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Some identities associated with theta functions and tenth order mock theta functions

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Abstract. The main object of this paper is to present some identities associated with theta functions and tenth order mock theta functions. Several closely-related identities such as (for example) q-product identities and Jacobi's triple-product identity are also considered.

1. Introduction

Three months before his death in early 1920, Ramanujan sent a letter to Hardy in which there were 17 functions, which he called mock theta functions. They are classified into three groups, four of order three, ten of order five and three of order seven. These mock theta functions are q-series converging for $|q| < 1$ and have certain properties as theta functions when q tends to a root of unity.

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ the set of positive integers, the set of integers and the set of complex numbers respectively. We also let

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}.$$

The q-shifted factorial $(a;q)_n$ is defined (for $|q| < 1$) by

$$(a;q)_n := \begin{cases} 1, & n = 0; \\ \prod_{k=0}^{n-1} (1 - aq^k), & n \in \mathbb{N}; \end{cases} \tag{1}$$

where $a, q \in \mathbb{C}$ with $a \neq q^{-m}$ $(m \in \mathbb{N}_0)$. We also write

$$(a;q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n) = \prod_{n=1}^{\infty} (1 - aq^{n-1}), \quad a, q \in \mathbb{C}; \ |q| < 1. \tag{2}$$

It should be noted that, when $a \neq 0$ and $|q| \geqq 1$, the infinite product in (2) diverges. So, whenever $(a;q)_\infty$ is involved in a given formula, the constraint

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$|q| < 1$ will be *tacitly* assumed to be satisfied. The following notation will be frequently used in our investigation:

(3) $(a_1, a_2, a_3, \ldots, a_k; q)_n = (a_1; q)_n (a_2; q)_n (a_3; q)_n \cdots (a_k; q)_n, \ n \in \mathbb{N} \cup \{\infty\}.$

Ramanujan (see [14, p. 13] and [13]) defined the general theta function $f(a, b)$ as follows:

$$\begin{aligned} f(a,b) &= 1 + \sum_{n=1}^{\infty} (ab)^{\frac{n(n-1)}{2}} (a^n + b^n) \\ &= \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = f(b,a), \quad |ab| < 1, \end{aligned}$$

where a and b are two complex numbers. The three most important special cases of $f(a, b)$ are defined (see [11]) by

$$\begin{aligned} \phi(q) &= f(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty \\ &= \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty (-q^2;q^2)_\infty}, \end{aligned} \tag{4}$$

$$\psi(q) = f(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{5}$$

and

$$\mathfrak{f}(-q) = f(-q, -q^2) = \sum_{n=-\infty}^{+\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty. \tag{6}$$

The last equality is known as *Euler's Pentagonal Number Theorem.*

For the remainder of this paper, we note

$$f_r := (q^r; q^r)_\infty, \tag{7}$$

in order to shorten all formulas. Then (see [8, Lemma 2])

$$\phi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \tag{8}$$

$$\phi(-q) = \frac{f_1^2}{f_2}, \tag{9}$$

$$\psi(q) = \frac{f_2^2}{f_1} \tag{10}$$

and

$$\chi(-q) = \frac{f_1}{f_2}. \tag{11}$$

Ramanujan also defined the function (see [15])

$$\chi(q) = (-q;q^2)_\infty = \frac{f_2^2}{f_1 f_4}.$$

Remarkably, the q-series identity

$$(-q;q)_\infty = \frac{1}{\chi(-q)}$$

provides the analytic equivalent form of Euler's famous theorem. We also have

$$(-q;q)_n = \frac{1}{(q;q^2)_n}, \tag{12}$$

since for all $k \in \mathbb{N}$

$$1 + q^k = \frac{1-q^{2k}}{1-q^k}$$

and

$$\begin{aligned}(-q;q)_n &= \prod_{k=1}^{n}(1+q^k) = \prod_{k=1}^{n}\frac{1-q^{2k}}{1-q^k}\\ &= \prod_{k=1}^{n}(1-q^{2k-1})^{-1}\\ &= \prod_{k=0}^{n}(1-q^{kn+1})^{-1} = \frac{1}{(q;q^2)_n}.\end{aligned}$$

Moreover,

$$\chi(q) = (-q;q^2)_\infty = \frac{1}{(q;q^2)_\infty(-q^2;q^2)_\infty} \tag{13}$$

holds, since

$$\begin{aligned}(-q;q^2)_\infty &= \prod_{n=1}^{\infty}(1+q^{2n-1}) = \prod_{n=1}^{\infty}\frac{1+q^n}{1+q^{2n}}\\ &= \prod_{n=1}^{\infty}\frac{1-q^{2n}}{(1-q^n)(1+q^{2n})}\\ &= \prod_{n=1}^{\infty}\frac{1}{(1-q^{2n-1})(1+q^{2n})}\\ &= \frac{1}{(q;q^2)_\infty(-q^2;q^2)_\infty}.\end{aligned}$$

Rogers-Ramanujan's identities. We recall the Rogers-Ramanujan functions

$$G(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q,q)_n}=\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}=\frac{(q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty}, \tag{14}$$

$$H(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n}=\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}=\frac{(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty} \tag{15}$$

and

$$R(q)=q^{1/5}\frac{H(q)}{G(q)}=\frac{q^{1/5}(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{16}$$

Remark 1.

- In the second equality of (14), the left-hand side can be interpreted as the number of partitions of n whose shares differ by at least 2, the right-hand side is the number of partitions of n in portions congruent to 1 or 4 modulo 5.
- In the second equality of (15), the left-hand side is the generating series of partitions of n into parts such that two adjacent differ by at least 2 and such that the smallest part is at minus 2, whereas the right-hand side $\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}$ is the generating series of the partitions of n such that each part is congruent to 2 or 3 modulo 5.

Andrews et al. [3, p. 5] introduced the general family $R(s,t,l,u,v,w)$,

$$R(s,t,l,u,v,w):=\sum_{n=0}^{\infty}q^{s\binom{n}{2}+tn}r(l,u,v,w:n), \tag{17}$$

where

$$r(l,u,v,w:n):=\sum_{j=0}^{[\frac{n}{u}]}(-1)^j\frac{q^{uv\binom{j}{2}+(w-ul)j}}{(q;q)_{n-uj}(q^{uv};q^{uv})_j}. \tag{18}$$

In the following proposition, we give three particular cases of double q-hypergeometric series R.

Proposition 1 ([3, p. 5]). *We have*

$$R_\alpha:=R(2,1,1,1,2,2)=(-q;q^2)_\infty, \tag{19}$$

$$R_\beta:=R(2,2,1,1,2,2)=(-q^2;q^2)_\infty \tag{20}$$

and

$$R_m := R(m,m,1,1,1,2) = \frac{(q^{2m};q^{2m})_\infty}{(q^m;q^{2m})_\infty}, \quad m \in \mathbb{N}^*. \tag{21}$$

Let us recall Ramanujan's celebrated congruences for the partition function $p(n)$ (see [16]):

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7$$

and

$$p(11n+6) \equiv 0 \pmod{11}.$$

In 1944 Dyson [6] defined the rank of a partition to be the largest part minus the number of parts of the partition. Let $N(m,n)$ denote the number of partitions of n with rank m and $N(m,t,n)$ be the number of partitions of n with rank congruent to m modulo t. Dyson conjectured that

$$N(k,5,5n+4) = \frac{p(5n+4)}{5}, \quad 0 \le k \le 4 \tag{a}$$

and

$$N(k,7,7n+5) = \frac{p(7n+5)}{7}, \quad 0 \le k \le 6. \tag{b}$$

Thus, if (a) and (b) are true, the partitions counted by $p(5n+4)$ and $p(7n+5)$ fall into five and seven equinumerous classes respectively. Furthermore, Dyson settled that the generating function for $N(m,n)$ satisfies

$$\sum_{m=-\infty}^{+\infty}\sum_{n=0}^{\infty} N(m,n)z^m q^n = \sum_{n=0}^{\infty}\frac{q^{n^2}}{T_n(z;q)}, \quad |q|<1,\ |q|<|z|<\frac{1}{|q|}, \tag{c}$$

where

$$T_n(z,q) := (zq;q)_n(z^{-1}q;q)_n, \tag{22}$$

where all symbols and notations are having their usual meanings.

2. Preliminaries

In this section, we record some basic results, which are given below.

Theorem 1. *Let*

$$T_\infty(z,q) := (zq;q)_\infty(z^{-1}q;q)_\infty, \tag{23}$$

R_α the quantity given by (19), *$\phi(q)$ is defined by* (4), *$R_n = \psi(q^n)$, f_n is defined by* (18) *and ψ is defined by* (5). *Then, we have*

$$T_\infty(-q,q^2)T_\infty(-1,q^2) = \frac{f_4}{f_1}(-q^3;q^2)_\infty, \tag{24}$$

$$\phi^2(q) - \psi(q)f_1R_\alpha^4 = 0, \tag{25}$$

$$\phi(q)f_4(q^2;q^4)_\infty - \psi^2(q) = 0 \tag{26}$$

and

$$\phi(q)-\psi(q)R_\alpha(q^2;q^4)_\infty=\phi(q)-R_1R_\alpha(q^2;q^4)_\infty=0. \tag{27}$$

Proof. In (23), substituting q by q^2 and letting $z=-q$, we get

$$T_\infty(-q,q^2)=(-q^3;q^2)_\infty(-q;q^2)_\infty.$$

Hence, from (13) and (22), we obtain

$$\begin{aligned}T_\infty(-q,q^2)=(-q^3;q^2)_\infty(-q;q^2)_\infty&=\frac{(-q^3;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}\\&=\frac{(-q^3;q^2)_\infty(-q^2;q^2)_\infty}{(q;q^2)_\infty T_\infty(-1,q^2)}\\&=\frac{(-q^3;q^2)_\infty(-q^2;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty T_\infty(-1,q^2)(q^2;q^2)_\infty}.\end{aligned}$$

A simple verification gives

$$(q^2;q^2)_\infty(-q^2;q^2)_\infty=(q^4;q^4)_\infty$$

and

$$(q;q^2)_\infty(q^2;q^2)_\infty=(q;q)_\infty.$$

Therefore

$$T_\infty(-q,q^2)=\frac{(-q^3;q^2)_\infty(q^4;q^4)_\infty}{T_\infty(-1,q^2)(q;q)_\infty}=\frac{(-q^3;q^2)_\infty f_4}{T_\infty(-1,q^2)f_1}.$$

Hence (24) holds.
To prove identity (25), we use (4)-(6) and (19). Then, we have

$$\begin{aligned}\phi^2(q)&=(-q;q^2)^4_\infty(q^2;q^2)^2_\infty=R^4_\alpha(q^2;q^2)^2_\infty\\&=R^4_\alpha(q^2;q^2)_\infty(q^2;q^2)_\infty\frac{(q;q^2)_\infty}{(q;q^2)_\infty}=R^4_\alpha\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}(q;q)_\infty\\&=R^4_\alpha\psi(q)f_1.\end{aligned}$$

Similarly, one has

$$\begin{aligned}\phi(q)f_4(q^2;q^4)_\infty&=(-q;q^2)^2_\infty(q^2;q^2)_\infty(q^4;q^4)_\infty(q^2;q^4)_\infty\\&=(-q;q)^2_\infty(q^2;q^2)^2_\infty=\psi^2(q).\end{aligned}$$

This gives (26).
Now, we have

$$\begin{aligned}\psi(q)R_\alpha(q^2;q^4)_\infty&=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}(-q;q^2)_\infty(q^2;q^4)_\infty\\&=(-q;q^2)_\infty(q^2;q^2)_\infty\frac{(q^2;q^4)_\infty}{(q;q^2)_\infty}\\&=(-q;q^2)_\infty(q^2;q^2)_\infty\frac{1}{(q;q^2)_\infty(-q^2;q^2)_\infty}\end{aligned}$$

$$= (-q;q^2)_\infty^2 (q^2;q^2)_\infty = \phi(q).$$

This completes the proof of Theorem 1. □

3. Tenth order mock theta functions

In his lost notebook [13, p. 9], Ramanujan recorded four mock theta functions:

$$\Phi(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(q;q)_{n+1}}, \tag{28}$$

$$\Psi(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{(n+1)(n+2)}{2}}}{(q;q)_{n+1}}, \tag{29}$$

$$D(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q^2)_n} \tag{30}$$

and

$$\chi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q^2)_{2n+1}}. \tag{31}$$

All these mock theta functions have been shown to be of order ten by Y. S. Choi [10, 12].

Let us define the q-binomial coefficient by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \begin{cases} \dfrac{(q;q)_n}{(q;q)_m (q;q)_{n-m}}, & 0 \leq m \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

We need some identities between the q-product and the q-binomial series. From [4, Equations (12-15), p. 616], we have

$$(z;q)_n = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q (-1)^m z^m q^{m(m-1)/2} \tag{32}$$

and

$$\frac{1}{(z;q)_n} = \sum_{m=0}^{\infty} \begin{bmatrix} n+m-1 \\ m \end{bmatrix}_q z^m. \tag{33}$$

Let us define some functions:

$$\Phi_z(q) = \sum_{n=0}^{\infty} \frac{z^n q^{\frac{n(n+1)}{2}}}{(q;q)_{n+1}}, \tag{34}$$

$$\Psi_z(q) = \sum_{n=0}^{\infty} \frac{z^n q^{\frac{(n+1)(n+2)}{2}}}{(q;q)_{n+1}} \tag{35}$$

and

$$D_z(q)=\sum_{n=0}^{\infty}\frac{z^nq^{n^2}}{(q;q^2)_n}. \tag{36}$$

Note that $\Phi_1(q)=\Phi(q)$, $\Psi_1(q)=\Psi(q)$ and $D_1(q)=D(q)$.

Theorem 2. *Under the above notation, we have*

$$\Phi_z(q)=\sum_{n=0}^{\infty}\frac{z^nq^{n(n+1)/2}}{(q;q)_{n+1}}=\sum_{n=0}^{\infty}q^n(-z;q)_n, \tag{A1}$$

$$\Psi_z(q)=q\sum_{n=0}^{\infty}\frac{z^nq^{n(n+3)/2}}{(q;q)_{n+1}}=q\sum_{n=0}^{\infty}q^n(-zq;q)_n \tag{A2}$$

and

$$D_z(q)=1+qz\sum_{n=0}^{\infty}\frac{z^nq^{n(n+2)}}{(q;q^2)_{n+1}}=1+qz\sum_{N=0}^{\infty}q^N(-zq^2;q^2)_N. \tag{A3}$$

Proof. To prove (A1), we use identities (32), (33) and (34). Then, we get

$$\begin{aligned}\Phi_z(q)&=\sum_{n=0}^{\infty}\frac{z^nq^{\frac{n(n+1)}{2}}}{(q;q)_{n+1}}\\&=\sum_{n=0}^{\infty}z^nq^{\frac{n(n+1)}{2}}\sum_{m=0}^{\infty}q^m\begin{bmatrix}n+m\\m\end{bmatrix}_q\\&=\sum_{m=0}^{\infty}q^m\sum_{n=0}^{\infty}z^nq^{\frac{n(n+1)}{2}}\begin{bmatrix}n+m\\m\end{bmatrix}_q.\end{aligned}$$

Let $k=m+n$. Hence, we obtain

$$\begin{aligned}\Phi_z(q)=\sum_{m=0}^{\infty}q^m\sum_{n=0}^{\infty}z^nq^{\frac{n(n+1)}{2}}\begin{bmatrix}n+m\\n\end{bmatrix}_q&=\sum_{k=0}^{\infty}q^k\sum_{n=0}^{k}z^nq^{\frac{n(n-1)}{2}}\begin{bmatrix}k\\n\end{bmatrix}_q\\&=\sum_{k=0}^{\infty}q^k\sum_{n=0}^{k}(-1)^n(-z)^nq^{\frac{n(n-1)}{2}}\begin{bmatrix}k\\n\end{bmatrix}_q\\&=\sum_{k=0}^{\infty}q^k(-z;q)_k.\end{aligned}$$

To prove (A2), we use identities (32), (33), (35). Therefore, we have

$$\begin{aligned}\Psi_z(q)&=q\sum_{n=0}^{\infty}\frac{z^nq^{\frac{n(n+3)}{2}}}{(q;q)_{n+1}}\\&=q\sum_{n=0}^{\infty}z^nq^{\frac{n(n+3)}{2}}\sum_{m=0}^{\infty}q^m\begin{bmatrix}n+m\\m\end{bmatrix}_q\end{aligned}$$

$$= q\sum_{m=0}^{\infty} q^m \sum_{n=0}^{\infty} z^n q^{\frac{n(n+3)}{2}} \begin{bmatrix} n+m \\ n \end{bmatrix}_q .$$

Taking $k = m+n$ yields

$$\begin{aligned}
\Psi_z(q) &= q\sum_{m=0}^{\infty} q^m \sum_{n=0}^{\infty} z^n q^{\frac{n(n+3)}{2}} \begin{bmatrix} n+m \\ n \end{bmatrix}_q \\
&= q\sum_{k=0}^{\infty} q^k \sum_{n=0}^{k} z^n q^{\frac{n(n+1)}{2}} \begin{bmatrix} k \\ n \end{bmatrix}_q \\
&= q\sum_{k=0}^{\infty} q^k \sum_{n=0}^{k} (-1)^n (-zq)^n q^{\frac{n(n-1)}{2}} \begin{bmatrix} k \\ n \end{bmatrix}_q \\
&= q\sum_{k=0}^{\infty} q^k (-zq;q)_k .
\end{aligned}$$

We use the same argument to obtain (A3). One has

$$\begin{aligned}
D_z(q) &= \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q;q^2)_n} = 1 + \sum_{n=1}^{\infty} \frac{z^n q^{n^2}}{(q;q^2)_n} \\
&= 1 + zq\sum_{n=0}^{\infty} \frac{z^n q^{n(n+2)}}{(q;q^2)_{n+1}},
\end{aligned}$$

Let us replace N by $n+m$, we deduce

$$\begin{aligned}
D_z(q) = \sum_{n=0}^{\infty} \frac{z^n q^{n(n+2)}}{(q;q^2)_{n+1}} &= \sum_{n=0}^{\infty} z^n q^{n(n+2)} \sum_{m=0}^{\infty} q^m \begin{bmatrix} n+m \\ n \end{bmatrix}_{q^2} \\
&= \sum_{m=0}^{\infty} q^m \sum_{n=0}^{\infty} z^n q^{n(n+2)} \begin{bmatrix} n+m \\ n \end{bmatrix}_{q^2} \\
&= \sum_{N=0}^{\infty} q^N \sum_{n=0}^{N} (-1)^n (-zq^2)^n q^{n^2-n} \begin{bmatrix} N \\ n \end{bmatrix}_{q^2} \\
&= \sum_{N=0}^{\infty} q^N (-zq^2;q^2)_N .
\end{aligned}$$

□

Application 1.

1. We have

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n} = 0.$$

Indeed, $z = -1$ in the identity (A1) yields

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n(n-1)/2}}{(q;q)_n} = \sum_{n=0}^{\infty} q^n (1;q)_n = 1.$$

Hence

$$\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q;q)_n} + 1 = 0.$$

So, we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q;q)_n} = 0.$$

2. One has

$$\sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q;q)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{(n+1)}}{(q;q^2)_n}. \qquad (c1)$$

Indeed, by setting $z = 1$ in the identity (A2) we obtain

$$\sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q;q)_{n+1}} = \sum_{n=0}^{\infty} q^{n+1}(-q;q)_n.$$

Then, (12) yields

$$\sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q;q)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{(n+1)}}{(q;q^2)_n}.$$

As a combinatorial interpretation to the identity (c1) (see [5, eq. (5), p. 6]), we have

$$\sum_{n=0}^{\infty} \frac{q^{(n+1)}}{(q;q^2)_n} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a(m,n) q^{n+1},$$

where $a(m,n)$ counts the number of partition of an integer n with odd parts and the number of parts equal m.

3. We have

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^n}{(q;q^2)_n(-q^2;q^2)_n}. \qquad (c2)$$

Indeed, identity (A3) with $z = 1/q$ gives

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} q^n(-q;q^2)_n.$$

Using identity (13), we get (c2).
A combinatorial interpretation to the identity (c2) (see [5, p. 12]), the left hand is the generating function of partition of n into the odd parts less twice the smallest part the right hand is the generating function of partition of n into odd and distinct parts.

Theorem 3. *For all $p \in \mathbb{N}$, we have*

$$\sum_{n=0}^{\infty} \frac{q^{n+1}}{(q^{p+n+1};q)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{np+n+1}}{(q^{n+1};q)_{n+1}}, \qquad (37)$$

(38)
$$\sum_{n=0}^{\infty} \frac{z^n q^{n(n+1+p)/2}}{(q;q)_{n+1}} = \sum_{n=0}^{\infty} q^n (-zq^p;q)_n$$

and

(39)
$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

Proof. From [2, eq. (2.5), p. 3], we have

$$\chi_0(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_n} = 1 + \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^{n+1};q)_{n+1}}.$$

So

$$1 + \sum_{n=1}^{\infty} \frac{q^n}{(q^{n+1};q)_n} = 1 + \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^{n+1};q)_{n+1}}$$

and then

$$\sum_{n=0}^{\infty} \frac{q^{n+1}}{(q^{n+2};q)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^{n+1};q)_{n+1}}.$$

We notice that (37) is true for $p \in \{0,1\}$. We assume that (37) holds up to order p, then

$$\sum_{n=0}^{\infty} \frac{q^{n+1}}{(q^{n+p+2};q)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^{n+p+1};q)_{n+1}}$$

or

$$\sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^{n+p+1};q)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{np} q^{2n+1}}{(q^{n+1};q)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{n(p+1)+n+1}}{(q^{n+1};q)_{n+1}}.$$

Hence, our result is true for $p+1$ and therefore for all $p \in \mathbb{N}$.
Let us prove (38). Using identities (32) and (33), we get

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{z^n q^{\frac{n(n+1+p)}{2}}}{(q;q)_{n+1}} &= \sum_{n=0}^{\infty} z^n q^{\frac{n(n+1+p)}{2}} \sum_{m=0}^{\infty} q^m \begin{bmatrix} n+m \\ m \end{bmatrix}_q \\ &= \sum_{m=0}^{\infty} q^m \sum_{n=0}^{\infty} z^n q^{\frac{n(n+1+p)}{2}} \begin{bmatrix} n+m \\ n \end{bmatrix}_q .\end{aligned}$$

Let $k = m+n$, i.e., $m = k-n$. Then, we obtain

$$\begin{aligned}\sum_{m=0}^{\infty} q^m \sum_{n=0}^{\infty} z^n q^{\frac{n(n+1+p)}{2}} \begin{bmatrix} n+m \\ n \end{bmatrix}_q &= \sum_{k=0}^{\infty} q^k \sum_{n=0}^{k} z^n q^{\frac{n(n+p-1)}{2}} \begin{bmatrix} k \\ n \end{bmatrix}_q \\ &= \sum_{k=0}^{\infty} q^k \sum_{n=0}^{k} (-1)^n (-zq^p)^n q^{\frac{n(n-1)}{2}} \begin{bmatrix} k \\ n \end{bmatrix}_q \\ &= \sum_{k=0}^{\infty} q^k (-zq^p;q)_k.\end{aligned}$$

Now, it remains to prove (39). We have (see [1, Equation (3.1)])

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} x^n}{(q;q)_n} = (x;q)_\infty. \tag{c}$$

Then, replacing x by $-q$ in (c) gives

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n} = (-q;q)_\infty$$

or

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = (-q;q)_\infty (q^2;q^2)_\infty.$$

Moreover

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

This proves identity (39). □

Application 2. 1. Let

(40) $$f_p(q) = \sum_{n=0}^{\infty} \frac{q^{np+n+1}}{(q^{n+1};q)_{n+1}}.$$

Equation (40) with $p = 0$ gives

$$f_0(q) = q\chi_1(q).$$

Hence, the function $f_0(q)$ is the generating function for partitions of n in which no part is as large as twice the smallest part.

2. By (40) with $p = 1$ yields

$$f_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^{n+1};q)_{n+1}}.$$

Then, the function $f_1(q)$ is the generating function for the partitions of n into odd parts and which no part is as large as twice the smallest part.

OPEN PROBLEMS

Based upon the work presented in this paper, we find it to be worthwhile to motivate the interested reader to consider the following related open problems.

a) For any function $g_n(q)$, if $|g_n(q)| < 1$, then show that:

$$\sum_{n=0}^{\infty} \frac{g_n(q)}{(q;q)_n} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} g_n(q). \tag{47}$$

b) Discuss the various properties for the identity given in (27).

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