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Fractional differentiation composition operators from S_p spaces to H_q spaces

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Abstract. Let S_p be the space of functions analytic on the unit disk and whose derivatives belong to the Hardy space. In this article, we investigate the boundedness and compactness of the fractional differentiation composition operators from S_p spaces into Hardy spaces. Furthermore, we derive a sufficient condition for the boundedness of the fractional differentiation composition operators on S_p spaces. These results extends some well-known results in literature.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane and $H(\mathbb{D})$ be the set of all analytic functions on the unit disk. For $1 \leq p \leq \infty$, the Hardy space H_p is defined as follows:

$$H_p = \left\{ f \in \ H(\mathbb{D}) : \|f\|_{H_p} = \lim_{r \to 1} M_p(f, r) < \infty \right\},$$

where

$$M_p(f,r) = \begin{cases} \left(\dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^p d\theta \right)^{\frac{1}{p}}, & \text{if } p \in (0,\infty); \\ \displaystyle\sup_{\theta \in [0,2\pi]} \{|f(re^{i\theta})|\}, & \text{if } p = \infty. \end{cases}$$

For $1 \leq p \leq \infty$, we denote by S_p the space of all analytic functions f on the unit disc $\mathbb{D}$ whose derivative f' lies in H_p, endowed with the norm

$$\|f\|_{S_p} = |f(0)| + \left\| f' \right\|_{H_p}. \tag{1}$$

It is clear that S_p is a Banach space with respect to this norm. See [2, 7] for more information on this space.

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The classical/Gaussian hyper-geometric series is defined by the power series expansion

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n,\quad |z|<1. \tag{2}$$

Here a,b,c are complex numbers such that $c\neq -m$, $m=0,1,2,\ldots$, and $(a)_n$ is Pochhammer's symbol/shifted factorial defined by Appel's symbol

$$(a)_n=a(a+1)\cdots(a+n-1)=\frac{\Gamma(a+n)}{\Gamma(a)},\quad n\in\mathbb{N},$$

and $(a)_0=1$ for $a\neq 0$. Obviously $F(a,b;c;z)\in H(\mathbb{D})$. Many properties of the hypergeometric series are found in standard textbooks such as [1,16].

For any two analytic functions f and g represented by their power series expansions,

$$f(z)=\sum_{n=0}^{\infty}a_nz^n,\qquad g(z)=\sum_{n=0}^{\infty}b_nz^n\in |z|<R,$$

the Hadamard product (or convolution) of f and g denoted by $f*g$ and is defined by

$$(f*g)(z)=\sum_{n=0}^{\infty}a_nb_nz^n. \tag{3}$$

In particular for $f,g\in H(\mathbb{D})$, we have

$$(f*g)(\rho z)=\frac{1}{2\pi}\int_0^{2\pi}f\left(\rho e^{it}\right)g\left(ze^{-it}\right)dt,\quad 0<\rho<1. \tag{4}$$

If $f(z)=\sum_{n=0}^{\infty}a_nz^n\in H(\mathbb{D})$ and $\beta>0$, then the fractional derivative $f^{[\beta]}$ (see [9]) of order β is defined by

$$f^{[\beta]}(z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+1+\beta)}{\Gamma(n+1)}a_nz^n. \tag{5}$$

In terms of convolution, we also have

$$f^{[\beta]}(z)=\Gamma(1+\beta)\left(f(z)*F(1,1+\beta;1;z)\right). \tag{6}$$

For $\beta=0$, we define

$$f^{[0]}(z)=f(z)$$

and for $n\in\mathbb{N}$ we have

$$f^{[n]}(z)=\frac{d^n}{dz^n}\left(z^nf(z)\right). \tag{7}$$

The fractional differentiation composition operator, denoted by $D^{\beta}_{\varphi,u}$ is defined as follows ([5,6])

$$D^{\beta}_{\varphi,u}f(z)=u(z)f^{[\beta]}\left(\varphi(z)\right),\ f\in H(\mathbb{D}), \tag{8}$$

where $u(z) \in H(\mathbb{D})$ and $\varphi(z)$ is nonconstant analytic self-map of $\mathbb{D}$.

This operator can be viewed as a generalization of a multiplication operator and a weighted composition operator. For $\beta = 0$, $D^{\beta}_{\varphi,u}$ equals the weighted composition operator defined by $(uC_\varphi)(f)(z) = u(z)f(\varphi(z))$, $z \in \mathbb{D}$, which reduces to the composition operator C_φ for $u(z) \equiv 1$. During the last century, composition operators were studied between different spaces of analytic functions. If $\beta = 1$, we get the operator $D^1_{\varphi,u} = M_u C_\varphi + M_{u\varphi} C_\varphi D$, which for $u(z) \equiv 1$ gives $D^1_{\varphi,1} = C_\varphi + \varphi C_\varphi D$, which for $\varphi(z) = z$ gives $D^1_{z,u} = M_u + zM_u D$ and $u(z) = \varphi'(z)$ gives $D^1_{\varphi,\varphi'} = M_{\varphi'} C_\varphi + \varphi D C_\varphi$. For particular choices of β, u and φ, we obtain many operators which is product, addition and composition of multiplication and differentiation operators. Weighted composition operators find its usefullness in many different ways. For example they are isometries of many Banach spaces of analytic functions. Novinger and Oberin proved that the isometries in S_p for $1 \leq p < \infty, p \neq 2$ are given by

$$Tf(z) = \lambda_1 \left[f(0) + \int_0^z W_{\varphi,\lambda_2(\varphi')^{1/p}}(f')(\xi)d\xi \right]$$

for $f \in S_p$ where $|\lambda_1| = |\lambda_2| = 1$ and φ is a self-analytic map of $\mathbb{D}$.

In 1978, the S_p spaces were introduced by Roan in [11], where he studied the boundedness and compactness of the composition operators $C_\varphi, (1 \leq p < \infty)$. Contreras and Hernández-Díaz characterized the boundedness, compactness of $W_{\varphi,\psi}$ from S_p into S_q, $1 \leq p, q < \infty$ in terms of weighted composition operators on Hardy spaces (See [7]). For some recent papers on different operators on S_p spaces one can see [3,7,10,12]. Recently in [17] Xie et al introduced a similar space B_p which consists of all $f \in H(\mathbb{D})$ such that f' belongs to the Bergman space. In that paper the authors investigated the boundedness and compactness of the weighted composition operators in B_p spaces.

The operator $D^{\beta}_{\varphi,u}$ was introduced by Naik and Borgohain in [5], where they studied the boundedness and compactness of this operator from mixed-norm spaces to weighted-type spaces. Recently the author of this paper studied the boundedness of the operator $D^{n}_{\varphi,u}$ from S_p spaces to weighted-type spaces.

In this paper we characterize the boundedness and compactness of the operator $D^{\beta}_{\varphi,u}$ from S_p to H_q spaces. Throughout this paper C is any constant which may vary for different lines.

2. Preliminary results

We collect some basic lemmas which are useful in the proof of the main results.

Lemma 1 ([13], Proposition 1.4.10). *For $\gamma > 1$ one has*

$$\int_0^{2\pi} \frac{d\theta}{|1-z|^\gamma} \leq C \frac{1}{(1-|z|)^{\gamma-1}}.$$

Lemma 2 ([8], Proposition 3.1). *If $f \in H_p$ $(0 < p < \infty)$, then*

$$|f(z)| \leq \; 2^{1/p} \; \frac{||f||_p}{(1-r)^{1/p}}, \quad r = |z|.$$

Lemma 1 is true for $f \in H_p$. Here we will give a similar result which involves fractional derivative $f^{[\beta]}$ of $f \in S_p$.

Proposition 1. *Suppose $f \in S_p$ for $0 < p \leq \infty$.*

(a) *For $1 \leq p \leq \infty$*

$$|f^{[0]}(z)| \leq C\|f\|_{S_p},$$

and for $\beta \geq 1$

$$|f^{[\beta]}(z)| \leq \frac{2^{1/p}\Gamma(1+\beta)}{1/p+\beta-1} \frac{\|f\|_{S_p}}{(1-|z|)^{1/p+\beta-1}}.$$

(b) *For $0 < p < 1$ and $0 \leq \beta < \infty$*

$$|f^{[\beta]}(z)| \leq \frac{2^{1/p}\Gamma(1+\beta)}{1/p-1} \frac{\|f\|_{S_p}}{(1-|z|)^{1/p+\beta-1}}.$$

Proof. For all $p \in (0, \infty]$, $f \in S_p$ implies $f' \in H_p$. Hence, Lemma 1 gives

$$|f'(z)| \leq 2^{1/p} \frac{\|f\|_{S_p}}{(1-|z|)^{1/p}} \tag{9}$$

Now, for any curve $\alpha(t) = x(t) + iy(t), 0 \leq t \leq 1$ from 0 to $z = x + iy$, we have

$$f(z) = \int_0^z f'(w)dw = \int_0^1 f'(\alpha(t))\,\alpha'(t)dt.$$

It is clear that

$$|f(z)| = \left| z \int_0^1 f'(tz)dt \right| \leq |z| \int_0^1 \left|f'(tz)\right| \; dt. \tag{10}$$

(a) The proof for the case $\beta = 0$ can be found in the proof of Theorem 2.1 of [7]. For $\beta \geq 1$ the definition of fractional derivative gives

$$\begin{aligned} |f^{[\beta]}(z)| &= |\Gamma(1+\beta)(f(z) * F(1, 1+\beta; 1; z))| \\ &= \frac{\Gamma(1+\beta)}{(2\pi)} \int_0^{2\pi} |f(ze^{-it})| \frac{1}{|1-\rho e^{it}|^{1+\beta}} dt. \end{aligned}$$

Lemma 1 and inequality (9) give

$$
\begin{aligned}
|f^{[\beta]}(z)| &= \frac{\Gamma(1+\beta)}{(2\pi)} \int_0^{2\pi} |f(ze^{-it})| \frac{1}{|1-\rho e^{it}|^{1+\beta}} dt \\
&\leq \frac{\Gamma(1+\beta)}{(2\pi)} \int_0^{2\pi} \left[\int_0^1 |ze^{-it}| 2^{1/p} \frac{\|f\|_{S_p}}{(1-\xi|z|)^{1/p}} d\xi \right] \frac{1}{|1-\rho e^{it}|^{1+\beta}} dt \\
&\leq \frac{\Gamma(1+\beta)}{(2\pi)} \int_0^1 |z| 2^{1/p} \frac{\|f\|_{S_p}}{(1-\xi|z|)^{1/p}} \frac{1}{(1-\rho)^{\beta}} d\xi.
\end{aligned}
$$

Taking $\rho = \frac{1+\xi|z|}{2}$, we have

$$
\begin{aligned}
|f^{[\beta]}(z)| &\leq 2^{1/p+\beta} \|f\|_{S_p} \frac{\Gamma(1+\beta)}{(2\pi)} \int_0^1 |z| \frac{1}{(1-\xi|z|)^{1/p+\beta}} d\xi \\
&\leq \frac{2^{1/p+\beta}\Gamma(1+\beta)}{1/p+\beta-1} \frac{\|f\|_{S_p}}{(1-\xi|z|)^{1/p+\beta-1}} \\
&\leq \frac{2^{1/p+\beta}\Gamma(1+\beta)}{1/p+\beta-1} \frac{\|f\|_{S_p}}{(1-|z|)^{1/p+\beta-1}}.
\end{aligned}
$$

(b) Equation (10) gives

$$
|f(z)| \leq \int_0^1 \frac{2^{1/p}\,\|f\|_{S_p}}{(1-t|z|)^{1/p}} dt \leq \frac{2^{1/p}}{1/p-1} \frac{\|f\|_{S_p}}{(1-|z|)^{1/p-1}}.
$$

Using the definition of fractional derivative, we have for $0 \leq \beta < \infty$

$$
\begin{aligned}
|f^{[\beta]}(z)| &\leq \int_0^{2\pi} |f(ze^{-it})| |F(1, 1+\beta; 1; \rho e^{it})| dt \\
&\leq \int_0^{2\pi} \frac{2^{1/p}}{1/p-1} \frac{\|f\|_{S_p}}{(1-|ze^{-it}|)^{1/p-1}} \frac{1}{|1-\rho e^{it}|^{\beta+1}} dt \\
&\leq \frac{2^{1/p}}{1/p-1} \int_0^{2\pi} \frac{\|f\|_{S_p}}{(1-|ze^{-it}|)^{1/p-1}} \frac{1}{|1-\rho e^{it}|^{\beta+1}} dt.
\end{aligned}
$$

Taking $\rho = \frac{1+|z|}{2}$ gives

$$
\begin{aligned}
|f^{[\beta]}(z)| &\leq \frac{2^{1/p}}{1/p-1} \frac{\|f\|_{S_p}}{(1-|z|)^{1/p-1}} \frac{1}{|1-\rho|^{\beta}} \\
&\leq \frac{2^{1/p}}{1/p-1} \frac{\|f\|_{S_p}}{(1-|z|)^{1/p+\beta-1}}.
\end{aligned}
$$

□

Theorem 1. *The set of polynomials is dense in S_p for $0 < p < \infty$.*

Proof. Suppose $f \in S_p$. Then $f' \in H_p$. Since polynomials are dense in H_p, therefore there exist a sequence of polynomials $\{p_n(z)\}_{n=1}^{\infty}$ in H_p such that $\lim_{n\to\infty} \|p_n - f'\|_{H_p} = 0$. Define

$$
P_n(z) = f(0) + \int_0^z p_n(w)dw.
$$

Then $P_n(z)$ is a polynomial and

$$\lim_{n\to\infty} \|P_n - f\|_{S_p} = \lim_{n\to\infty} \left[|P_n(0) - f(0)| + \|P_n' - f'\|_{H_p}\right]$$
$$= \lim_{n\to\infty} \|p_n - f'\|_{H_p} = 0. \qquad \square$$

3. Boundedness of the operator $D^{\beta}_{\varphi,u} : S_p \to H_q$

In this section we characterize the boundedness of the operator $D^{\beta}_{\varphi,u}$ from S_p spaces to H_q spaces.

Theorem 2. *Let $0 < q < \infty$ and $f \in S_p$ for $0 < p \le \infty$, $u(z) \in H(\mathbb{D})$ and $\varphi(z)$ is an analytic self-map of $\mathbb{D}$. Then for $1 \le p \le \infty$ and $\beta \ge 1$ or $0 < p < 1$ and $\beta \ge 0$ the operator $D^{\beta}_{\varphi,u} : S_p \to H_q$ is bounded if and only if*

$$\sup_{0<r<1} \left[\frac{1}{2\pi}\int_0^{2\pi} \frac{|u(z)|^q}{(1-|\varphi(z)|^2)^{q/p+q(\beta-1)}} d\theta\right]^{1/q} < \infty. \tag{11}$$

Proof. Suppose (11) holds. Then

$$\|D^{\beta}_{\varphi,u} f\|_{H_q} = \sup_{0<r<1} \left[\frac{1}{2\pi}\int_0^{2\pi} |D^{\beta}_{\varphi,u} f(z)|^q d\theta\right]^{1/q}$$
$$= \sup_{0<r<1} \left[\frac{1}{2\pi}\int_0^{2\pi} |u(z) f^{[\beta]}(\varphi(z))|^q d\theta\right]^{1/q}.$$

Proposition 1 gives

$$\|D^{\beta}_{\varphi,u} f\|_{H_q} \le C\|f\|_{S_p} \sup_{0<r<1} \left[\frac{1}{2\pi}\int_0^{2\pi} \frac{|u(z)|^q}{(1-|\varphi(z)|^2)^{q/p+q(\beta-1)}} d\theta\right]^{1/q} < C\|f\|_{S_p}.$$

Hence, $D^{\beta}_{\varphi,u} : S_p \to H_q$ is bounded. Conversely, suppose $D^{\beta}_{\varphi,u} : S_p \to H_q$ is bounded and for any $w \in \mathbb{D}$ define the test function

$$f_w(z) = (1-|w|^2) F\left(\frac{1}{p} + \beta, 1; 1+\beta; \overline{w}z\right).$$

Then

$$f_w'(z) = (1-|w|^2)\frac{\frac{1}{p}+\beta}{1+\beta} F\left(\frac{1}{p} + \beta + 1, 2; 2+\beta; \overline{w}z\right)$$
$$= (1-|w|^2)\frac{\frac{1}{p}+\beta}{1+\beta} \frac{1}{(1-\overline{w}z)^{1/p+1}} F\left(1 - \frac{1}{p}, \beta; 2+\beta; \overline{w}z\right).$$

Since $2+\beta - \left(1-\frac{1}{p}\right) - \beta = 1 + \frac{1}{p} > 0$, therefore $F\left(1-\frac{1}{p}, \beta; 2+\beta; \overline{w}z\right)$ is bounded in $\mathbb{D}$. Hence, there is a constant C such that

$$|f_w'(z)| \le C\frac{(1-|w|^2)}{(1-\overline{w}z)^{1/p+1}}.$$

Therefore,

$$M_p^p(f'_w, r) = \frac{1}{2\pi}\int_0^{2\pi} |f'_w(re^{\theta})|d\theta$$
$$\leq \frac{1}{2\pi}\int_0^{2\pi} C\frac{(1-|w|^2)^p}{|1-\overline{w}re^{i\theta}|^{1+p}}d\theta.$$

Lemma 1 gives

$$M_p^p(f'_w, r) \leq C\frac{(1-|w|^2)^p}{(1-|w|r)^p} < \infty.$$

Hence $f'_w \in H_p$ and therefore $f_w \in S_p$. The fractional derivative of f_w is

$$\begin{aligned}
f_w^{[\beta]}(z) &= \Gamma(1+\beta)\left(f_w * F(1, 1+\beta; 1; z)\right)\\
&= \Gamma(1+\beta)\left((1-|w|^2)F\left(\frac{1}{p}+\beta, 1; 1+\beta; \overline{w}z\right) * F(1, 1+\beta; 1; z)\right)\\
&= \Gamma(1+\beta)(1-|w|^2)\left(\sum_{n=0}^{\infty}\frac{(1/p+\beta)_n(1)_n}{(1)_n(1+\beta)_n}\overline{w}^n z^n * \sum_{n=0}^{\infty}\frac{(1+\beta)_n(1)_n}{(1)_n(1)_n}z^n\right)\\
&= \Gamma(1+\beta)(1-|w|^2)\left(\sum_{n=0}^{\infty}\frac{(1/p+\beta)_n}{(1)_n}\overline{w}^n z^n\right)\\
&= \Gamma(1+\beta)\frac{(1-|w|^2)}{(1-\overline{w}z)^{1/p+\beta}}.
\end{aligned}$$

This gives

$$\begin{aligned}
M_q^q(D_{\varphi,u}^{\beta}f_w, r) &= \frac{1}{2\pi}\int_0^{2\pi} |D_{\varphi,u}^{\beta}f_w(z)|^q d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi} |u(z)f_w^{[\beta]}(\varphi(z))|^q d\theta\\
&= \Gamma(1+\beta)\frac{1}{2\pi}\int_0^{2\pi}\frac{|u(z)|^q(1-|w|^2)}{|1-\bar{w}\varphi(z)|^{q/p+q\beta}}d\theta.
\end{aligned}$$

Taking $w = \varphi(z)$ gives

$$M_q^q(D_{\varphi,u}^{\beta}f_w, r) = \Gamma(1+\beta)\frac{1}{2\pi}\int_0^{2\pi}\frac{|u(z)|^q}{(1-|\varphi(z)|^2)^{q/p+q(\beta-1)}}d\theta.$$

Since $D_{\varphi,u}^{\beta} : S_p \to H_q$ is bounded, we have

$$\sup_{0<r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{|u(z)|^q}{(1-|\varphi(z)|^2)^{q/p+q(\beta-1)}}d\theta\right)^{1/q} < C\|f\|_{S_p} < \infty. \qquad \square$$

In the next result, we give a sufficient condition for the operator $D_{\varphi,u}^{\beta} : S_p \to S_q$ to be bounded.

Theorem 3. *Let $0 < p, q < \infty, u(z) \in H(\mathbb{D})$ and φ is an analytic self-map of $\mathbb{D}$. In addition suppose $f(z)$ satisfies the following properties:*

$$\sup_{0<r<1}\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{|u'(z)|^q}{(1-|\varphi(z)|^2)^{q/p+q(\beta-1)}}d\theta\right]^{1/q}<\infty \tag{12}$$

and

$$\sup_{0<r<1}\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{|u(z)\varphi'(z)|^q}{(1-|\varphi(z)|^2)^{q/p+q\beta}}d\theta\right]^{1/q}<\infty. \tag{13}$$

Then the operator $D^{\beta}_{\varphi,u}: S_p \to S_q$ is bounded.

Proof. Suppose (12) and (13) hold. Then Proposition 1 gives

$$\begin{aligned}
M_q((D^{\beta}_{\varphi,u}f)',r) &= \left(\int_0^{2\pi}|(D^{\beta}_{\varphi,u}f)'(z)|^q d\theta\right)^{1/q}\\
&\leq \left(\int_0^{2\pi}|u'(z)f^{[\beta]}(z)|^q d\theta\right)^{1/q} + \left(\int_0^{2\pi}|u(z)\varphi'(z)f^{[\beta]+1}(z)|^q d\theta\right)^{1/q}\\
&\leq C\left(\int_0^{2\pi}\frac{|u'(z)|^q\|f\|^q_{S_p}}{(1-|\varphi(z)|^2)^{q/p+q(\beta-1)}}d\theta\right)^{1/q}\\
&\quad + \left(\int_0^{2\pi}\frac{|u(z)\varphi'(z)|^q\|f\|^q_{S_p}}{(1-|\varphi(z)|^2)^{q/p+q\beta}}d\theta\right)^{1/q}\\
&\leq C\Bigg[\left(\int_0^{2\pi}\frac{|u'(z)|^q}{(1-|\varphi(z)|^2)^{q/p+q(\beta-1)}}d\theta\right)^{1/q}\\
&\quad + \left(\int_0^{2\pi}\frac{|u(z)\varphi'(z)|^q}{(1-|\varphi(z)|^2)^{q/p+q\beta}}d\theta\right)^{1/q}\Bigg]\|f\|_{S_p}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|(D^{\beta}_{\varphi,u}f)'\|_{H_q} \leq & C\Bigg[\sup_{0<r<1}\left(\int_0^{2\pi}\frac{|u'(z)|^q}{(1-|\varphi(z)|^2)^{q/p+q(\beta-1)}}d\theta\right)^{1/q}\\
&+\sup_{0<r<1}\left(\int_0^{2\pi}\frac{|u(z)\varphi'(z)|^q}{(1-|\varphi(z)|^2)^{q/p+q\beta}}d\theta\right)^{1/q}\Bigg]\|f\|_{S_p}\\
&\leq C\|f\|_{S_p}
\end{aligned}$$

Hence, $D^{\beta}_{\varphi,u}: S_p \to S_q$ is bounded. □

4. COMPACTNESS OF THE OPERATOR $D^{\beta}_{\varphi,u}$

In this section we find conditions for the compactness of the operator $D^{\beta}_{\varphi,u}$ from S_p spaces to H_q spaces. Now we state the following result whose proof can be obtained by adapting the proof of Lemma 9 of [5].

Lemma 3. *Suppose $\beta \geq 0$, $0 < p < \infty, u(z) \in H(\mathbb{D})$ and φ is an analytic self-map of $\mathbb{D}$. Then the operator $D^{\beta}_{\varphi,u} : S_p \to H_q$ is compact if and only if $D^{\beta}_{\varphi,u} : S_p \to H_q$ is bounded and for any bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in S_p which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have*

$$\|D^{\beta}_{\varphi,u} f_k\|_{H_q} \to 0,$$

as $k \to \infty$.

Theorem 4. *Suppose $\beta \geq 0$, $0 < p, q \leq \infty, u(z) \in H(\mathbb{D})$ and $\varphi(z)$ is an analytic self-map of $\mathbb{D}$. Then the operator $D^{\beta}_{\varphi,u} : S_p \to H_q$ is compact if and only if $D^{\beta}_{\varphi,u} : S_p \to H_q$ is bounded and*

$$\lim_{|\varphi(z)|\to 1} \int_0^{2\pi} \frac{|u(z)|^q}{(1-|\varphi(z)|^2)^{q/p+q(\beta-1)}} d\theta = 0. \tag{14}$$

Proof. Suppose, $D^{\beta}_{\varphi,u} : S_p \to H_q$ is bounded and (14) holds. This implies that for every $\varepsilon > 0$ there exist $\rho \in (0,1)$ such that when $\rho < |\varphi(z)| < 1$,

$$\int_0^{2\pi} \frac{|u(z)|^q}{(1-|\varphi(z)|^2)^{\frac{q}{p}+q(\beta-1)}} d\theta < \varepsilon.$$

Suppose $g(z) = \frac{1}{\Gamma(1+\beta)}$. Obviously, $g(z) \in S_p$. This implies that $u(z) \in H_q$.

Assume that $\{h_k\}_{k\in\mathbb{N}}$ is bounded sequence in S_p and converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. For $\rho < |\varphi(z)| < 1$,

$$\int_0^{2\pi} |u(z) h_k^{[\beta]}(\varphi(z))|^q d\theta \leq C\|h_k\|_{S_p} \int_0^{2\pi} \frac{|u(z)|^q}{(1-|\varphi(z)|^2)^{\frac{1}{p}+\beta-1}} d\theta < LC\varepsilon. \tag{15}$$

For $|\varphi(z)| \leq \rho$, we have

$$h_k^{[\beta]}(\varphi(z)) = \frac{\Gamma(1+\beta)}{2\pi i} \int_{|w|=\xi} h_k(w) F\left(1, 1+\beta; 1; \frac{\varphi(z)}{w}\right) \cdot \frac{1}{w} dw, \quad |\varphi(z)| < |w|.$$

Hence, $w = \xi e^{i\psi}$ gives us

$$\begin{aligned} h_k^{[\beta]}(\varphi(z)) &= \frac{\Gamma(1+\beta)}{2\pi} \int_0^{2\pi} h_k(w) F\left(1, 1+\beta; 1; \frac{\varphi(z)}{w}\right) d\psi \\ &= \frac{\Gamma(1+\beta)}{2\pi} \int_0^{2\pi} h_k(w) \frac{1}{\left(1-\frac{\varphi(z)}{w}\right)^{1+\beta}} d\psi. \end{aligned}$$

A simple calculation gives us

$$\begin{aligned} |h_k^{[\beta]}(\varphi(z))| &\leq \frac{\Gamma(1+\beta)}{2\pi} \int_0^{2\pi} |h_k(w)| \frac{1}{\left|1-\frac{\varphi(z)}{w}\right|^{1+\beta}} d\psi \\ &\leq \frac{\Gamma(1+\beta)}{2\pi(\xi - r)^{1+\beta}} \int_0^{2\pi} |h_k(w)| d\psi, \end{aligned}$$

where $|\varphi(z)| = r$. Therefore,

$$|h_k^{[\beta]}(\varphi(z))| \leq C \int_0^{2\pi} |h_k(w)| d\psi$$

in compact subsets of $\mathbb{D}$. Therefore, we have

$$\left[\int_0^{2\pi} |u(z)|^q |h_k^{[\beta]}(\varphi(z))|^q d\theta\right]^{1/q} \leq \left[\int_0^{2\pi} \left|u(z)\int_0^{2\pi} h_k(w) d\psi\right|^q d\theta\right]^{1/q}$$

Minkowski's inequality gives

$$\begin{aligned}\left[\int_0^{2\pi} |u(z)|^q |h_k^{[\beta]}(\varphi(z))|^q d\theta\right]^{1/q} &\leq \int_0^{2\pi} \left[\int_0^{2\pi} |u(z) h_k(w)|^q \, d\theta\right]^{1/q} d\psi \\ &\leq \|u\|_{H_q} \int_0^{2\pi} \left[\int_0^{2\pi} |h_k(w)|^q d\theta\right]^{1/q} d\psi \to 0\end{aligned}$$

on compact subsets of $\mathbb{D}$. It follows that $\|D^{\beta}_{\varphi,u} h_k\|_{H_q} \to 0$ as $k \to \infty$. Therefore, the operator $D^{\beta}_{\varphi,u} : S_p \to H_q$ is compact.

Conversely suppose, $D^{\beta}_{\varphi,u} : S_p \to H_q$ is compact. Then it is bounded. Suppose (14) is not true. Then there is a sequence $\{z_k\}_{k\in\mathbb{N}}$ such that $\varphi(z_k) \to 1$ as $z_k \to \infty$ and $\delta > 0$ such that

$$\int_0^{2\pi} \frac{|u(z)|^q}{(1 - |\varphi(z)|^2)^{q/p + q(\beta-1)}} d\theta \geq \delta, \quad k \in \mathbb{N}.$$

Let f_w be the test function defined in converse of Theorem 2 and let $g_k(z) = f_{\varphi(z_k)}$, $k \in \mathbb{N}$.

Clearly

$$|g_k(z)| = |f_{\varphi(z_k)}| \leq C \frac{(1 - |\varphi(z_k)|^2)}{(1 - \overline{\varphi(z_k)} z)^{\frac{1}{p} + \beta}}.$$

As $k \to \infty$, we have $\varphi(z_k) \to 1$. Hence, $|g_k|$, that is $g_k \to 0$ as $k \to \infty$ uniformly on compact subsets of $\mathbb{D}$. Therefore,

$$\lim_{k\to\infty} \|D^{\beta}_{\varphi,u} g_k\|_{H_q} = 0.$$

But, from our assumption we have

$$\begin{aligned}\|D^{\beta}_{\varphi,u} g_k\|_{H_q} &= \sup_{0<r<1} \left[\int_0^{2\pi} |u(z) g_k^{[\beta]}(\varphi(z))|^q d\theta\right]^{1/q} \\ &= \sup_{0<r<1} \left[\int_0^{2\pi} \frac{|u(z)(1 - |\varphi(z_k)|^2)|^q}{(1 - \overline{\varphi(z_k)}\varphi(z_k))^{q/p + q\beta}} d\theta\right]^{1/q} \\ &\geq \sup_{0<r<1} \left[\int_0^{2\pi} \frac{|u(z)|^q}{(1 - |\varphi(z_k)|^2)^{q/p + q(\beta-1)}} d\theta\right]^{1/q} \geq C\delta > 0,\end{aligned}$$

when $k \to \infty$, which is a contradiction.

Hence, (14) must be true. $\square$

5. Conclusion

Fractional derivatives play a significant role in various fields of engineering and allied sciences. On the other hand, the study of operators involving fractional derivatives leads the use of fractional derivatives in a new direction, which involves generalizing well-known results in analytic function spaces. In this paper, we studied the boundedness of the operator $D^{\beta}_{\varphi,u} : S_p \to H_q$. A sufficient condition for the boundedness of the operator $D^{\beta}_{\varphi,u} : S_p \to S_q$ is given. The necessary condition is open for evaluation.

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