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Inequalities for the normalized determinant of positive operators in Hilbert spaces via some inequalities in terms of Kantorovich ratio

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Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the normalized determinant by $\Delta_x(A) := \exp \langle \ln Ax, x\rangle$. In this paper we prove among others that, if $0 < mI \le A \le MI$, then

$$1 \le K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}-\frac{1}{M-m}\left\langle \left|A-\frac{1}{2}(m+M)I\right|x,x\right\rangle\right]}$$
$$\le \frac{\Delta_x(A)}{m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}}$$
$$\le \left[K\left(\frac{M}{m}\right)\right]^{\left[\frac{1}{2}+\frac{1}{M-m}\left\langle \left|A-\frac{1}{2}(m+M)I\right|x,x\right\rangle\right]} \le K\left(\frac{M}{m}\right),$$

for $x \in H$, $\|x\| = 1$, where $K(\cdot)$ is *Kantorovich's ratio.*

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \ge 0$) if $\langle Ax, x\rangle \ge 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \ge B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [1, 2], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp \langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [1].

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For each unit vector $x \in H$, see also [5], we have:

(i) *continuity*:

$$\text{the map } A \to \Delta_x(A) \text{ is norm continuous;}$$

(ii) *bounds*:

$$\left\langle A^{-1}x, x\right\rangle^{-1} \le \Delta_x(A) \le \langle Ax, x\rangle\,;$$

(iii) *continuous mean*:

$$\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A) \text{ for } p \downarrow 0,$$
$$\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A) \text{ for } p \uparrow 0;$$

(iv) *power equality:*

$$\Delta_x(A^t) = \Delta_x(A)^t, \quad \text{for all } t > 0;$$

(v) *homogeneity*:

$$\begin{matrix} \Delta_x(tA) = t\Delta_x(A), \\ \Delta_x(tI) = t, \end{matrix} \quad \text{for all } t > 0;$$

(vi) *monotonicity*:

$$0 < A \le B \quad \text{implies} \quad \Delta_x(A) \le \Delta_x(B);$$

(vii) *multiplicativity*:

$$\Delta_x(AB) = \Delta_x(A)\Delta_x(B) \text{ for commuting } A \text{ and } B;$$

(viii) *Ky Fan type inequality*:

$$\Delta_x((1-\alpha)\,A + \alpha B) \ge \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}, \quad \text{for } 0 < \alpha < 1.$$

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \dfrac{b-a}{\ln b - \ln a}, & \text{if } b \ne a; \\ a, & \text{if } b = a. \end{cases}$$

In [1] the authors obtained the following additive reverse inequality for the operator A which satisfies the condition $0 < mI \le A \le MI$, where m, M are positive numbers,

$$\begin{aligned} 0 &\le \langle Ax, x\rangle - \Delta_x(A) \\ &\le L\,(m,M)\left[\ln L\,(m,M) + \frac{M \ln m - m \ln M}{M-m} - 1\right], \end{aligned} \tag{1}$$

for all $x \in H$, $\|x\| = 1$.

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu} b^{\nu} \leq (1-\nu)\, a + \nu b \tag{2}$$

with equality if and only if $a = b$. The inequality (2) is also called *ν-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [7]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)}, & \text{if } h \in (0,1) \cup (1,\infty); \\ 1, & \text{if } h = 1. \end{cases} \tag{3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [2], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right), \tag{4}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

Since $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$, then by (4) for A^{-1} we get

$$1 \leq \frac{\langle A^{-1}x, x\rangle}{\Delta_x(A^{-1})} \leq S\left(\frac{m^{-1}}{M^{-1}}\right) = S\left(\left(\frac{m}{M}\right)^{-1}\right) = S\left(\frac{M}{m}\right),$$

which is equivalent to

$$1 \leq \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \leq S\left(\frac{M}{m}\right), \tag{5}$$

for $x \in H$, $\|x\| = 1$.

We consider the *Kantorovich's ratio* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \quad h > 0. \tag{6}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$\left(a^{1-\nu} b^{\nu}\right) \leq K^r\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu} \leq (1-\nu)\, a + \nu b \leq K^R\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu}, \tag{7}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

The first inequality in (7) was obtained by Zou et al. in [9] and the second by Liao et al. [6].

2. Main results

Our first main result is as follows.

Theorem 1. *If* $0 < mI \le A \le MI$ *for positive numbers* m, M*, then*

$$1 \le K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}-\frac{1}{M-m}\left\langle\left|A-\frac{1}{2}(m+M)I\right|x,x\right\rangle\right]} \tag{8}$$

$$\le \frac{\Delta_x(A)}{m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}}$$

$$\le \left[K\left(\frac{M}{m}\right)\right]^{\left[\frac{1}{2}+\frac{1}{M-m}\left\langle\left|A-\frac{1}{2}(m+M)I\right|x,x\right\rangle\right]} \le K\left(\frac{M}{m}\right)$$

for $x \in H$, $\|x\| = 1$.

Proof. Assume that $t \in [m, M]$ and consider $\nu = \frac{t-m}{M-m} \in [0, 1]$. Then

$$\min\{1-\nu,\nu\} = \frac{1}{2} - \left|\nu - \frac{1}{2}\right| = \frac{1}{2} - \left|\frac{t-m}{M-m} - \frac{1}{2}\right|$$

$$= \frac{1}{2} - \frac{1}{M-m}\left|t - \frac{1}{2}(m+M)\right|,$$

$$\max\{1-\nu,\nu\} = \frac{1}{2} + \left|\nu - \frac{1}{2}\right| = \frac{1}{2} + \left|\frac{t-m}{M-m} - \frac{1}{2}\right|$$

$$= \frac{1}{2} + \frac{1}{M-m}\left|t - \frac{1}{2}(m+M)\right|,$$

$$(1-\nu)m + \nu M = \frac{M-t}{M-m}m + \frac{t-m}{M-m}M = t,$$

$$m^{1-\nu}M^{\nu} = m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}}.$$

By using (7) we get

$$m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}} \le \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|} m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}} \tag{9}$$

$$\le t \le \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}+\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|} m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}},$$

for $t \in [m, M]$.

By taking the log in (9) we get

$$\begin{aligned}
&\frac{M-t}{M-m}\ln m+\frac{t-m}{M-m}\ln M \\
&\le\left[\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|\right]\ln K\left(\frac{M}{m}\right) \\
&\quad+\frac{M-t}{M-m}\ln m+\frac{t-m}{M-m}\ln M \\
&\le \ln t\le\left[\frac{1}{2}+\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|\right]\ln K\left(\frac{M}{m}\right) \\
&\quad+\frac{M-t}{M-m}\ln m+\frac{t-m}{M-m}\ln M \\
&\le \ln K\left(\frac{M}{m}\right)+\frac{M-t}{M-m}\ln m+\frac{t-m}{M-m}\ln M
\end{aligned} \tag{10}$$

for $t\in[m,M]$.

If $0<mI\le A\le MI$, then by using the continuous functional calculus for selfadjoint operators we get from (10) that

$$\begin{aligned}
&\ln m\frac{MI-A}{M-m}+\ln M\frac{A-mI}{M-m} \\
&\le\left[\frac{1}{2}I-\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)I\right|\right]\ln K\left(\frac{M}{m}\right) \\
&\quad+\ln m\frac{MI-A}{M-m}+\ln M\frac{A-mI}{M-m} \\
&\le\ln A\le\left[\frac{1}{2}I+\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)I\right|\right]\ln K\left(\frac{M}{m}\right) \\
&\quad+\ln m\frac{MI-A}{M-m}+\ln M\frac{A-mI}{M-m} \\
&\le\ln K\left(\frac{M}{m}\right)I+\ln m\frac{MI-A}{M-m}+\ln M\frac{A-mI}{M-m},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\ln m\frac{M-\langle Ax,x\rangle}{M-m}+\ln M\frac{\langle Ax,x\rangle-m}{M-m} \\
&\le\ln K\left(\frac{M}{m}\right)\left[\frac{1}{2}-\frac{1}{M-m}\left\langle\left|A-\frac{1}{2}(m+M)I\right|x,x\right\rangle\right] \\
&\quad+\ln m\frac{M-\langle Ax,x\rangle}{M-m}+\ln M\frac{\langle Ax,x\rangle-m}{M-m} \\
&\le\langle\ln Ax,x\rangle \\
&\le\ln K\left(\frac{M}{m}\right)\left[\frac{1}{2}+\frac{1}{M-m}\left\langle\left|A-\frac{1}{2}(m+M)I\right|x,x\right\rangle\right]
\end{aligned}$$

$$\ln m \frac{M - \langle Ax, x\rangle}{M-m} + \ln M \frac{\langle Ax, x\rangle - m}{M-m}$$
$$\le \ln K\left(\frac{M}{m}\right) + \ln m \frac{M - \langle Ax, x\rangle}{M-m} + \ln M \frac{\langle Ax, x\rangle - m}{M-m},$$

for $x \in H$, $\|x\| = 1$.

This inequality can also be written as

$$\ln\left(m^{\frac{M-\langle Ax,x\rangle}{M-m}} M^{\frac{\langle Ax,x\rangle - m}{M-m}}\right) \tag{11}$$
$$\le \ln\left[K\left(\frac{M}{m}\right)\right]^{\left[\frac{1}{2} - \frac{1}{M-m}\left\langle \left|A - \frac{1}{2}(m+M)I\right| x, x\right\rangle\right]}$$
$$+ \ln\left(m^{\frac{M-\langle Ax,x\rangle}{M-m}} M^{\frac{\langle Ax,x\rangle - m}{M-m}}\right)$$
$$\le \langle \ln Ax, x\rangle$$
$$\le \ln\left[K\left(\frac{M}{m}\right)\right]^{\left[\frac{1}{2} + \frac{1}{M-m}\left\langle \left|A - \frac{1}{2}(m+M)I\right| x, x\right\rangle\right]}$$
$$+ \ln\left(m^{\frac{M-\langle Ax,x\rangle}{M-m}} M^{\frac{\langle Ax,x\rangle - m}{M-m}}\right)$$
$$\le \ln K\left(\frac{M}{m}\right) + \ln\left(m^{\frac{M-\langle Ax,x\rangle}{M-m}} M^{\frac{\langle Ax,x\rangle - m}{M-m}}\right),$$

for $x \in H$, $\|x\| = 1$.

If we take the exponential in (11), then we get

$$m^{\frac{M-\langle Ax,x\rangle}{M-m}} M^{\frac{\langle Ax,x\rangle - m}{M-m}}$$
$$\le \left(m^{\frac{M-\langle Ax,x\rangle}{M-m}} M^{\frac{\langle Ax,x\rangle - m}{M-m}}\right) K\left(\frac{M}{m}\right)^{\left[\frac{1}{2} - \frac{1}{M-m}\left\langle \left|A - \frac{1}{2}(m+M)I\right| x, x\right\rangle\right]}$$
$$\le \exp\langle \ln Ax, x\rangle$$
$$\le \left(m^{\frac{M-\langle Ax,x\rangle}{M-m}} M^{\frac{\langle Ax,x\rangle - m}{M-m}}\right) \ln\left[K\left(\frac{M}{m}\right)\right]^{\left[\frac{1}{2} + \frac{1}{M-m}\left\langle \left|A - \frac{1}{2}(m+M)I\right| x, x\right\rangle\right]}$$
$$\le \left(m^{\frac{M-\langle Ax,x\rangle}{M-m}} M^{\frac{\langle Ax,x\rangle - m}{M-m}}\right) K\left(\frac{M}{m}\right),$$

and the inequality (8) is proved. $\square$

Corollary 1. *With the assumption of Theorem 1, we have the alternative inequality*

$$
\begin{aligned}
(12)\qquad 1 &\le K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}-\frac{1}{m^{-1}-M^{-1}}\left\langle\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|x,x\right\rangle\right]}\\
&\le \frac{M^{\frac{m^{-1}-\left\langle A^{-1}x,x\right\rangle}{m^{-1}-M^{-1}}}m^{\frac{\left\langle A^{-1}x,x\right\rangle-M^{-1}}{m^{-1}-M^{-1}}}}{\Delta_x(A)}\\
&\le \left[K\left(\frac{M}{m}\right)\right]^{\left[\frac{1}{2}+\frac{1}{m^{-1}-M^{-1}}\left\langle\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|x,x\right\rangle\right]}\\
&\le K\left(\frac{M}{m}\right),
\end{aligned}
$$

for $x \in H$, $\|x\| = 1$.

Proof. If we write the inequality for A^{-1} that satisfies the condition $0 < M^{-1}I \le A^{-1} \le m^{-1}I$, then

$$
\begin{aligned}
1 &\le K\left(\frac{m^{-1}}{M^{-1}}\right)^{\left[\frac{1}{2}-\frac{1}{m^{-1}-M^{-1}}\left\langle\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|x,x\right\rangle\right]}\\
&\le \frac{\Delta_x(A^{-1})}{M^{-\frac{m^{-1}-\langle A^{-1}x,x\rangle}{m^{-1}-M^{-1}}}m^{-\frac{\langle A^{-1}x,x\rangle-M^{-1}}{m^{-1}-M^{-1}}}}\\
&\le \left[K\left(\frac{m^{-1}}{M^{-1}}\right)\right]^{\left[\frac{1}{2}+\frac{1}{m^{-1}-M^{-1}}\left\langle\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|x,x\right\rangle\right]}\\
&\le K\left(\frac{m^{-1}}{M^{-1}}\right),
\end{aligned}
$$

namely

$$
\begin{aligned}
1 &\le K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}-\frac{1}{m^{-1}-M^{-1}}\left\langle\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|x,x\right\rangle\right]}\\
&\le \frac{\left[\Delta_x(A)\right]^{-1}}{\left(M^{\frac{m^{-1}-\langle A^{-1}x,x\rangle}{m^{-1}-M^{-1}}}m^{\frac{\langle A^{-1}x,x\rangle-M^{-1}}{m^{-1}-M^{-1}}}\right)^{-1}}\\
&\le \left[K\left(\frac{M}{m}\right)\right]^{\left[\frac{1}{2}+\frac{1}{m^{-1}-M^{-1}}\left\langle\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|x,x\right\rangle\right]}\\
&\le K\left(\frac{M}{m}\right),
\end{aligned}
$$

which is equivalent to the desired result (12). □

Corollary 2. *If* $0 < mI \leq A$ *and* $B \leq MI$ *for positive numbers* m *and* M*, then*

$$\frac{m^{\frac{M-1}{M-m}}M^{\frac{1-m}{M-m}}}{\ln\left(\frac{M}{m}\right)}\Theta\left(A,B,m,M,x\right) \tag{13}$$

$$\leq \int_0^1 \Delta_x((1-t)\,A+tB)dt$$

$$\leq K\left(\frac{M}{m}\right)\frac{m^{\frac{M-1}{M-m}}M^{\frac{1-m}{M-m}}}{\ln\left(\frac{M}{m}\right)}\Theta\left(A,B,m,M,x\right),$$

where

$$\Theta\left(A,B,m,M,x\right) := \begin{cases} \dfrac{\left(\frac{M}{m}\right)^{\frac{\langle(B-A)x,x\rangle}{M-m}}-1}{\frac{\langle(B-A)x,x\rangle}{M-m}}, & \text{if } \langle(B-A)\,x,x\rangle \neq 0; \\ 1, & \text{if } \langle(B-A)\,x,x\rangle = 0, \end{cases}$$

for $x \in H$, $\|x\| = 1$.

Proof. From (12) we get

$$m^{\frac{M-\langle[(1-t)A+tB]x,x\rangle}{M-m}}M^{\frac{\langle[(1-t)A+tB]x,x\rangle-m}{M-m}}$$

$$\leq \Delta_x((1-t)\,A+tB)$$

$$\leq K\left(\frac{M}{m}\right)m^{\frac{M-\langle[(1-t)A+tB]x,x\rangle}{M-m}}M^{\frac{\langle[(1-t)A+tB]x,x\rangle-m}{M-m}}$$

for $t \in [0,1]$.

If we take the integral over $t \in [0,1]$, then we get

$$\int_0^1 m^{\frac{M-\langle[(1-t)A+tB]x,x\rangle}{M-m}}M^{\frac{\langle[(1-t)A+tB]x,x\rangle-m}{M-m}}dt \tag{14}$$

$$\leq \int_0^1 \Delta_x((1-t)\,A+tB)dt$$

$$\leq K\left(\frac{M}{m}\right)\int_0^1 m^{\frac{M-\langle[(1-t)A+tB]x,x\rangle}{M-m}}M^{\frac{\langle[(1-t)A+tB]x,x\rangle-m}{M-m}}dt.$$

Observe that

$$\int_0^1 m^{\frac{M-\langle[(1-t)A+tB]x,x\rangle}{M-m}} M^{\frac{\langle[(1-t)A+tB]x,x\rangle-m}{M-m}} dt$$

$$= m^{\frac{M}{M-m}} M^{\frac{-m}{M-m}} \int_0^1 \left(\frac{M}{m}\right)^{\frac{\langle[(1-t)A+tB]x,x\rangle}{M-m}} dt$$

$$= m^{\frac{M}{M-m}} M^{\frac{-m}{M-m}} \left(\frac{M}{m}\right)^{\frac{1}{M-m}} \int_0^1 \left(\frac{M}{m}\right)^{t\frac{\langle(B-A)x,x\rangle}{M-m}} dt$$

$$= m^{\frac{M-1}{M-m}} M^{\frac{1-m}{M-m}} \int_0^1 \left(\frac{M}{m}\right)^{t\frac{\langle(B-A)x,x\rangle}{M-m}} dt.$$

Since for $a > 0$, $a \neq 1$ and $b \in \mathbb{R}$, we have

$$\int_0^1 a^{bx} dx = \frac{a^b - 1}{b \ln a},$$

then for $\langle (B-A)\, x, x\rangle \neq 0$

$$\int_0^1 \left(\frac{M}{m}\right)^{t\frac{\langle(B-A)x,x\rangle}{M-m}} dt = \frac{\left(\frac{M}{m}\right)^{\frac{\langle(B-A)x,x\rangle}{M-m}} - 1}{\frac{\langle(B-A)x,x\rangle}{M-m} \ln\left(\frac{M}{m}\right)}$$

and by (14) we derive (13). □

3. Related Results

We also have.

Theorem 2. *With the assumption of Theorem 1, we get*

$$1 \le \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\langle \ln Ax,x\rangle-\frac{\ln M+\ln m}{2}\right|}$$
$$\le \frac{\frac{\ln M-\langle \ln Ax,x\rangle}{\ln M-\ln m}m+\frac{\langle \ln Ax,x\rangle-\ln m}{\ln M-\ln m}M}{\Delta_x(A)}$$
$$\le \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}+\frac{1}{\ln M-\ln m}\left|\langle \ln Ax,x\rangle-\frac{\ln M+\ln m}{2}\right|}$$
$$\le K\left(\frac{M}{m}\right), \tag{15}$$

for $x \in H$, $\|x\| = 1$.

Proof. Assume that $m^{1-\nu}M^{\nu} = \exp s$, then $s = (1-\nu)\ln m + \nu \ln M \in [\ln m, \ln M]$, which gives that

$$\nu = \frac{s - \ln m}{\ln M - \ln m}.$$

Also

$$\min\{1-\nu, \nu\} = \frac{1}{2} - \frac{1}{\ln M - \ln m}\left|s - \frac{\ln M + \ln m}{2}\right|,$$
$$\max\{1-\nu, \nu\} = \frac{1}{2} + \frac{1}{\ln M - \ln m}\left|s - \frac{\ln M + \ln m}{2}\right|.$$

From (7) we have

$$\begin{aligned}\exp s &\leq \exp s \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2} - \frac{1}{\ln M - \ln m}\left|s - \frac{\ln M + \ln m}{2}\right|} \\ &\leq \frac{\ln M - s}{\ln M - \ln m} m + \frac{s - \ln m}{\ln M - \ln m} M \\ &\leq \exp s \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2} + \frac{1}{\ln M - \ln m}\left|s - \frac{\ln M + \ln m}{2}\right|},\end{aligned}$$

namely

$$\begin{aligned}1 &\leq \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2} - \frac{1}{\ln M - \ln m}\left|s - \frac{\ln M + \ln m}{2}\right|} \\ &\leq \frac{\frac{\ln M - s}{\ln M - \ln m} m + \frac{s - \ln m}{\ln M - \ln m} M}{\exp s} \\ &\leq \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2} + \frac{1}{\ln M - \ln m}\left|s - \frac{\ln M + \ln m}{2}\right|},\end{aligned}$$

for $s \in [\ln m, \ln M]$.

If $0 < m \leq A \leq M$ and $x \in H$, $\|x\| = 1$, then $\ln m \leq \langle \ln Ax, x\rangle \leq \ln M$ and for $s = \langle \ln Ax, x\rangle$, we deduce

$$\begin{aligned}1 &\leq \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2} - \frac{1}{\ln M - \ln m}\left|\langle \ln Ax, x\rangle - \frac{\ln M + \ln m}{2}\right|} \\ &\leq \frac{\frac{\ln M - \langle \ln Ax, x\rangle}{\ln M - \ln m} m + \frac{\langle \ln Ax, x\rangle - \ln m}{\ln M - \ln m} M}{\exp \langle \ln Ax, x\rangle} \\ &\leq \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2} + \frac{1}{\ln M - \ln m}\left|\langle \ln Ax, x\rangle - \frac{\ln M + \ln m}{2}\right|},\end{aligned}$$

and the inequality (15) is proved. □

Corollary 3. *With the assumption of Theorem 1, we have*

$$
\begin{aligned}
(16)\qquad 1 &\le \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\langle \ln Ax,x\rangle-\frac{\ln M+\ln m}{2}\right|} \\
&\le \frac{\Delta_x(A)}{\left(\frac{\langle \ln Ax,x\rangle-\ln m}{\ln M-\ln m}M^{-1}+\frac{\ln M-\langle \ln Ax,x\rangle}{\ln M-\ln m}m^{-1}\right)^{-1}} \\
&\le \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\langle \ln Ax,x\rangle+\frac{\ln M+\ln m}{2}\right|} \le K\left(\frac{M}{m}\right),
\end{aligned}
$$

for $x \in H$, $\|x\| = 1$.

Proof. If we write the inequality (15) for A^{-1} that satisfies the condition $0 < M^{-1}I \le A^{-1} \le m^{-1}I$, then we obtain

$$
\begin{aligned}
1 &\le K\left(\frac{m^{-1}}{M^{-1}}\right)^{\frac{1}{2}-\frac{1}{\ln m^{-1}-\ln M^{-1}}\left|\left\langle \ln A^{-1}x,x\right\rangle-\frac{\ln m^{-1}+\ln M^{-1}}{2}\right|} \\
&\le \frac{\frac{\ln m^{-1}-\left\langle \ln A^{-1}x,x\right\rangle}{\ln m^{-1}-\ln M^{-1}}M^{-1}+\frac{\left\langle \ln A^{-1}x,x\right\rangle-\ln M^{-1}}{\ln m^{-1}-\ln M^{-1}}m^{-1}}{\Delta_x(A^{-1})} \\
&\le K\left(\frac{m^{-1}}{M^{-1}}\right)^{\frac{1}{2}+\frac{1}{\ln m^{-1}-\ln M^{-1}}\left|\left\langle \ln A^{-1}x,x\right\rangle-\frac{\ln m^{-1}+\ln M^{-1}}{2}\right|} \le M\left(\frac{m^{-1}}{M^{-1}}\right),
\end{aligned}
$$

namely

$$
\begin{aligned}
1 &\le K\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\langle \ln Ax,x\rangle-\frac{\ln m+\ln M}{2}\right|} \\
&\le \frac{\frac{\langle \ln Ax,x\rangle-\ln m}{\ln M-\ln m}M^{-1}+\frac{\ln M-\langle \ln Ax,x\rangle}{\ln M-\ln m}m^{-1}}{\Delta_x(A^{-1})} \\
&\le K\left(\frac{M}{m}\right)^{\frac{1}{2}+\frac{1}{\ln M-\ln m}\left|\langle \ln Ax,x\rangle-\frac{\ln m+\ln M}{2}\right|} \le K\left(\frac{M}{m}\right),
\end{aligned}
$$

for $x \in H$, $\|x\| = 1$.

This proves (16). □

4. Conclusion

In this paper, we obtained between others, various upper and lower bounds for the *normalized determinant* $\Delta_x(A)$ under the natural assumption that $0 < mI \le A \le MI$ for some positive numbers m, M. These bounds are expressed in terms of the *Kantorovich's ratio* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \quad h > 0.$$

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