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Accuracy of analytical approximation formula for bond prices in a three-factor convergence model of interest rates

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Abstract. We consider a convergence model of interest rates, in which the behaviour of the domestic instantaneous interest rate (so called short rate) depends on the short rate in a monetary union that the country is going to join. The short rate in the monetary union is modelled by a two-factor model, which leads to a three-factor model for the domestic rate. In this setting, term structures of interest rates are computed from bond prices, which are obtained as solutions to a parabolic partial differential equation. A closed-form solution is known only in special cases. An analytical approximation formula for the domestic bond prices has been proposed, with the error estimate only for certain parameter values, when the solution has a separable form. In this paper, we derive the order of accuracy in the general case. We also study a special case, which makes it possible to model the phenomenon of negative interest rates that were observed in the previous years. It turns out that it leads to a higher accuracy than the one achieved in the general case without restriction on parameters.

1. Introduction

Short rate models constitute a framework for modelling interest rates and interest rate derivatives, in which an instantaneous interest rate (called short rate) is modelled by a stochastic differential equation or a system of such equations. Derivative prices are then given by a solution to a parabolic partial differential equation. In particular, bond prices are obtained in this way. A discount zero coupon bond is a financial security which pays a unit amount money at the specified time in the future, called maturity of the

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bond. The bond prices are connected with interest rates by the formula

$$P(t,T) = e^{-R(t,T)(T-t)} \quad \Rightarrow \quad R(t,T) = -\frac{P(t,T)}{T-t},$$

where t is the current time, T is the maturity of the bond, P is the bond price and R is the corresponding interest rate. Therefore, we need to solve the bond-pricing partial differential equation in order to obtain the interest rates implied by the model, as well as to discount any future cash flows. See for example [3] or [9] for more details on interest rate modelling and partial differential approach to pricing derivatives in short rate models.

There are several ways for modelling the short rate. We consider a convergence model, which models the interest rates in a country which is going to join a monetary union and its interest rates are influenced by interest rates in the monetary union. A pioneering paper in this regard is the Corzo-Schwartz model [6], its modifications and generalizations were studied in [1, 7, 12, 14]. In this paper we consider the model suggested in [12] and extend the analysis of accuracy of the analytical approximation formula for bond prices. The bond prices are solutions to a parabolic partial differential equation. Proposing the approximate analytical formulae and their mathematical analysis in various short rate models is an ongoing and current research, see for example [4, 10, 11] for different approaches.

The paper is organized as follows. In Section 2 we describe the model, partial differential equation for the bond prices and the approximation to its solution proposed in [12]. In Section 3 we derive the order of its accuracy in the general case, extending the result from [12] where it was computed in a special case, when the bond-pricing equation has a separable solution. In Section 4 we deal with a phenomenon of negative interest rates (cf. [15]) by considering parameter values which allow negative rates in the model, and the order of the approximation formula accuracy in this setting. We conclude the paper in Section 5.

2. A three-factor convergence model of interest rates and approximation formula for the bond prices

In this paper we consider the model based on [12]. We recall that the motivation comes from the pioneering convergence model by [6]. In this paper, the European short rate is modelled by the Vasicek model [13] and the domestic short rate is reverting to the European short rate with a possible minor divergence. This model has been generalized in [12]. Firstly, by considering nonconstant volatilites of the processes in the form of [5]. Secondly, by considering a more general model for the European short rate, which is modelled as a sum of two processes, as proposed in [2].

In the risk neutral measure, the European short rate r_e is a sum of r_1 and r_2, given by the stochastic differential equations:

$$
\begin{aligned}
\mathbf{d}r_1 &= (b_1 + b_2 r_1)\mathbf{d}t + \sigma_1 r_1^{\gamma_1}\mathbf{d}w_1,\\
\mathbf{d}r_2 &= (c_1 + c_2 r_2)\mathbf{d}t + \sigma_2 r_2^{\gamma_2}\mathbf{d}w_2.
\end{aligned}
$$

The stochastic differential equation for the domestic short rate r_d reads as:

$$\mathbf{d}r_d = (a_1 + a_2 r_d + a_3 r_1 + a_4 r_2)\mathbf{d}t + \sigma_d r_d^{\gamma_d}\mathbf{d}w_d.$$

The increments of the Wiener processes w can be correlated; we denote by $\rho_{ij} \in (-1, 1)$ the correlation between the increments $\mathbf{d}w_i$ and $\mathbf{d}w_j$ (where $i, j \in \{d, 1, 2\}$). We note that ρ_{ij} cannot be arbitrary, but the resulting correlation matrix has to be positive definite. The parameters γ_1, γ_2 are assumed to be nonnegative.

In this setting, the bond price P is a function of the stochastic factors r_1, r_2, r_d and of the time $\tau \geq 0$ remaining to its maturity. The partial differential equation for $P(\tau, r_1, r_2, r_d)$ is given by:

$$
\begin{aligned}
&-\frac{\partial P}{\partial \tau} + \mu_d \frac{\partial P}{\partial r_d} + \mu_1 \frac{\partial P}{\partial r_1} + \mu_2 \frac{\partial P}{\partial r_2} + \frac{1}{2}\sigma_d^2 r_d^{2\gamma_d} \frac{\partial^2 P}{\partial r_d^2} + \frac{1}{2}\sigma_1^2 r_1^{2\gamma_1} \frac{\partial^2 P}{\partial r_1^2}\\
&+ \frac{1}{2}\sigma_2^2 r_2^{2\gamma_2} \frac{\partial^2 P}{\partial r_2^2} + \rho_{1d}\sigma_1 r_1^{\gamma_1} \sigma_d r_d^{\gamma_d} \frac{\partial^2 P}{\partial r_1 \partial r_d} + \rho_{2d}\sigma_2 r_2^{\gamma_2} \sigma_d r_d^{\gamma_d} \frac{\partial^2 P}{\partial r_2 \partial r_d}\\
&+ \rho_{12}\sigma_1 r_1^{\gamma_1} \sigma_2 r_2^{\gamma_2} \frac{\partial^2 P}{\partial r_1 \partial r_2} - r_d P = 0, \qquad (1)
\end{aligned}
$$

where we have denoted by μ the risk neutral drifts

$$\mu_d = a_1 + a_2 r_d + a_3 r_1 + a_4 r_2, \mu_1 = b_1 + b_2 r_1, \mu_2 = c_1 + c_2 r_2.$$

The initial condition is given by $P(r_d, r_1, r_2, 0) = 1$ for all r_d, r_1, r_2 (their range depends on the values of $\gamma_1, \gamma_2, \gamma_d$: if any of them is equal to zero, the corresponding r variable can attain any real value; a positive γ does not allow negative values of the corresponding r). We refer the reader to [9] for details on derivation of the partial differential equation for interest rate derivatives in multifactor short rate models.

If $\gamma_d = \gamma_1 = \gamma_2 = 0$ (i.e., the volatilities of the processes are constant; so called Vasicek-type of the model, since in the paper [13] a popular one-factor model with this property was presented), a closed form solution in a separate form

$$P(r_d, r_1, r_2, \tau) = \exp(A(\tau)r_d + B(\tau)r_1 + C(\tau)r_2 + D(\tau)), \tag{2}$$

exists. This closed form solution was a base for the approximation formula suggested for the general model in [12], which, for the generic case $a_2 \neq b_2$ and $a_2 \neq c_2$, has the following form:

$$P^{ap}(r_d, r_1, r_2, \tau) = \exp(A(\tau)r_d + B(\tau)r_1 + C(\tau)r_2 + D(r_d, r_1, r_2, \tau)), \tag{3}$$

where

$$A(\tau) = \frac{1 - e^{a_2\tau}}{a_2}, \tag{4}$$

$$B(\tau) = \frac{a_3(b_2(1 - e^{a_2\tau}) - a_2(1 - e^{b_2\tau}))}{a_2 b_2 (a_2 - b_2)}, \tag{5}$$

$$C(\tau) = \frac{a_4(c_2(1 - e^{a_2\tau}) - a_2(1 - e^{c_2\tau}))}{a_2 c_2 (a_2 - c_2)}, \tag{6}$$

$$\begin{aligned} D(r_d, r_1, r_2, \tau) = \int_0^\tau & a_1 A(s) + b_1 B(s) + c_1 C(s) + \frac{1}{2}\sigma_d^2 r_d^{2\gamma_d} A^2(s) \\ & + \frac{1}{2}\sigma_1^2 r_1^{2\gamma_1} B^2(s) + \frac{1}{2}\sigma_2^2 r_2^{2\gamma_2} C^2(s) \rho_{1d}\sigma_1 r_1^{\gamma_1}\sigma_d r_d^{\gamma_d} A(s)B(s) \\ & + \rho_{2d}\sigma_2 r_2^{\gamma_2}\sigma_d r_d^{\gamma_d} A(s)C(s) + \rho_{12}\sigma_1 r_1^{\gamma_1}\sigma_2 r_2^{\gamma_2} B(s)C(s)\mathbf{d}s. \end{aligned} \tag{7}$$

The function $D(\tau)$ can be expressed in a closed form as well, but we leave it in the integral form for the sake of brevity.

Another special case studied in [12] is the Cox-Ingersoll-Ross type (referring to one-factor model [8] with the same form of volatilities) with $\gamma_d = \gamma_1 = \gamma_2 = 1/2$ and $\rho_{12} = \rho_{1d} = \rho_{2d} = 0$ (i.e., the increments of the Wiener processes are uncorrelated). In this case, the bond-pricing equation has again the solution in the form (2) and the resulting system of ordinary differential equations for A, B, C, D was used in [12] to derive the order of accuracy of the approximation (3) with A, B, C, D given by (4)-(7) as $\tau \to 0^+$. If we denote by $P^{cir,ex}$ the exact solution and by $P^{cir,ap}$ the approximation given above, we have

$$\log P^{cir,ap} - \log P^{cir,ex} = -\frac{1}{24}\sigma_d^2(a_1 + a_2 r_d + a_3 r_1 + a_4 r_2)\tau^4 + O(\tau^5) \tag{8}$$

as $\tau \to 0^+$.

3. ORDER OF ACCURACY OF THE APPROXIMATION FORMULA IN THE GENERAL CASE

In this section we generalize the estimate (8) to the case of general powers $\gamma_d, \gamma_1, \gamma_2$ and correlations $\rho_{1d}, \rho_{2d}, \rho_{12}$.

Firstly, let us denote

$$f^{ex}(r_d, r_1, r_2, \tau) = \log P(r_d, r_1, r_2, \tau),$$
$$f^{ap}(r_d, r_1, r_2, \tau) = \log P^{ap}(r_d, r_1, r_2, \tau),$$

where P^{ex} is the exact solution of (1) and P^{ap} is its approximation given by (3) and (4)-(7). Then, f^{ex} satisfies

$$\begin{aligned}
&\frac{\partial f^{ex}}{\partial \tau} + \mu_d \frac{\partial f^{ex}}{\partial r_d} + \mu_1 \frac{\partial f^{ex}}{\partial r_1} + \mu_2 \frac{\partial f^{ex}}{\partial r_2} + \frac{1}{2}\sigma_d^2 r_d^{2\gamma_d} \left(\left(\frac{\partial f^{ex}}{\partial r_d} \right)^2 + \frac{\partial^2 f^{ex}}{\partial r_d^2} \right) \\
&\quad + \frac{1}{2}\sigma_1^2 r_1^{2\gamma_1} \left(\left(\frac{\partial f^{ex}}{\partial r_1} \right)^2 + \frac{\partial^2 f^{ex}}{\partial r_1^2} \right) + \frac{1}{2}\sigma_2^2 r_2^{2\gamma_2} \left(\left(\frac{\partial f^{ex}}{\partial r_2} \right)^2 + \frac{\partial^2 f^{ex}}{\partial r_2^2} \right) \\
&\quad + \rho_{1d}\sigma_d r_d^{\gamma_d} \sigma_1 r_1^{\gamma_1} \left(\frac{\partial f^{ex}}{\partial r_d}\frac{\partial f^{ex}}{\partial r_1} + \frac{\partial^2 f^{ex}}{\partial r_d \partial r_1} \right) \\
&\quad + \rho_{2d}\sigma_d r_d^{\gamma_d} \sigma_2 r_2^{\gamma_2} \left(\frac{\partial f^{ex}}{\partial r_d}\frac{\partial f^{ex}}{\partial r_2} + \frac{\partial^2 f^{ex}}{\partial r_d \partial r_2} \right) \\
&\quad + \rho_{12}\sigma_1 r_1^{\gamma_1} \sigma_2 r_2^{\gamma_2} \left(\frac{\partial f^{ex}}{\partial r_1}\frac{\partial f^{ex}}{\partial r_2} + \frac{\partial^2 f^{ex}}{\partial r_1 \partial r_2} \right) - r_d = 0. \qquad (9)
\end{aligned}$$

If we substitute f^{ap} into the left hand side of (9) in place of f^{ex}, we obtain a nontrivial right hand side, which we denote by $h(r_d, r_1, r_2, \tau)$. Next, we define a function

$$g(r_d, r_1, r_2, \tau) = f^{ap}(r_d, r_1, r_2, \tau) - f^{ex}(r_d, r_1, r_2, \tau)$$

and note that in order to assess the accuracy of our approximation we need to find the order of the function g. The function g satisfies

$$\begin{aligned}
&-\frac{\partial g}{\partial \tau} + \mu_d \frac{\partial g}{\partial r_d} + \mu_1 \frac{\partial g}{\partial r_1} + \mu_2 \frac{\partial g}{\partial r_2} + \frac{1}{2}\sigma_d^2 r_d^{2\gamma_d} \left(\left(\frac{\partial g}{\partial r_d} \right)^2 + \frac{\partial^2 g}{\partial r_d^2} \right) \\
&+ \frac{1}{2}\sigma_1^2 r_1^{2\gamma_1} \left(\left(\frac{\partial g}{\partial r_1} \right)^2 + \frac{\partial^2 g}{\partial r_1^2} \right) + \frac{1}{2}\sigma_2^2 r_2^{2\gamma_2} \left(\left(\frac{\partial g}{\partial r_2} \right)^2 + \frac{\partial^2 g}{\partial r_2^2} \right) \\
&+ \rho_{1d}\sigma_d r_d^{\gamma_d} \sigma_1 r_1^{\gamma_1} \left(\frac{\partial g}{\partial r_d}\frac{\partial g}{\partial r_1} + \frac{\partial^2 g}{\partial r_d \partial r_1} \right) \\
&+ \rho_{2d}\sigma_d r_d^{\gamma_d} \sigma_2 r_2^{\gamma_2} \left(\frac{\partial g}{\partial r_d}\frac{\partial g}{\partial r_2} + \frac{\partial^2 g}{\partial r_d \partial r_2} \right)
\end{aligned}$$

$$+ \rho_{12}\sigma_1 r_1^{\gamma_1}\sigma_2 r_2^{\gamma_2}\left(\frac{\partial g}{\partial r_1}\frac{\partial g}{\partial r_2} + \frac{\partial^2 g}{\partial r_1 \partial r_2}\right) - r_d$$

$$= h(r_d, r_1, r_2, \tau) + \frac{1}{2}\sigma_d^2 r_d^{2\gamma_d}\left[\left(\frac{\partial f^{ex}}{\partial r_d}\right)^2 - \frac{\partial f^{ap}}{\partial r_d}\frac{\partial f^{ex}}{\partial r_d}\right]$$

$$+ \frac{1}{2}\sigma_1^2 r_1^{2\gamma_1}\left[\left(\frac{\partial f^{ex}}{\partial r_1}\right)^2 - \frac{\partial f^{ap}}{\partial r_1}\frac{\partial f^{ex}}{\partial r_1}\right]$$

$$+ \frac{1}{2}\sigma_2^2 r_2^{2\gamma_2}\left[\left(\frac{\partial f^{ex}}{\partial r_2}\right)^2 - \frac{\partial f^{ap}}{\partial r_2}\frac{\partial f^{ex}}{\partial r_2}\right]$$

$$+ \rho_{1d}\sigma_1 r_1^{\gamma_1}\sigma_d r_d^{\gamma_d}\left(2\frac{\partial f^{ex}}{\partial r_1}\frac{\partial f^{ex}}{\partial r_d} - \frac{\partial f^{ap}}{\partial r_1}\frac{\partial f^{ex}}{\partial r_d} - \frac{\partial f^{ex}}{\partial r_1}\frac{\partial f^{ap}}{\partial r_d}\right)$$

$$+ \rho_{2d}\sigma_2 r_2^{\gamma_2}\sigma_d r_d^{\gamma_d}\left(2\frac{\partial f^{ex}}{\partial r_2}\frac{\partial f^{ex}}{\partial r_d} - \frac{\partial f^{ap}}{\partial r_2}\frac{\partial f^{ex}}{\partial r_d} - \frac{\partial f^{ex}}{\partial r_2}\frac{\partial f^{ap}}{\partial r_d}\right)$$

$$+ \rho_{12}\sigma_1 r_1^{\gamma_1}\sigma_2 r_2^{\gamma_2}\left(2\frac{\partial f^{ex}}{\partial r_1}\frac{\partial f^{ex}}{\partial r_2} - \frac{\partial f^{ap}}{\partial r_1}\frac{\partial f^{ex}}{\partial r_2} - \frac{\partial f^{ex}}{\partial r_1}\frac{\partial f^{ap}}{\partial r_2}\right). \tag{10}$$

Since the function f^{ap} is given in a closed form, we are able to compute the Taylor expansion of h (alternatively, we can conveniently use the system of ordinary equations, from which the functions A, B, C, D in the Vasicek case are obtained, cf. [12] for the resulting system):

$$h(r_d, r_1, r_2, \tau) = k_3(r_d, r_1, r_2)\tau^3 + k_4(r_d, r_1, r_2)\tau^4 + O(\tau^5) \tag{11}$$

with

$$k_3 = \frac{1}{6}\sigma_d^2\gamma_d\left((2\gamma_d - 1)\sigma_d^2 r_d^{4\gamma_d - 2} + 2r_d^{2\gamma_d - 1}(a_1 + a_2 r_d + a_3 r_1 + a_4 r_2)\right).$$

We write also the function g in the form of its Taylor expansion:

$$g(r_d, r_1, r_2, \tau) = \sum_{k=0}^{\infty} c_k(r_d, r_1, r_2)\tau^k = \sum_{k=\omega}^{\infty} c_k(r_d, r_1, r_2)\tau^k, \tag{12}$$

i.e., the first nonzero term is $c_\omega(r_d, r_1, r_2)\tau^\omega$. We know that $\omega \neq 0$, since $g(r_d, r_1, r_2, 0) = \log P^{ap}(r_d, r_1, r_2, 0) - \log P^{ex}(r_d, r_1, r_2, 0) = 0$. Therefore, the lowest order term on the left hand side of (10) is $-c_\omega(r_d, r_1, r_2)\omega\tau^{\omega-1}$. What remains, is the derivation of the lowest order term on its right hand side.

We know that $f^{ex}(r_d, r_1, r_2, 0) = 0$, therefore f^{ex} is $O(\tau)$ as well as its partial derivatives $\frac{\partial f^{ex}}{\partial r_d}$, $\frac{\partial f^{ex}}{\partial r_1}$, $\frac{\partial f^{ex}}{\partial r_2}$. Using the explicit form of f^{ap} we get $\frac{\partial f^{ap}}{\partial r_d} = O(\tau)$, $\frac{\partial f^{ap}}{\partial r_1} = O(\tau^2)$ and $\frac{\partial f^{ap}}{\partial r_2} = O(\tau^2)$. Therefore the right hand side of (10) is at least of order 2, from which it follows that $\omega \geq 3$. Using this information, that $f^{ap} - f^{ex} = O(\tau^3)$, we make the estimates of orders more precise. We get

$$\left(\frac{\partial f^{ex}}{\partial r_d}\right)^2 - \frac{\partial f^{ap}}{\partial r_d}\frac{\partial f^{ex}}{\partial r_d} = \frac{\partial f^{ex}}{\partial r_d}\left(\frac{\partial f^{ex}}{\partial r_d} - \frac{\partial f^{ap}}{\partial r_d}\right) = O(\tau)O(\tau^3) = O(\tau^4)$$

and in the same way for r_1 and r_2. Also,

$$2\frac{\partial f^{ex}}{\partial r_1}\frac{\partial f^{ex}}{\partial r_d} - \frac{\partial f^{ap}}{\partial r_1}\frac{\partial f^{ex}}{\partial r_d} - \frac{\partial f^{ex}}{\partial r_1}\frac{\partial f^{ap}}{\partial r_d} = \frac{\partial f^{ex}}{\partial r_1}\left(\frac{\partial f^{ex}}{\partial r_d} - \frac{\partial f^{ap}}{\partial r_d}\right)$$
$$+ \frac{\partial f^{ex}}{\partial r_d}\left(\frac{\partial f^{ex}}{\partial r_1} - \frac{\partial f^{ap}}{\partial r_1}\right) = O(\tau)O(\tau^3) + O(\tau)O(\tau^3) = O(\tau^4)$$

and again similarly the remaining terms. Therefore, we can conclude that the right hand side of (10) is $O(\tau^3)$ and the $O(\tau^3)$ term comes only from the function $h(r_d, r_1, r_2, \tau)$ and hence it is equal to $k_3(r_d, r_1, r_2)\tau^3$. It follows that ω in (12) is $\omega = 4$ (if the term c_4 does not vanish for a particular choice of parameters) and $-4c_4(r_d, r_1, r_2) = k_3(r_d, r_1, r_2)$, from which follows

$$c_4(r_d, r_1, r_2) = -\frac{1}{4}k_3(r_d, r_1, r_2).$$

In this way we have proved the following theorem.

Theorem 1. *Let P^{ex} be the exact solution of* (1) *and P^{ap} be the approximation given by* (3) *and* (4)-(7)*. Then*

$$\log P^{ap}(r_d.r_1, r_1, \tau) - \log P^{ex}(r_d.r_1, r_1, \tau) = c_4(r_d.r_1, r_1)\tau^4 + o(\tau^4),$$

where

$$c_4 = -\frac{1}{24}\sigma_d^2\gamma_d\left((2\gamma_d - 1)\sigma_d^2 r_d^{4\gamma_d - 2} + 2r_d^{2\gamma_d - 1}(a_1 + a_2 r_d + a_3 r_1 + a_4 r_2)\right).$$

We note that for $\gamma_1 = \gamma_2 = \gamma_d = 1/2$ and $\rho_{1d} = \rho_{2d} = \rho_{12} = 0$ we get the same expression as in [12] and we remark that the same estimate holds for the CIR-type model $\gamma_1 = \gamma_2 = \gamma_d = 1/2$ without any restrictions on correlation values.

4. A convergence model allowing negative interest rates

We note that the results from the previous section hold also in the case of some γ_i ($i \in \{d, 1, 2\}$) being equal to zero. In such a case, the volatility is constant, which implies a possibility of negative values of the corresponding process. In the case of the processes r_1, r_2 we obtain a well known Ornstein-Uhlenbeck process with normal conditional distribution which has a variance independent of the level of the process. At the time when many of the interest rate models were introduced, the negative interest rates were considered to be impossible and their positive probability was seen as a disadvantage of the Vasicek model of interest rates. However, they were widely observed in the past years, see the historical data for example at [15] for a large set of countries. Therefore it is reasonable to allow the processes of the domestic and European short rates to achieve also negative values.

With this motivation we consider the case when the factor r_1 is modelled by the Ornstein-Uhlenbeck process (i.e., $\gamma_1 = 0$), while $\gamma_2 > 0$. This allows the conditional variance of the European short rate to be nonconstant, since it is a sum of the processes $r_1 + r_2$. In the stochastic differential equation for the domestic short rate we take $\gamma_d = 0$, thus allowing negative domestic rates too. Therefore, we consider the model

$$\begin{aligned}
\mathbf{d}r_d &= (a_1 + a_2 r_d + a_3 r_1 + a_4 r_2)\mathbf{d}t + \sigma_d \mathbf{d}w_d,\\
\mathbf{d}r_1 &= (b_1 + b_2 r_1)\mathbf{d}t + \sigma_1 \mathbf{d}w_1,\\
\mathbf{d}r_2 &= (c_1 + c_2 r_2)\mathbf{d}t + \sigma_2 r_2^{\gamma_2} \mathbf{d}w_2,
\end{aligned}$$

where ρ_{ij} is again the correlation between the increments of the Wiener processes w_i and w_j.

From Theorem 1 we immediately see that in this case the term $c_4(r_d, r_1, r_2)$, i.e., the $O(\tau^4)$ term of the error of the logarithms of the bond prices, vanishes. Returning back to the derivation, using the information that this error (and therefore the function g) is now $O(\tau^5)$ we repeat the same reasoning as before and conclude that also the $O(\tau^4)$ term of the right hand side of (10) comes only from the $h(r_d, r_1, r_2, \tau)$ term. Using the notation of (11) we have

$$k_4(r_d, r_1, r_2) = \frac{1}{16}\gamma_2 \sigma_2 \sigma_d \rho_{2d} a_4 \left[(\gamma_2 - 1)\sigma_2^2 r_2^{3\gamma_2 - 2} + 2 r_2^{\gamma_2 - 1}(c_1 + c_2 r_2) \right].$$

Furthermore, using the new estimate of g we conclude that the only $O(\tau^4)$ term on the left hand side comes from the derivative $\frac{\partial g}{\partial \tau}$. Therefore the term $c_5(r_d, r_1, r_2)$ from (12), which is now the leading term, can be expressed as

$$c_5(r_d, r_1, r_2) = -\frac{1}{5} k_4(r_d, r_1, r_2).$$

Therefore we can state the following theorem:

Theorem 2. *Let P^{ex} be the exact solution of* (1) *with $\gamma_d = 0$ and $\gamma_1 = 0$, and P^{ap} be the approximation given by* (3) *and* (4)-(7). *Then*

$$\log P^{ap}(r_d, r_1, r_2, \tau) - \log P^{ex}(r_d, r_1, r_2, \tau) = c_5(r_d, r_1, r_2)\tau^5 + O(\tau^6),$$

where

$$c_5 = -\frac{1}{80}\gamma_2 \sigma_2 \sigma_d \rho_{2d} a_4 \left[(\gamma_2 - 1)\sigma_2^2 r_2^{3\gamma_2 - 2} + 2 r_2^{\gamma_2 - 1}(c_1 + c_2 r_2) \right].$$

5. Conclusions

We considered a three-factor convergence model of interest rates studied in [12] and extended the result on the accuracy of the approximate analytical solution for the bond-pricing partial differential equation. We derived the order of accuracy in a general case, while before it was known only in a special case which leads to a solution in a separated form. Afterwards, we studied the model with parameters which allow interest rates to become negative, as this situation currently occurs on some markets. The approximation turns out to be more precise by one order.

We stress that the approximation formula is easy to compute and therefore it can be conveniently used in calibration of the model which will be the next step in our research. Since a calibration requires a computation of the bond prices for many parameter values, their quick evaluation is essential. The approximation studied in this paper provides a suitable way and the theorems which we have proved give a justification for its use.

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