a particular solution of (7).

Let $F(x,t)$ be the generating function of the polynomials given by (17), we have:

$$F(x,t) = \sum_{n=0}^{+\infty} \frac{1}{n!} \tilde{y}_n(x) t^n, \tag{18}$$

where the polynomials $\tilde{y}_n$ are obtained by putting in (10), $\beta_n = 1$.

Finally, the general expression of the generating functions of hypergeometric polynomials, given in [26], is

$$F(x,t) = \frac{\rho(s)}{\rho(x)} \left. \frac{1}{1 - A'(s)t} \right|_{s=\xi(x,t)}, \tag{19}$$

with $s = \xi(x,t)$, solution of the equation $s - x - A(s)t = 0$.

3. Generation of polynomials' families starting from a general problematic

3.1. **The general problematic.** Let us consider the following integrals:

$$S_n(x) = \int x^n \sin x \, dx; \; C_n(x) = \int x^n \cos x \, dx; \; E_n(x) = \int x^n e^x \, dx, \tag{20}$$

with n, a natural integer. In the following, we will respectively designate these problems by the acronym SCE. There are tables showing the SCE expressions, see for example Gradshteyn *et al.* [17] and Spiegel *et al.* [33]. In most documents presenting these results, recurrence relations are often mentioned, which by iteration make it possible to generate the sequence of solutions. This method becomes tedious when the exponent n becomes very large. Hence the importance of having general results to determine SCE integrals for any value of n, no matter how large. We present the results of the SCE integrals in the form of a theorem and we give the proof.

Theorem 1. *Let the sequences above* (20). *We can show that they can, respectively, take the following forms:*

$$S_n(x) = s_n(x) \; \cos x + \hat{s}_{n-1}(x) \sin x + \alpha_n, \tag{21}$$

$$C_n(x) = c_n(x) \; \sin x + \hat{c}_{n-1}(x) \; \cos x + \beta_n, \tag{22}$$

$$E_n(x) = e_n(x) \; e^x + \gamma_n, \tag{23}$$

where c_n, s_n and e_n are polynomials of degrees $(n \in \mathbb{N})$; $\hat{s}_{n-1}$ and $\hat{c}_{n-1}$ polynomials of degrees $n-1$ with $n \in \mathbb{N}^\star$.

The α_n, β_n and γ_n are, respectively, integration constants linked to each problem of order n.

By hypothesis, we must have $s_{-n} = \hat{s}_{-n} = c_{-n} = \hat{c}_{-n} = e_{-n} = 0$, $\forall n \in \mathbb{N}^\star$.

Proof. To this end, we will use inductive proof on the "S integral", the other proofs can be deduced from it by an adequate change of variables $x = \frac{\pi}{2} \pm y$ or $x = iy$, respectively for "C" and "E integrals". Apart from the variable change, the S and C integrals can be related. It is enough to carry out an integration by parts, one obtains:

$$S_n(x) = -x^n \cos x + nC_{n-1}(x), \tag{24}$$

$$C_n(x) = x^n \sin x - nS_{n-1}(x). \tag{25}$$

Let us show that the proposal (21) is true for $n = 0$. An integration of (20), for $n = 0$, gives $S_0(x) = -\cos x + cst$. This corresponds to (21) with $s_0(x) = -1$, $\hat{s}_{-1}(x) = 0$ and $\alpha_0 = cst$. The polynomial s_0 is of degree zero and the associated integration constant is of zero order.

Suppose that the proposition (21) is true for the index m. Let us try to establish its veracity for $m+1$. From (21), we have to check that

$$S_{m+1}(x) = s_{m+1}(x)\cos x + \hat{s}_m(x)\sin x + \alpha_{m+1}. \tag{26}$$

The polynomials s_{m+1} and $\hat{s}_m$, being of degrees respectively $m+1$ and m.

From (20), we have

$$dS_m = x^m \ \sin x \ dx \text{ and } dC_m = x^m \ \cos x \ dx. \tag{27}$$

We can write

$$dS_{m+1} = x \ dS_m = x\left(s'_m + \hat{s}_{m-1}\right)\cos x \ dx + x\left(-s_m + \hat{s}'_{m-1}\right)\sin x \ dx.$$

Let $\lambda_m = x\left(s'_m + \hat{s}_{m-1}\right)$ and $\mu_{m+1} = x\left(-s_m + \hat{s}'_{m-1}\right)$. The polynomials λ_m and μ_{m+1} are of degrees respectively m and $m+1$. We then write

$$dS_{m+1} = \sum_{k=0}^{m} \tilde{\lambda}_m(k) \ dC_k + \sum_{k=0}^{m+1} \tilde{\mu}_{m+1}(k) \ dS_k, \tag{28}$$

with dS_k and dC_k given by (27).

The real $\tilde{\lambda}_m(k)$ are the order-k coefficients of the polynomial λ_m and the $\tilde{\mu}_{m+1}(k)$ are the order-k coefficients of the polynomial μ_{m+1}.

By integrating (28), we have:

$$S_{m+1}(x) = \sum_{k=0}^{m} \tilde{\lambda}_m(k) \ C_k(x) + \sum_{k=0}^{m+1} \tilde{\mu}_{m+1}(k) \ S_k(x) + K, \tag{29}$$

where K is an integration constant.

In order to express the relation (29) as a function of S_n, let us use the relation (25), which finally allows us to write

$$S_{m+1}(x) = \tau_{m+1}(x)\cos x + \kappa_m(x)\sin x + \theta_{m+1}. \tag{30}$$

with

$$\tau_{m+1}(x) = -\sum_{k=0}^{m} k \ \tilde{\lambda}_m(k) \ s_{k-1}(x) + \sum_{k=0}^{m+1} \tilde{\mu}_{m+1}(k) \ s_k(x),$$

$$\kappa_m(x) = \sum_{k=0}^{m} \tilde{\lambda}_m(k) \ \left[x^k - k \ \hat{s}_{k-2}(x)\right] + \sum_{k=0}^{m+1} \tilde{\mu}_{m+1}(k) \ \hat{s}_{k-1}(x)$$

and the integration constant of $m+1$ order taken as

$$\theta_{m+1} = -\sum_{k=0}^{m} k \ \tilde{\lambda}_m(k) \ \alpha_{k-1} + \sum_{k=0}^{m+1} \tilde{\mu}_{m+1}(k) \ \alpha_k + K.$$

The polynomials τ_{m+1} and κ_m, are respectively of degree $m+1$ and m. Of course $\tau_{m+1} = s_{m+1}, \kappa_m = \hat{s}_m$ and $\theta_{m+1} = \alpha_{m+1}$ to recover (26). This proves theorem 1. $\square$

3.2. **Differential equations related to "SCE integrals".** The SCE problems can be put into differential form, in the sense that one can substitute for each of them a differential equation. The problem can be taken in another sense, that of the resolution of a differential equation. We will explicitly make this transformation for the S integral.

By deriving the two members of (20) and (21) and identifying them, we obtain the following relations: $s'_n(x)+\hat{s}_{n-1}(x)=0$ and $\hat{s}'_{n-1}(x)-s_n(x)=x^n$, then by eliminating from these relations the polynomials $\hat{s}_{n-1}$, we obtain the differential equation to which the polynomials s_n must obey $s''_n+s_n=-x^n$. Thus, the S integral is reduced to the determination of a single family of polynomials $\{s_n; n\in\mathbb{N}\}$. For the C integral, the differential equation is: $c''_n+c_n=x^n$. The same procedure applied to the E integral leads to:

$$e'_n+e_n=x^n. \tag{31}$$

3.3. **Explicit Determination of the e_n Polynomials.** The $\tilde{e}_n$ are polynomial solutions of the differential equation (31). Let us find the general solution. We begin by writing (31) in this form $y'+y=x^n$, where y is a function of the variable x and $n\in\mathbb{N}$.

To find the general solution of the previous equation, we use the method of variation of the integration constant. One finds

$$y(x)=\tilde{e}_n(x)+C\ e^{-x}, \tag{32}$$

with C, an integration constant.

After this, let us find the polynomial solutions of (31). In order to find them, let use the adapted Frobenius method, writing

$$\tilde{e}_n(x)=\sum_{l=0}^{n} a_l\ x^l. \tag{33}$$

In addition, one obtains the following recurrence formula $a_l=\dfrac{-a_{l-1}}{l}$ for $l\in[1,n]$, which can take the form $a_l=\dfrac{(-1)^l}{l!}a_0$. Since $a_n=1$, we can find the independent term of the polynomials $\tilde{e}_n$: $a_0=n!(-1)^n$. Thus, the coefficient of the term in x^l is $a_l=(-1)^{n+l}\,\dfrac{n!}{l!}$. Finally, the $\tilde{e}_n$ have the form:

$$\tilde{e}_n(x)=\sum_{l=0}^{n}(-1)^{l+n}\,\frac{n!}{l!}x^l. \tag{34}$$

We can write the solution (34) in the form:

$$\tilde{e}_n(x)=n!\ (-1)^n\ e_n(-x), \tag{35}$$

where e_n are the *truncated-exponential polynomials*. These polynomials and their generalizations have been intensely studied by Dattoli *et al.* [10–13], Duran *et al.* [14], Srivastava *et al.* [34], ...

It is important to point out at this stage that the truncated-exponential polynomials were used by Gori [16] and Dattoli *et al.* [12] to study flattened gaussian beams.

The polynomials $\tilde{e}_n$ obey the following recursion relations:

$$\begin{aligned} \tilde{e}_n(x) &= x^n - n\ \tilde{e}_{n-1}(x); \\ \tilde{e}'_n(x) &= n\ \tilde{e}_{n-1}(x). \end{aligned} \tag{36}$$

The relations (36) show us that the $\tilde{e}_n$ polynomials form an *Appel sequence* [5].

Since the polynomials $\tilde{e}_n$ are completely determined, they serve to deduce the polynomials of the other two problems. To do this, in (20), we successively put $x = iy$ and $x = -iy$. The obtained expressions are linear combinations of the S_n and C_n, this allows, by theorem 1 to express the following relations:

$$\begin{aligned} c_n(x) &= \frac{i^n}{2}\left[(-1)^n\,\tilde{e}_n(ix) + \tilde{e}_n(-ix)\right], \\ c'_n(x) &= \frac{i^{n+1}}{2}\left[(-1)^{n+1}\,\tilde{e}_n(ix) + \tilde{e}_n(-ix)\right], \\ s_n(x) &= \frac{i^n}{2}\left[(-1)^{n+1}\,\tilde{e}_n(ix) - \tilde{e}_n(-ix)\right], \\ s'_n(x) &= \frac{i^{n+1}}{2}\left[(-1)^n\,\tilde{e}_n(ix) - \tilde{e}_n(-ix)\right]. \end{aligned} \tag{37}$$

From (37), we can see that:

$$c_n(x) = -s_n(x);\ \hat{s}_n(x) = \hat{c}_n(x);\ \forall n \in \mathbb{N} \tag{38}$$

and with (34) and (37), the explicit expressions of the s_n are

$$\begin{aligned} s_n^{even}(x) &= -x^n + \sum_{l=0}^{n-2}(-1)^{\frac{l+n}{2}+1}\frac{n!}{l!}x^l,\ n = 2, 4, \cdots \\ s_n^{odd}(x) &= -x^n + \sum_{l=0}^{n-2}(-1)^{\frac{l+n}{2}}\frac{n!}{l!}x^l,\ n = 3, 5, \cdots \end{aligned} \tag{39}$$

with $s_0(x) = -1$ and $s_1(x) = -x$.

3.4. **The Generating functions.** Let us denote the generating functions of the polynomials s_n, c_n and $\tilde{e}_n$, respectively by S, C and E:

$$\begin{aligned} S(x,t) &= \sum_{n=0}^{+\infty}\frac{1}{n!}\ s_n(x)\ t^n;\ C(x,t) \\ &= \sum_{n=0}^{+\infty}\frac{1}{n!}\ c_n(x)\ t^n;\ E(x,t) = \sum_{n=0}^{+\infty}\frac{1}{n!}\tilde{e}_n(x)t^n. \end{aligned} \tag{40}$$

We will start by determining the generating function S. The generating function C can be deduced from S, by virtue of the relation (38). To find its explicit form, we start from the differential equation of the s_n. This multiplied by $\frac{1}{n!}\, t^n$, allows to obtain the differential equation to which the generating function must obey

$$\frac{\partial^2 S}{\partial x^2} + S = -e^{xt}.$$

The general solution of the previous equation is

$$S(x,t) = f(t)\cos x + g(t)\sin x - \frac{e^{xt}}{1+t^2},$$

where f and g are arbitrary functions. For a polynomial solution, the generating function is the particular solution: $S(x,t) = \dfrac{-e^{xt}}{1+t^2}$. Let us now determine the generating function of $\tilde{e}_n$. The differential equation (31) allow to find

$$E(x,t) = \frac{e^{xt}}{1+t}. \tag{41}$$

Finally, the generating functions (40) are connected by the relation

$$-2S(x,t) = 2C(x,t) = E(ix,-it) + E(-ix,it). \tag{42}$$

4. Application of the NU method to the e_n polynomials

4.1. The $\tilde{e}_n$ polynomials are hypergeometric. Consider the differential equation (31). By differentiating it, we obtain the equation $\tilde{e}''_n + \tilde{e}'_n = nx^{n-1}$. Using this new equation and (31), we obtain the following differential equation:

$$x\tilde{e}''_n + (x-n)\,\tilde{e}'_n - n\tilde{e}_n = 0. \tag{43}$$

This equation is of hypergeometric type [27], it is of the same type as (7) with $A(x) = x, B(x) = x - n$ and $\lambda = -n$. This allows us to use the NU method in the study of this family of polynomials.

Another point of view is to use the results of S. M. Zagorodnyuk [36], in his study of the family of Hypergeometric Sobolev Orthogonal Polynomials on the unit Circle. He shows that this family is related to truncated-exponential polynomials which in turn are expressed as a function of hypergeometric functions ${}_2F_0$.

4.2. The Rodrigues formula for the $\tilde{e}_n$ polynomials. The NU method [27] provides the general Rodrigues formula suitable for all hypergeometric polynomials (see (10)). The $\tilde{e}_n$ being hypergeometric, this formula will be used. For the family of polynomials $\tilde{e}_n$, the function ρ is

$$\rho(x) = x^{-n-1}\, e^{x}. \tag{44}$$

The relation (44) replaced in (10) gives the Rodrigues formula sought

$$\tilde{e}_n(x) = \beta_n \; x^{n+1} \; e^{-x} \; \frac{d^n}{dx^n} \left(x^{-1} \; e^x \right)$$

with the constants β_n to be determined. To do this, we use the fact that $\tilde{e}_n(0) = a_0 = n! \; (-1)^n$. We have to determine the constant term provided by (10) and compare it to $\tilde{e}_n(0)$. The binomial formula applied to the nth derivative in (10) makes it possible to rewrite this expression in the form:

$$\tilde{e}_n(x) = \beta_n \; x^n \sum_{k=0}^{n} \; (-1)^k \; \frac{n!}{(n-k)!} \; x^{-k}, \tag{45}$$

the constant term is obtained for $k = n$, in (45). So, $\tilde{e}_n(0) = n! \, (-1)^n \, \beta_n \Longrightarrow \beta_n = 1$. Finally, the Rodrigues formula for the $\tilde{e}_n$ is

$$\tilde{e}_n(x) = x^{n+1} \; e^{-x} \; \frac{d^n}{dx^n} \left(x^{-1} \; e^x \right). \tag{46}$$

One can, at this point, find an expression of the polynomials s_n by using the relation of Rodrigues (46), one obtains

$$s_n(x) = \frac{-i^n}{2} \; x^{n+1} \Big[(-1)^n \; e^{-ix} \; \frac{d^n}{dx^n} \left(x^{-1} \; e^{ix} \right) + e^{ix} \; \frac{d^n}{dx^n} \left(x^{-1} \; e^{-ix} \right) \Big]. \tag{47}$$

4.3. **Generating function of the $\tilde{e}_n$.** The integral representation (17) makes it possible to determine the generating functions of the hypergeometric polynomials. Indeed, the NU method proposes the relation (19). The case of the $\tilde{e}_n$ polynomials is a particular one. The determination of the generating function, by the NU method, requires sustained attention. Then, although hypergeometric, the $\tilde{e}_n$ polynomials do not have the same behavior as the classical one known in the literature. Indeed, by observing the proof of (19), provided by Nikiforov *et al.* [27], the numerator of (17), contains the expression (13), such as the definition (18) used provides the result (19). The functions ρ_n, in the case of the polynomials $\tilde{e}_n$, are independent of the integer n. The relation (19) can not be applied because the conditions of its use are not fulfilled. Thus, it is necessary to find an expression exactly reproducing the generating function of the $\tilde{e}_n$, taking into account this peculiarity: *the case where the function ρ_n would be independent of the integer n.* In the case of the classical hypergeometric polynomials, the problem does not arise. Thus, it is necessary to write a result, similar to that of Nikiforov-Uvarov and adapted to hypergeometric polynomials having the same peculiarity as the $\tilde{e}_n$. The fact that the ρ_n function does not depend on n, leads us to pose that $\rho_n(x) = \sigma(x)$.

Before stating the new result, let us study the behavior of the function σ, in the case of hypergeometric differential equations of the same kind as (43). Let us consider the hypergeometric differential equation (7), with

$A(x) = \alpha x + \beta$; $B(x) = \gamma x + \delta$. The function ρ, in this case is given by $\rho(x) = (\alpha x + \beta)^{\frac{-\beta\gamma+\alpha(\delta-\alpha)}{\alpha^2}} \exp\left(\frac{\gamma}{\alpha}x\right)$, and the function σ by

$$\sigma(x) = (\alpha x + \beta)^{n+\frac{-\beta\gamma+\alpha(\delta-\alpha)}{\alpha^2}} \exp\left(\frac{\gamma}{\alpha}x\right). \tag{48}$$

The condition for not using the NU formula (19), in cases similar to the $\tilde{e}_n$, is

$$n + \frac{-\beta\gamma + \alpha(\delta - \alpha)}{\alpha^2} = constant. \tag{49}$$

Applied to Laguerre polynomials (or associated Laguerre) for $\alpha = 1, \beta = 0, \gamma = -1$ and $\delta = n$ (or $m + 1$), the condition (49) is not verified, therefore the generating function can be determined by (19). In the case of $\tilde{e}_n$, it is not difficult to check that (49) is respected. This is also the case, for example, for $\alpha = 1, \beta = 1, \gamma = -1$ and $\delta = -n$. This is the case of the family of polynomials $\{(-1)^n\ (x+1)^n\,; n \in \mathbb{N}\}$.

With all these previous considerations, we have to rewrite the integral representation of the $\tilde{e}_n$ (See (17)) in order to reach the equivalent of (19) in the case of hypergeometric polynomials having the same behaviour as the $\tilde{e}_n$(See (49)). Notice that the statement given below is analogous to the one in provided by Nikiforov *et al.* [27] (See (17)). The lemma below is entirely inspired by the theorem, from Nikiforov *et al.* [27], giving the integral representation of hypergeometric polynomials.

Lemma 1. *Let us consider the hypergeometric differential equation* (7) *with the first-order polynomials A and B and the constant λ defined in* (9). *If the condition* (49) *is respected, then there is a function u given by*

$$u(x) = \int_{(C)} \frac{\sigma(s)}{(s-x)^{n+1}} ds, \tag{50}$$

such that the equation (7) *admits particular solutions of the form:*

$$\begin{aligned} \tilde{e}_n(x) &= \gamma_n \frac{A^n(x)}{\sigma(x)} \int_{(C)} \frac{\sigma(s)}{(s-x)^{n+1}} ds, \\ \gamma_n &= \frac{n!}{2\pi i}\beta_n. \end{aligned} \tag{51}$$

Equipped with the result (51), we can show that the $\tilde{e}_n$ polynomials are solutions of the differential equation (7) and the generating function is given by (41). Let us now give the adapted theorem:

Theorem 2. *Let us consider the case of hypergeometric polynomials $\tilde{e}_n$ as evoked by Lemma 1. One can show that their generating function E is given by:*

$$E(x,t) = \left.\frac{\sigma(s)}{\sigma(x)}\right|_{s=\xi(x,t)}, \tag{52}$$

with the function σ *given by* (13) *and* $s = \xi(x,t)$*, solution of* $s - x - A(x)t = 0$.

Proof. To prove this theorem, we use the expression of the polynomials $\tilde{e}_n$ given by (51) and the general definition of the generating functions (18), which leads to

$$E(x,t) = \frac{1}{2\pi i \sigma(x)} \int_{(C)} \frac{\sigma(s)}{s-x} \sum_{n=0}^{+\infty} \left(\frac{A(x)t}{s-x}\right)^n ds. \tag{53}$$

Using the fact that the serie in (53) is geometric, we obtain

$$E(x,t) = \frac{1}{2\pi i \sigma(x)} \int_{(C)} \frac{\sigma(s)}{s - x - A(x)t} ds. \tag{54}$$

The value of the integral (54) can be obtained by using the residue theorem. Indeed, the pole is the solution $s = \xi(x,t)$ of the equation $s - x - A(x)t = 0$. The integral representation (51) is often known, in the literature, as "the Schlaefli integral" [6]. It coincides with the polynomial representation if the contour (C) surrounds the pole, and must be such that the function σ is analytical everywhere on and in (C) (See Arfken *et al.* [6]). This is the case for the pole $s = \xi(x,t)$. So, by virtue of the residue theorem, we obtain the formula (52). □

5. Complete solution of the hypergeometric differential equation (43)

5.1. **General solution of the equation** (43)**.** Consider the differential equation $xy'' + (x-n)\,y' - ny = 0$. Let $y(x) = e^{-x}\ u(x)$; the equation above becomes $xu'' - (x+n)\,u' = 0$. By integrating this equation, we find $u(x) = C_1 + C_2 \displaystyle\int x^n\ e^x dx$, or

$$y(x) = C_1\ e^{-x} + C_2\ \tilde{e}_n(x). \tag{55}$$

This result proves that the family of polynomials $\tilde{e}_n$ is a solution of (43). In addition, the second solution is the function $y = e^{-x}$. We can see the analogy with the results obtained before (See (32)).

5.2. **The $\tilde{e}_n$ polynomials and the associated Laguerre polynomials "of a particular type".** The associated Laguerre polynomials and the $\tilde{e}_n$ are hypergeometric polynomials. A careful observation shows similarities between their differential equations. We are led to think that there is a relationship between these two families. To achieve this relation, we start by comparing the differential equation of the associated Laguerre polynomials [6, 10–12, 20, 24, 25, 27, 30–32, 34]:

$$x \frac{d^2}{dx^2} L_n^{(\alpha)}(x) + (\alpha + 1 - x)\frac{d}{dx} L_n^{(\alpha)}(x) + n\ L_n^{(\alpha)}(x) = 0 \tag{56}$$

and (43). We notice that two changes must take place on (56). First, we have to change the variable x to $-x$, which leads to:

$$x\frac{d^2}{dx^2}L_n^{(\alpha)}(-x) + (\alpha+1+x)\frac{d}{dx}L_n^{(\alpha)}(-x) - n\ L_n^{(\alpha)}(-x) = 0. \tag{57}$$

The second change follows from the comparison of the differential equations (43) and (57). From this, we find that the order of the associated Laguerre polynomials must be $\alpha = -n-1$. Thus, we conclude that the two families of polynomials must be proportional. We write

$$\tilde{e}_n(x) = \lambda\ L_n^{(-n-1)}(-x), \tag{58}$$

with λ a non-zero constant.

This result is not surprising because the two families being hypergeometric, they can be deduced from hypergeometric and confluent hypergeometric functions which are general solutions of (7). Indeed, it is possible to express a number of elementary and special functions in terms of hypergeometric and confluent hypergeometric functions [1, 6, 7, 17, 27, 35, 36].

To determine the proportionality constant, we write the Rodrigues formula of $L_n^{(-n-1)}(-x)$ [6–8, 26, 27]:

$$L_n^{(-n-1)}(-x) = \frac{e^{-x}}{n!}\ x^{n+1}\ \frac{d^n}{dx^n}\left(x^{-1}\ e^x\right) \tag{59}$$

that we compare with the $\tilde{e}_n$'s Rodrigues formula(See (46)), we find

$$\tilde{e}_n(x) = n!\ L_n^{(-n-1)}(-x). \tag{60}$$

It is easy to verify this, knowing the expressions of the Laguerre polynomials of order α, that from (60), we get, for example, $\tilde{e}_0(x) = 1, \tilde{e}_1(x) = x - 1, \tilde{e}_2(x) = x^2 - 2x + 2, \ldots$

The relation (60) between these two families of polynomials can be verified by comparing the generating function of the associated Laguerre polynomials, obtained by H. J. Weber [35] and that of the $\tilde{e}_n$ polynomials given in (41).

CONCLUSION

This paper aims at constructing families of polynomials to solve the SCE problems. After investigation, we found relationships between the three problems. Thus, the search for these polynomial sequences can simply be summarized in the determination of only one of them. We focus on the sequence $\{\tilde{e}_n; n \in \mathbb{N}\}$.

First, one can see that this sequence is an Appel one [5]. Second, they are hypergeometric polynomials. As a result, the NU method can be used to demonstrate or verify some results: Rodrigues formula, generating function, ... As for the generating function, the formula (19) of Nikiforov-Uvarov can not be used without care. Thus, we have established a new result, inspired by Nikiforov *et al.* [27] (Lemma 1), taking into account the peculiarity of $\tilde{e}_n$

polynomials. This property is the singular presence of the integer n in the expression of the polynomial B of the differential equation (See (43)). The corollary of this property is that the function $\sigma(x) \equiv \rho_n(x)$(See (12) and (13)), for the $\tilde{e}_n$, is independent of the integer n. This result does not occur for classical hypergeometric polynomials (See (48) and (49)).

Third, the $\tilde{e}_n$ polynomials can be deduced from hypergeometric functions. Indeed, the differential equation (7) is a confluent hypergeometric equation [1, 6, 7]. This result leads to a relation between the $\tilde{e}_n$ and the associated Laguerre polynomials with negative orders (See (60)).

The use of the Frobenius method for the determination of the three families of polynomials, related to SCE problems, made it possible to find the results of Dattoli *et al.* [10–13], Srivastava *et al.* [34], Gori [16] and Zagorodnyuk [36]. The rest of the work focused on the link with the NU method, the determination of the Rodrigues formula and the study of the particular properties of the hypergeometric polynomials of the same type as the $\tilde{e}_n$. Indeed, the generating functions of such families can not be deduced, directly, from the result (19) of the NU method. This last fact led to the development of Theorem 4.2 adapted to the situation.

At the end of our investigation, we obtained five procedures for determining the $\tilde{e}_n$: firstly, we obtain a polynomial serie given by (34) using Frobenius method applied to (31); secondly, the Rodrigues formula (46) provided by the NU method; thirdly, the expression of the generating function (41). Fourthly and fifthly, the expressions of the $\tilde{e}_n$ in term of truncated-exponential polynomials and associated Laguerre polynomials. The present study shows how some mathematical problems can lead to results connected to other ones obtained by various methods and procedures.

A natural generalization going above the SCE problems will treat non trivial dependences of the funtions A and B on a natural integer n. The path followed here already brings fruitfull results. We are working on it and hope to present the corresponding results soon.

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Appendix A. Some relations between polynomials from SCE integrals

The polynomials of SC problems are linked, we have:

$$\hat{s}_{n-1}(x) = -s'_n(x), \;\; \hat{c}_{n-1}(x) = c'_n(x), \tag{61}$$

$$c_n = x^n + ns'_{n-1},\ c'_n = -ns_{n-1}. \tag{62}$$

Other relationships can be found between the families. The first group of these relations is obtained by using the relation of recurrence (24) and by substituting the relations (21), one finds

$$s_n = -x^n + n\ \hat{c}_{n-2},\ \hat{s}_n = (n+1)\, c_n. \tag{63}$$

The second group is obtained by integrating once again (24) by parts, this leads to

$$\begin{aligned} S_n(x) &= -x^n\ \cos x + nx^{n-1}\ \sin x - n\,(n-1)\, S_{n-2}(x); \\ C_n(x) &= x^n\ \sin x + nx^{n-1}\ \cos x - n\,(n-1)\, C_{n-2}(x), \end{aligned} \tag{64}$$

then substituting (21), we have the second group

$$\begin{aligned} & s_n = -x^n - n\,(n-1)\, s_{n-2},\ \hat{s}_n = (n+1)\, x^n - n\,(n+1)\, \hat{s}_{n-2}; \\ & c_n = x^n - n\hat{s}_{n-2},\ \hat{c}_n = -\,(n+1)\, s_n; \\ & c_n = x^n - n\,(n-1)\, c_{n-2},\ \hat{c}_n = (n+1)\, x^n - n\,(n+1)\, \hat{c}_{n-2}. \end{aligned} \tag{65}$$

APPENDIX B. GENERALIZATION OF THE E INTEGRAL

The polynomials of SC problems are linked, we have:

$$\hat{s}_{n-1}(x) = -s'_n(x),\ \hat{c}_{n-1}(x) = c'_n(x), \tag{66}$$

$$c_n = x^n + ns'_{n-1},\ c'_n = -ns_{n-1}. \tag{67}$$

Other relationships can be found between the families. The first group of these relations is obtained by using the relation of recurrence (24) and by substituting the relations (21), one finds

$$s_n = -x^n + n\ \hat{c}_{n-2},\ \hat{s}_n = (n+1)\, c_n. \tag{68}$$

The second group is obtained by integrating once again (24) by parts, this leads to

$$\begin{aligned} S_n(x) &= -x^n\ \cos x + nx^{n-1}\ \sin x - n\,(n-1)\, S_{n-2}(x); \\ C_n(x) &= x^n\ \sin x + nx^{n-1}\ \cos x - n\,(n-1)\, C_{n-2}(x), \end{aligned} \tag{69}$$

then substituting (21), we have the second group

$$\begin{aligned} & s_n = -x^n - n\,(n-1)\, s_{n-2},\ \hat{s}_n = (n+1)\, x^n - n\,(n+1)\, \hat{s}_{n-2}; \\ & c_n = x^n - n\hat{s}_{n-2},\ \hat{c}_n = -\,(n+1)\, s_n; \\ & c_n = x^n - n\,(n-1)\, c_{n-2},\ \hat{c}_n = (n+1)\, x^n - n\,(n+1)\, \hat{c}_{n-2}. \end{aligned} \tag{70}$$

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