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Operator upper bounds for Davis-Choi-Jensen's difference in Hilbert spaces

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Abstract. In this paper we obtain several operator inequalities providing upper bounds for the Davis-Choi-Jensen's Difference

$$\Phi(f(A)) - f(\Phi(A))$$

for any convex function $f : I \to \mathbb{R}$, any selfadjoint operator A in H with the spectrum $\operatorname{Sp}(A) \subset I$ and any linear, positive and normalized map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$, where H and K are Hilbert spaces. Some examples for convex and operator convex functions are also provided.

1. Introduction

Let H be a complex Hilbert space and $\mathcal{B}(H)$, the Banach algebra of bounded linear operators acting on H. We denote by $\mathcal{B}_h(H)$ the semi-space of all selfadjoint operators in $\mathcal{B}(H)$. We denote by $\mathcal{B}^+(H)$ the convex cone of all positive operators on H and by $\mathcal{B}^{++}(H)$ the convex cone of all positive definite operators on H.

Let H, K be complex Hilbert spaces. Following [1] (see also [9, p. 18]) we can introduce the following definition.

Definition 1. A map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear if it is additive and homogeneous, namely

$$\Phi(\lambda A + \mu B) = \lambda \Phi(A) + \mu \Phi(B)$$

for any λ, $\mu \in \mathbb{C}$ and A, $B \in \mathcal{B}(H)$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is positive if it preserves the operator order, i.e. if $A \in \mathcal{B}^+(H)$ then $\Phi(A) \in \mathcal{B}^+(K)$. We write $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is normalized if it preserves the identity operator, i.e., $\Phi(1_H) = 1_K$. We write $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

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We observe that a positive linear map Φ preserves the *order relation*, namely

$$A \leq B \text{ implies } \Phi(A) \leq \Phi(B)$$

and preserves the adjoint operation $\Phi(A^*) = \Phi(A)^*$.
If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $\alpha 1_H \leq A \leq \beta 1_H$, then $\alpha 1_K \leq \Phi(A) \leq \beta 1_K$.

If the map $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear, positive and $\Psi(1_H) \in \mathcal{B}^{++}(K)$, then by putting $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ we get $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, namely it is also normalized.

A real valued continuous function f on an interval I is said to be *operator convex (concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda)f(A) + \lambda f(B)$$

for all $\lambda \in [0,1]$ and for every selfadjoint operators A, $B \in \mathcal{B}(H)$ whose spectra are contained in I.

The following Jensen's type result is well known [9, p. 22]:

Theorem 1 (Davis-Choi-Jensen's Inequality)**.** *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then for any selfadjoint operator A whose spectrum is contained in I we have*

$$f(\Phi(A)) \leq \Phi(f(A)). \tag{1}$$

We observe that if $\Psi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$, then by taking $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ in (1) we get

$$f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right) \leq \Psi^{-1/2}(1_H)\Psi(f(A))\Psi^{-1/2}(1_H).$$

If we multiply both sides of this inequality by $\Psi^{1/2}(1_H)$ we get the following *Davis-Choi-Jensen's inequality for general positive linear maps*

$$\Psi^{1/2}(1_H) f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right)\Psi^{1/2}(1_H) \leq \Psi(f(A)). \tag{2}$$

Let $C_j \in \mathcal{B}(H)$, $j = 1, ..., k$ be contractions with

$$\sum_{j=1}^{k} C_j^* C_j = 1_H. \tag{3}$$

The map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ defined by [9, p. 19]

$$\Phi(A) := \sum_{j=1}^{k} C_j^* A C_j$$

is a normalized positive linear map on $\mathcal{B}(H)$.

For more results on inequlities for selfadjoint operators in Hilbert spaces, see [2, 3, 6–8] and the references therein.

In this paper we obtain several operator inequalities providing upper bounds for the Davis-Choi-Jensen's Difference

$$\Phi(f(A)) - f(\Phi(A))$$

for any convex function $f : I \to \mathbb{R}$, any selfadjoint operator A in H with the spectrum $\operatorname{Sp}(A) \subset I$ and any linear, positive and normalized map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$, where H and K are Hilbert spaces. Some examples for convex and operator convex functions are also provided.

2. Main results

We use the following result that was obtained in [4].

Lemma 1. *If $f : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$, then*

$$0 \leq \frac{(b-t) f(a) + (t-a) f(b)}{b-a} - f(t) \leq (b-t)(t-a) \frac{f'_-(b) - f'_+(a)}{b-a} \leq \frac{1}{4}(b-a)\left[f'_-(b) - f'_+(a)\right] \tag{4}$$

for any $t \in [a,b]$.

If the lateral derivatives $f'_-(b)$ and $f'_+(a)$ are finite, then the second inequality and the constant $1/4$ are sharp.

We have:

Theorem 2. *Let $f : [m,M] \to \mathbb{R}$ be a convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A) \subset [m,M]$.*

If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then

$$\begin{aligned} &\Phi(f(A)) - f(\Phi(A)) \\ &\leq \frac{f'_-(M) - f'_+(m)}{M-m}(M1_K - \Phi(A))(\Phi(A) - m1_K) \\ &\leq \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right] 1_K. \end{aligned} \tag{5}$$

Proof. Utilizing the continuous functional calculus for a selfadjoint operator T with $0 \leq T \leq 1_H$ and the convexity of f on $[m,M]$, we have

$$f(m(1_H - T) + MT) \leq f(m)(1_H - T) + f(M)T \tag{6}$$

in the operator order.

If we take in (6)

$$0 \leq T = \frac{A - m1_H}{M-m} \leq 1_H,$$

then we get

$$\begin{aligned} &f\left(m\left(1_H - \frac{A - m1_H}{M-m}\right) + M\frac{A - m1_H}{M-m}\right) \\ &\leq f(m)\left(1_H - \frac{A - m1_H}{M-m}\right) + f(M)\frac{A - m1_H}{M-m}. \end{aligned} \tag{7}$$

Observe that

$$m\left(1_H - \frac{A - m1_H}{M - m}\right) + M\frac{A - m1_H}{M - m}$$
$$= \frac{m(M1_H - A) + M(A - m1_H)}{M - m} = A$$

and

$$f(m)\left(1_H - \frac{A - m1_H}{M - m}\right) + f(M)\frac{A - m1_H}{M - m}$$
$$= \frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m}$$

and by (7) we get the following inequality of interest

$$f(A) \le \frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m}. \tag{8}$$

If we take the map Φ in (8), then we get

$$\begin{aligned}\Phi(f(A)) &\le \Phi\left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m}\right]\\ &= \frac{f(m)\Phi(M1_H - A) + f(M)\Phi(A - m1_H)}{M - m}\\ &= \frac{f(m)(M\Phi(1_H) - \Phi(A)) + f(M)(\Phi(A) - m\Phi(1_H))}{M - m}\\ &= \frac{f(m)(M1_K - \Phi(A)) + f(M)(\Phi(A) - m1_K)}{M - m},\end{aligned}$$

which implies that

$$\begin{aligned}&\Phi(f(A)) - f(\Phi(A))\\ &\quad\le \frac{f(m)(M1_K - \Phi(A)) + f(M)(\Phi(A) - m1_K)}{M - m} - f(\Phi(A)).\end{aligned} \tag{9}$$

Since $m1_K \le \Phi(A) \le M1_K$, then by using (4) for $a = m$, $b = M$ and the continuous functional calculus, we have

$$\begin{aligned}&\frac{f(m)(M1_K - \Phi(A)) + f(M)(\Phi(A) - m1_K)}{M - m} - f(\Phi(A))\\ &\le \frac{f'_-(M) - f'_+(m)}{M - m}(M1_K - \Phi(A))(\Phi(A) - m1_K)\\ &\le \frac{1}{4}(M - m)\left[f'_-(M) - f'_+(m)\right]1_K.\end{aligned} \tag{10}$$

By making use of (9) and (10) we get the desired result (5). □

Corollary 1. *Let $f:[m,M]\to\mathbb{R}$ be an operator convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$.*

If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then

$$\begin{aligned}(11)\qquad 0&\le\Phi(f(A))-f(\Phi(A))\\&\le\frac{f'_-(M)-f'_+(m)}{M-m}(M1_K-\Phi(A))(\Phi(A)-m1_K)\\&\le\frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]1_K.\end{aligned}$$

We also have the following scalar inequality of interest:

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ and $t\in[0,1]$, then*

$$\begin{aligned}(12)\qquad &2\min\{t,1-t\}\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right]\\&\le(1-t)f(a)+tf(b)-f((1-t)a+tb)\\&\le2\max\{t,1-t\}\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right].\end{aligned}$$

The proof follows, for instance, by Corollary 1 from [5] for $n=2$, $p_1=1-t$, $p_2=t$, $t\in[0,1]$ and $x_1=a$, $x_2=b$.

Theorem 3. *Let $f:[m,M]\to\mathbb{R}$ be a convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$.*

If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then

$$\begin{aligned}(13)\qquad &2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\&\quad\times\left(\frac{1}{2}(M-m)1_K-\left|\Phi(A)-\frac{1}{2}(m+M)1_K\right|\right)\\&\le\frac{f(m)(M1_K-\Phi(A))+f(M)(\Phi(A)-m1_K)}{M-m}-f(\Phi(A))\\&\le2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\&\quad\times\left(\frac{1}{2}(M-m)1_K+\left|\Phi(A)-\frac{1}{2}(m+M)1_K\right|\right)\end{aligned}$$

and

$$\begin{aligned}(14)\qquad &2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\&\quad\times\left(\frac{1}{2}(M-m)1_K-\Phi\left(\left|A-\frac{1}{2}(m+M)1_K\right|\right)\right)\\&\le\frac{f(m)(M1_K-\Phi(A))+f(M)(\Phi(A)-m1_K)}{M-m}-\Phi(f(A))\end{aligned}$$

$$\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)1_K+\Phi\left(\left|A-\frac{1}{2}(m+M)1_H\right|\right)\right).$$

Proof. We have from (12) that

$$2\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \tag{15}$$
$$\leq(1-t)f(m)+tf(M)-f((1-t)m+tM)$$
$$\leq 2\left(\frac{1}{2}+\left|t-\frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right],$$

for all $t\in[0,1]$.

Utilizing the continuous functional calculus for a selfadjoint operator T with $0\leq T\leq 1_H$ we get from (15) that

$$2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2}1_H-\left|T-\frac{1}{2}1_H\right|\right) \tag{16}$$
$$\leq(1-T)f(m)+Tf(M)-f((1-T)m+TM)$$
$$\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2}1_H+\left|T-\frac{1}{2}1_H\right|\right),$$

in the operator order.

If we take in (16)

$$0\leq T=\frac{A-m1_H}{M-m}\leq 1_H,$$

then, like in the proof of Theorem 2, we get

$$2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \tag{17}$$
$$\times\left(\frac{1}{2}(M-m)1_H-\left|A-\frac{1}{2}(m+M)1_H\right|\right)$$
$$\leq\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}-f(A)$$
$$\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)1_H+\left|A-\frac{1}{2}(m+M)1_H\right|\right).$$

Since $m1_K\leq\Phi(A)\leq M1_K$, then by writing the inequality (17) for $\Phi(A)$ instead of A we get (13).

If we take Φ in (17), then we get

$$\begin{aligned}
&2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\
&\quad\times\Phi\left(\frac{1}{2}(M-m)1_H-\left|A-\frac{1}{2}(m+M)1_H\right|\right)\\
&\leq\Phi\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]-\Phi f(A)\\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\
&\quad\times\Phi\left(\frac{1}{2}(M-m)1_H+\left|A-\frac{1}{2}(m+M)1_H\right|\right),
\end{aligned}$$

which is equivalent to (14). □

Corollary 2. *Let $f:[m,M]\to\mathbb{R}$ be an operator convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$.*

If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then

$$\begin{aligned}
(18)\qquad &2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\
&\quad\times\left(\frac{1}{2}(M-m)1_K-\Phi\left(\left|A-\frac{1}{2}(m+M)1_H\right|\right)\right)\\
&\leq\frac{f(m)(M1_K-\Phi(A))+f(M)(\Phi(A)-m1_K)}{M-m}-\Phi(f(A))\\
&\leq\frac{f(m)(M1_K-\Phi(A))+f(M)(\Phi(A)-m1_K)}{M-m}-f(\Phi(A))\\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\
&\quad\times\left(\frac{1}{2}(M-m)1_K+\left|\Phi(A)-\frac{1}{2}(m+M)1_K\right|\right).
\end{aligned}$$

We also have:

Corollary 3. *Let $f:[m,M]\to\mathbb{R}$ be a convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$.*

If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then

$$\begin{aligned}
(19)\qquad \Phi(f(A))-f(\Phi(A))&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\
&\quad\times\left(\frac{1}{2}(M-m)1_K+\left|\Phi(A)-\frac{1}{2}(m+M)1_K\right|\right)\\
&\leq 2(M-m)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]1_K.
\end{aligned}$$

Proof. From (9) we have

$$\begin{aligned}
&\Phi(f(A)) - f(\Phi(A)) \\
&\le \frac{f(m)(M1_K - \Phi(A)) + f(M)(\Phi(A) - m1_K)}{M-m} - f(\Phi(A))
\end{aligned}$$

and from (14) we have

$$\begin{aligned}
&\frac{f(m)(M1_K - \Phi(A)) + f(M)(\Phi(A) - m1_K)}{M-m} - f(\Phi(A)) \\
&\le 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\
&\quad \times \left(\frac{1}{2}(M-m)1_K + \left|\Phi(A) - \frac{1}{2}(m+M)1_K\right|\right),
\end{aligned}$$

which produce the desired result (19). □

Remark 1. If $f:[m,M]\to\mathbb{R}$ is an operator convex function on $[m,M]$, A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$ and $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then

$$\begin{aligned}
0 &\le \Phi(f(A)) - f(\Phi(A)) \\
&\le 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\
&\quad \times \left(\frac{1}{2}(M-m)1_K + \left|\Phi(A) - \frac{1}{2}(m+M)1_K\right|\right) \\
&\le 2(M-m)\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]1_K.
\end{aligned} \tag{20}$$

We also have [4]:

Lemma 3. *Assume that $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$. If f' is K-Lipschitzian on $[a,b]$, then*

$$\begin{aligned}
&|(1-t)f(a) + tf(b) - f((1-t)a+tb)| \\
&\le \frac{1}{2}K(b-t)(t-a) \le \frac{1}{8}K(b-a)^2
\end{aligned} \tag{21}$$

for all $t\in[0,1]$.

The constants $1/2$ and $1/8$ are the best possible in (21).

Remark 2. If $f:[a,b]\to\mathbb{R}$ is twice differentiable and $f''\in L_\infty[a,b]$, then

$$\begin{aligned}
&|(1-t)f(a) + tf(b) - f((1-t)a+tb)| \\
&\le \frac{1}{2}\|f''\|_{[a,b],\infty}(b-t)(t-a) \le \frac{1}{8}\|f''\|_{[a,b],\infty}(b-a)^2,
\end{aligned} \tag{22}$$

where $\|f''\|_{[a,b],\infty} := \operatorname{essup}_{t\in[a,b]}|f''(t)| < \infty$. The constants $1/2$ and $1/8$ are the best possible in (22).

We have:

Theorem 4. *Let $f:[m,M]\to\mathbb{R}$ be a twice differentiable convex function on $[m,M]$ with $\|f''\|_{[m,M],\infty}:=\operatorname{essup}_{t\in[m,M]} f''(t)<\infty$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$. If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then*

$$\begin{aligned}\Phi(f(A))-f(\Phi(A))&\\ \le\frac{1}{2}\|f''\|_{[m,M],\infty}&(M1_K-\Phi(A))(\Phi(A)-m1_K)\\ \le\frac{1}{8}\|f''\|_{[m,M],\infty}&(M-m)^2 1_K.\end{aligned}\tag{23}$$

Proof. From (22) and the continuous functional calculus, we get

$$\begin{aligned}0&\le\frac{f(m)(M1_H-B)+f(M)(B-m1_H)}{M-m}-f(B)\\ &\le\frac{1}{2}\|f''\|_{[m,M],\infty}(M1_H-B)(B-m1_H)\\ &\le\frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_H\end{aligned}\tag{24}$$

where B is a selfadjoint operator with the spectrum $\operatorname{Sp}(B)\subset[m,M]$.

If we use (24) for $\Phi(A)$ we get

$$\begin{aligned}0&\le\frac{f(m)(M1_K-\Phi(A))+f(M)(\Phi(A)-m1_K)}{M-m}-f(\Phi(A))\\ &\le\frac{1}{2}\|f''\|_{[m,M],\infty}(M1_K-\Phi(A))(\Phi(A)-m1_K)\\ &\le\frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_K.\end{aligned}\tag{25}$$

Since

$$\begin{aligned}&\Phi(f(A))-f(\Phi(A))\\ &\le\frac{f(m)(M1_K-\Phi(A))+f(M)(\Phi(A)-m1_K)}{M-m}-f(\Phi(A)),\end{aligned}$$

hence by (25) we get (23). □

Corollary 4. *Let $f:[m,M]\to\mathbb{R}$ be an operator convex function on $[m,M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A)\subset[m,M]$.*

If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then

$$\begin{aligned}0&\le\Phi(f(A))-f(\Phi(A))\\ &\le\frac{1}{2}\|f''\|_{[m,M],\infty}(M1_K-\Phi(A))(\Phi(A)-m1_K)\\ &\le\frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_K.\end{aligned}\tag{26}$$

3. Some examples

We consider the exponential function $f(x) = \exp(\alpha x)$ with $\alpha \in \mathbb{R} \setminus \{0\}$. This function is convex but not operator convex on $\mathbb{R}$. If A is selfadjoint with $\mathrm{Sp}(A) \subset [m, M]$ for some $m < M$ and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then by (5), (19) and (23) we have

$$\begin{aligned}
(27) \qquad & \Phi(\exp(\alpha A)) - \exp(\alpha \Phi(A)) \\
& \leq \alpha \frac{\exp(\alpha M) - \exp(\alpha m)}{M - m} (M 1_K - \Phi(A))(\Phi(A) - m 1_K) \\
& \leq \frac{1}{4} \alpha (M - m) [\exp(\alpha M) - \exp(\alpha m)] 1_K,
\end{aligned}$$

$$\begin{aligned}
(28) \qquad & \Phi(\exp(\alpha A)) - \exp(\alpha \Phi(A)) \\
& \leq 2 \left[\frac{\exp(\alpha m) + f(\alpha M)}{2} - \exp\left(\alpha \frac{m + M}{2}\right) \right] \\
& \quad \times \left(\frac{1}{2}(M - m) 1_K + \left| \Phi(A) - \frac{1}{2}(m + M) 1_K \right| \right) \\
& \leq 2(M - m) \left[\frac{\exp(\alpha m) + f(\alpha M)}{2} - \exp\left(\alpha \frac{m + M}{2}\right) \right] 1_K
\end{aligned}$$

and

$$\begin{aligned}
(29) \qquad & \Phi(\exp(\alpha A)) - \exp(\alpha \Phi(A)) \\
& \leq \frac{1}{2} \alpha^2 \begin{cases} \exp(\alpha M) & \text{if } \alpha > 0 \\ \exp(\alpha m) & \text{if } \alpha < 0 \end{cases} \times (M 1_K - \Phi(A))(\Phi(A) - m 1_K) \\
& \leq \frac{1}{8} \alpha^2 (M - m)^2 \begin{cases} \exp(\alpha M) & \text{if } \alpha > 0 \\ \exp(\alpha m) & \text{if } \alpha < 0 \end{cases} \times 1_K.
\end{aligned}$$

The function $f(x) = -\ln x$, $x > 0$ is operator convex on $(0, \infty)$. If A is selfadjoint with $\mathrm{Sp}(A) \subset [m, M]$ for some $0 < m < M$ and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then by (11), (20) and (26) we have

$$\begin{aligned}
(30) \qquad & 0 \leq \ln(\Phi(A)) - \Phi(\ln(A)) \\
& \leq \frac{1}{mM} (M 1_V - \Phi(A))(\Phi(A) - m 1_K) \leq \frac{1}{4mM} (M - m)^2 1_K,
\end{aligned}$$

$$\begin{aligned}
(31) \qquad & 0 \leq \ln(\Phi(A)) - \Phi(\ln(A)) \\
& \leq 2 \ln\left(\frac{m + M}{2\sqrt{mM}}\right) \left(\frac{1}{2}(M - m) 1_K + \left| \Phi(A) - \frac{1}{2}(m + M) 1_K \right| \right) \\
& \leq 2(M - m) \ln\left(\frac{m + M}{2\sqrt{mM}}\right) 1_K
\end{aligned}$$

and

$$0 \leq \ln(\Phi(A)) - \Phi(\ln(A)) \leq \frac{1}{2m^2}(M1_K - \Phi(A))(\Phi(A) - m1_K) \leq \frac{1}{8m^2}(M-m)^2 1_K. \tag{32}$$

We observe that if $M > 2m$ then the bound in (30) is better than the one from (32). If $M < 2m$, then the conclusion is the other way around.

The function $f(x) = x\ln x$, $x > 0$ is operator convex on $(0,\infty)$. If A is selfadjoint with $\mathrm{Sp}(A) \subset [m, M]$ for some $0 < m < M$ and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then by (11), (20) and (26) we have

$$0 \leq \Phi(A\ln(A)) - \Phi(A)\ln(\Phi(A)) \leq \frac{\ln(M) - \ln(m)}{M-m}(M1_K - \Phi(A))(\Phi(A) - m1_K) \leq \frac{1}{4}(M-m)[\ln(M) - \ln(m)]1_K, \tag{33}$$

(34)

$$\begin{aligned} 0 &\leq \Phi(A\ln(A)) - \Phi(A)\ln(\Phi(A)) \\ &\leq 2\left[\frac{m\ln(m) + M\ln(M)}{2} - \left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right] \\ &\quad\times\left(\frac{1}{2}(M-m)1_K + \left|\Phi(A) - \frac{1}{2}(m+M)1_K\right|\right) \\ &\leq 2(M-m)\left[\frac{m\ln(m) + M\ln(M)}{2} - \left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right]1_K \end{aligned}$$

and

$$0 \leq \Phi(A\ln(A)) - \Phi(A)\ln(\Phi(A)) \leq \frac{1}{2m}(M1_K - \Phi(A))(\Phi(A) - m1_K) \leq \frac{1}{8m}(M-m)^2 1_K. \tag{35}$$

Consider the power function $f(x) = x^r$, $x \in (0,\infty)$ and r a real number. If $r \in (-\infty, 0] \cup [1,\infty)$, then f is convex and for $r \in [-1, 0] \cup [1, 2]$ is operator convex. If we use the inequalities (5), (19) and (23) we have for $r \in (-\infty, 0] \cup [1,\infty)$ that

$$\Phi(A^r) - (\Phi(A))^r \leq r\frac{M^{r-1} - m^{r-1}}{M-m}(M1_K - \Phi(A))(\Phi(A) - m1_K) \leq \frac{1}{4}r(M-m)\left(M^{r-1} - m^{r-1}\right)1_K, \tag{36}$$

$$\begin{aligned}\Phi\left(A^{r}\right)-(\Phi(A))^{r} & \\ \leq 2\left[\frac{m^{r}+M^{r}}{2}-\left(\frac{m+M}{2}\right)^{r}\right] & \\ \times\left(\frac{1}{2}(M-m) 1_{K}+\left|\Phi(A)-\frac{1}{2}(m+M) 1_{K}\right|\right) & \\ \leq 2(M-m)\left[\frac{m^{r}+M^{r}}{2}-\left(\frac{m+M}{2}\right)^{r}\right] 1_{K} &\end{aligned} \tag{37}$$

and

$$\begin{aligned}&\Phi\left(A^{r}\right)-(\Phi(A))^{r}\\ &\leq \frac{1}{2} r(r-1)\begin{cases}M^{r-2} \text { for } r \geq 2 \\ m^{r-2} \text { for } r \in(-\infty, 0] \cup[1,2)\end{cases}\\ &\times\left(M 1_{K}-\Phi(A)\right)\left(\Phi(A)-m 1_{K}\right)\\ &\leq \frac{1}{8} r(r-1)(M-m)^{2}\begin{cases}M^{r-2} \text { for } r \geq 2 \\ m^{r-2} \text { for } r \in(-\infty, 0] \cup[1,2)\end{cases} \times 1_{K},\end{aligned} \tag{38}$$

where A is selfadjoint with $\operatorname{Sp}(A) \subset [m, M]$ for some $0 < m < M$ and $\Phi \in \mathfrak{P}_N [\mathcal{B}(H), \mathcal{B}(K)]$.

If $r \in [-1, 0] \cup [1, 2]$, then we also have $0 \leq \Phi(A^r) - (\Phi(A))^r$ in the inequalities (36)-(38).

For $r = -1$ we have the inequalities

$$\begin{aligned}0 & \leq \Phi\left(A^{-1}\right)-(\Phi(A))^{-1} \\ & \leq \frac{M+m}{M^{2} m^{2}}\left(M 1_{K}-\Phi(A)\right)\left(\Phi(A)-m 1_{K}\right) \\ & \leq \frac{1}{4}(M-m)^{2} \frac{M+m}{M^{2} m^{2}} 1_{K},\end{aligned} \tag{39}$$

$$\begin{aligned}0 & \leq \Phi\left(A^{-1}\right)-(\Phi(A))^{-1} \\ & \leq \frac{(M-m)^{2}}{m M(m+M)}\left(\frac{1}{2}(M-m) 1_{K}+\left|\Phi(A)-\frac{1}{2}(m+M) 1_{K}\right|\right) \\ & \leq \frac{(M-m)^{3}}{m M(m+M)} 1_{K}\end{aligned} \tag{40}$$

and

$$\begin{aligned}0 & \leq \Phi\left(A^{-1}\right)-(\Phi(A))^{-1} \\ & \leq \frac{1}{m^{3}}\left(M 1_{K}-\Phi(A)\right)\left(\Phi(A)-m 1_{K}\right) \leq \frac{1}{4 m^{3}}(M-m)^{2} 1_{K},\end{aligned} \tag{41}$$

where A is selfadjoint with $\operatorname{Sp}(A) \subset [m, M]$ for some $0 < m < M$ and $\Phi \in \mathfrak{P}_N [\mathcal{B}(H), \mathcal{B}(K)]$.

4. CONCLUSION

In this paper we obtained several operator inequalities providing upper bounds for the celebrated Davis-Choi-Jensen's Difference for any convex function $f : I \to \mathbb{R}$, any selfadjoint operator A in H with the spectrum $\operatorname{Sp}(A) \subset I$ and any linear, positive and normalized map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$, where H and K are Hilbert spaces. Some examples for fundamental convex and operator convex functions of interest, to ilustrate the main results, were also provided.

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