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$k-$regular decomposable incidence structure of maximum degree

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Abstract. This paper discusses incidence structures and their rank. The aim of this paper is to prove that there exists a regular decomposable incidence structure $\mathcal{J} = (\mathbb{P}, \mathcal{B})$ of maximum degree depending on the size of the set and a predetermined rank. Furthermore, an algorithm for construction of this structures is given. In the proof of the main result, the points of the set $\mathbb{P}$ are shown by Euler's formula of complex number. Two examples of construction the described incidence structures of maximum degree 6 and maximum degree 30 are given.

1. Introduction

An incidence structure is a triple $(\mathbb{P}, \mathcal{B}, I)$, where $\mathbb{P}$ and $\mathcal{B}$ are two disjoint sets and I is a subset of $\mathbb{P} \times \mathcal{B}$. The elements of $\mathbb{P}$ are called points, the elements of $\mathcal{B}$ are called blocks and $\mathcal{J}$ is called incidence relation. Let $\mathbb{P}_b \subseteq \mathbb{P}$ be a set of points that are incidence with the block b. If the implication $b \neq b' \Rightarrow \mathbb{P}_b \neq \mathbb{P}_{b'}$ holds, the incidence structure is said to be simple. Accordingly, if $\mathcal{B}$ is a family of nonempty subset of $\mathbb{P}\,(\mathcal{B} \subseteq \mathcal{P}(\mathbb{P}) \setminus \{\emptyset\})$ and I defines the following incidence relation: $(P, b) \in I \Leftrightarrow P \in b\,(\forall P \in \mathbb{P}, \forall b \in \mathcal{B})$, then $(\mathbb{P}, \mathcal{B}, I)$ is the simple incidence structure.

In this paper we will talk about the previously described simple incidence structure. For the sake of simplicity, we will denote $\mathcal{J} = (\mathbb{P}, \mathcal{B})$ and we will just call it incidence structure.

The incidence structure $\mathcal{J} = (\mathbb{P}, \mathcal{B})\,(\mathcal{B} \subseteq \mathcal{P}(\mathbb{P}) \setminus \{\emptyset\})$ is called regular of degree k, or $k-$regular if every point of $\mathbb{P}$ is in exactly k blocks.

The rank of the incidence structure $\mathcal{J} = (\mathbb{P}, \mathcal{B})$ is number $d = d(\mathcal{J}) = \max\{|b|, b \in \mathcal{B}\}$. The incidence structure is called uniform if and only if every block of $\mathcal{B}$ has a cardinality $d(\mathcal{J})$. The cardinality of the block $b\,(|b|)$ can also be called the length of the block b.

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A decomposition of an incidence structure $\mathcal{J} = (\mathbb{P}, \mathcal{B})$ is a partition of $\mathbb{P}$ into point classes together with a partition of $\mathcal{B}$ into block classes. If a partition has k subset, it is called $k-$partition.

In [1], [5], [7] and [8], nets, k-nets, seminets, 3-seminets and their corresponding groupoids are investigated.

Analogous to the concept of $k-$seminets, that are introduced by Janez Ušan in [9], a $k-$regular decomposable incidence structure can be defined.

Definition 1. Let $(\mathbb{P}, \mathcal{B})$ be a regular incidence structure and let $\Pi = \{X_1, \ldots, X_k\}, k \in \mathbb{N} \setminus \{1,2\}$ be a partition of set $\mathcal{B}$ such that:

(1a)
(1b)
$$\left(\forall x \in X_i\right)\left(\forall x' \in X_j\right) \begin{cases} \left|x \cap x'\right| = 0, & i = j; \\ \left|x \cap x'\right| \leq 1, & i \neq j. \end{cases}$$

Then, for an ordered pair $\mathcal{J}_k = (\mathbb{P}, \Pi)$ or $\mathcal{J} = (\mathbb{P}, \{X_1, \ldots, X_k\})$ we say that is a $k-$regular decomposable incidence structure.

The elements of the set Π are called classes. It is easy to check that, due to regularity and equation (1a), each of the classes $X_1, \ldots, X_k$ is a partition of the set $\mathbb{P}$.

The incidence structures and partitions of the set are discussed in more detail in the [3] and [4]. Thus it is known that a Stirling number of the second kind

$$\Pi\left(n,l\right) = \frac{1}{l!}\sum_{j=0}^{l}\left(-1\right)^{j}\binom{l}{j}\left(l-j\right)^{n}$$

gives the number of partitions of a set $\mathbb{P}$ with cardinality n and l non-empty subsets, and the Bell number

$$\Pi\left(n\right) = \sum_{l=1}^{n}\pi\left(n,l\right)$$

gives the number of all partitions of set $\mathbb{P}$ with cardinality n [2], [6].

2. Main results

In [3], Galić defines U-k-seminets of maximal degree and shows the existence and construction depending on the set over which one constructs a k-seminets. Further, in [4], it is shown how many U-k-seminets of maximal degree can be constructed over the set for the given t-order.

In this paper, we will show the existence and construction of a regular decomposable incidence structure of maximum degree k depending on the size of the set $\mathbb{P}$ and a predetermined rank d. This is described and proved in such a way that the points of the set $\mathbb{P}$ are shown by Euler's formula of complex number, located on two concentric circles. We talk about this in the following theorem.

Theorem 1. *Let $\mathbb{P}$ be a set of points such that $|\mathbb{P}| = t,\ t \in \{3, 4, \ldots\} \subset \mathbb{N}$. Then for*

$$2 \le d < \frac{t+2}{2}$$

there is a family $\mathcal{B}$ of subsets of $\mathbb{P}$ whose rank is d and there is a partition $\pi = \{X_1, \ldots, X_k\}$ of $\mathcal{B}$ so that $(\mathbb{P}, \mathcal{B})$ is a regular decomposable incidence structure of degree $k \le t - d + 2$.

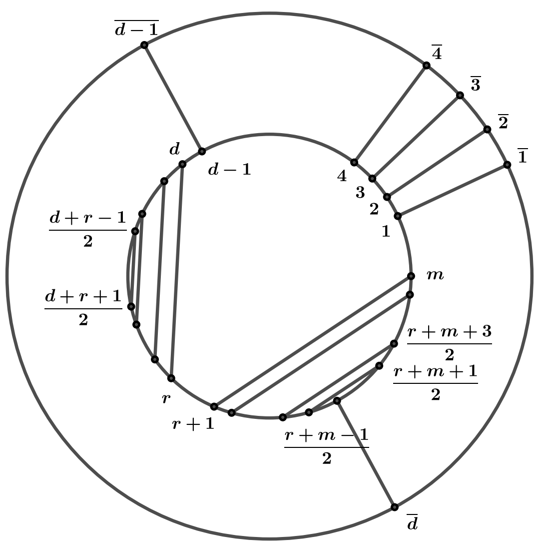
Proof. We will present the proof according to the following cases:

(1) $t = 2r$:
- a) $d = 2j,\quad j \in \mathbb{N}$ and $r = odd$;
- b) $d = 2j,\quad j \in \mathbb{N}$ and $r = even$;
- c) $d = 2j+1,\ j \in \mathbb{N}$ and $r = odd$;
- d) $d = 2j+1,\ j \in \mathbb{N}$ and $r = even$;

(2) $t = 2r+1$:
- a) $d = 2j,\quad j \in \mathbb{N}$ and $r = odd$;
- b) $d = 2j,\quad j \in \mathbb{N}$ and $r = even$;
- c) $d = 2j+1,\ j \in \mathbb{N}$ and $r = odd$;
- d) $d = 2j+1,\ j \in \mathbb{N}$ and $r = even$;

Proof of case (1)a). We will show the points of set $\mathbb{P}$ using the Euler's formula of complex numbers located on two concentric circles. Let $\overline{b}$ be the block with the maximum length $d = |\overline{b}|$ and arrangement of points on the outer circle, and let $\overline{b}^c = \mathbb{P} \setminus \overline{b}$ be the set of points on the inner circle, so that $|\overline{b}^c| = m = t - d$, where the points are arranged as in Figure 1.

Figure 1.



Then the set $\mathbb{P}$ will have the following form:

$$\mathbb{P} = \Big\{2e^{i\frac{2\pi}{m}},\ 2e^{i\frac{2\pi}{m}\cdot 2},\ \ldots,\ 2e^{i\frac{2\pi}{m}\cdot(d-1)},\ 2e^{i\frac{2\pi}{m}\cdot\frac{r+m+1}{2}},\ e^{i\frac{2\pi}{m}},\ e^{i\frac{2\pi}{m}\cdot 2},\ \ldots$$
$$e^{i\frac{2\pi}{m}\cdot(d-1)},\ e^{i\frac{2\pi}{m}\cdot d},\ \ldots,\ e^{i\frac{2\pi}{m}\cdot\frac{d+r-1}{2}},\ e^{i\frac{2\pi}{m}\cdot\frac{d+r+1}{2}},\ \ldots,\ e^{i\frac{2\pi}{m}\cdot r},$$
$$e^{i\frac{2\pi}{m}\cdot(r+1)},\ \ldots,\ e^{i\frac{2\pi}{m}\cdot\frac{r+m-1}{2}},\ e^{i\frac{2\pi}{m}\cdot\frac{r+m+1}{2}},\ e^{i\frac{2\pi}{m}\cdot\frac{r+m+3}{2}},\ \ldots,\ e^{i\frac{2\pi}{m}\cdot m}\Big\}.$$

The partition X_1 of the set $\mathbb{P}$ will consist of blocks composed of pairs of points located at the vertices of the chord or at the vertices of the radius sections connecting the two concentric circles (see Figure 1). It follows that class X_1 consists of the following blocks:

$$X_1 = \Big\{\Big\{e^{i\frac{2\pi}{m}}, 2e^{i\frac{2\pi}{m}}\Big\}, \Big\{e^{i\frac{2\pi}{m}\cdot 2}, 2e^{i\frac{2\pi}{m}\cdot 2}\Big\},\ \ldots,\ \Big\{e^{i\frac{2\pi}{m}\cdot(d-1)},\ 2e^{i\frac{2\pi}{m}\cdot(d-1)}\Big\},$$
$$\Big\{e^{i\frac{2\pi}{m}\cdot\frac{r+m+1}{2}}, 2e^{i\frac{2\pi}{m}\cdot\frac{r+m+1}{2}}\Big\}, \ldots,\ \Big\{e^{i\frac{2\pi}{m}\cdot\frac{d+r-1}{2}}, e^{i\frac{2\pi}{m}\cdot\frac{d+r+1}{2}}\Big\},$$
$$\Big\{e^{i\frac{2\pi}{m}\cdot d},\ e^{i\frac{2\pi}{m}\cdot r}\Big\}, \ldots,\ \Big\{e^{i\frac{2\pi}{m}\cdot\frac{r+m-1}{2}}, e^{i\frac{2\pi}{m}\cdot\frac{r+m+3}{2}}\Big\}, \ldots,\ \Big\{e^{i\frac{2\pi}{m}\cdot(r+1)}, 1\Big\}\Big\}.$$

Partition $X_s,\ s \in \{1, 2, \ldots, m\}$ will be obtained from class X_1 by rotating the complex points of the smaller circle around the center by angle $\alpha = -\dfrac{2\pi}{m}(s-1)$, and then for the blocks we take pairs of points located at the vertices of the tendons and the vertices of the radius sections as in the case of class X_1.

In this way, general class $X_s,\ s \in \{1, 2, \ldots, m\}$ will have the following form:

$$X_s = \Big\{\Big\{e^{i\frac{2\pi}{m}\cdot s}, 2e^{i\frac{2\pi}{m}}\Big\}, \Big\{e^{i\frac{2\pi}{m}\cdot(s+1)}, 2e^{i\frac{2\pi}{m}\cdot 2}\Big\}, \ldots, \Big\{e^{i\frac{2\pi}{m}\cdot(d+s-2)}, 2e^{i\frac{2\pi}{m}\cdot(d-1)}\Big\},$$
$$\Big\{e^{i\frac{2\pi}{m}\cdot\frac{r+m+1}{2}+s-1}, 2e^{i\frac{2\pi}{m}\cdot\frac{r+m+1}{2}}\Big\}, \ldots, \Big\{e^{i\frac{2\pi}{m}\cdot\frac{d+r-1}{2}+s-1}, e^{i\frac{2\pi}{m}\cdot\frac{d+r+1}{2}+s-1}\Big\},$$
$$\Big\{e^{i\frac{2\pi}{m}\cdot(d+s-1)}, e^{i\frac{2\pi}{m}\cdot(r+s-1)}\Big\}, \ldots, \Big\{e^{i\frac{2\pi}{m}\cdot\frac{r+m-1}{2}+s-1}, e^{i\frac{2\pi}{m}\cdot\frac{r+m+3}{2}+s-1}\Big\},$$
$$\ldots, \Big\{e^{i\frac{2\pi}{m}\cdot(r+s)}, e^{i\frac{2\pi}{m}\cdot(m+s-1)}\Big\}\Big\}.$$

Since each pair of points set $\overline{b}^c$ belonging to the same block has a different arc distance, which is $\alpha = \dfrac{2\pi}{m}\cdot l$, where $l \in \{1, 2, \ldots, r-d\}$, this ensures that statements (1)a) and (1)b) hold for partitions $X_1, \ldots, X_m$.

Partition X_{m+1} will consist exclusively of blocks with cardinality one and will have the following form:

$$X_{m+1} = \Big\{\Big\{2e^{i\frac{2\pi}{m}}\Big\}, \ldots, \Big\{e^{i\frac{2\pi}{m}}\Big\}, \ldots, \Big\{e^{i\frac{2\pi}{m}\cdot(m-1)}\Big\}, \{1\}\Big\}.$$

It remains to form partition X_0 which will contain block

$$\overline{b}=\left\{2e^{i\frac{2\pi}{m}},\ldots,2e^{i\frac{2\pi}{m}\cdot(d-1)},2e^{i\frac{2\pi}{m}\cdot\frac{r+m+1}{2}}\right\}$$

with the maximum number of elements. We will arrange the remaining elements from $\overline{b}^c$ $\left(|\overline{b}^c|=2\,(r-j)\right)$ into blocks with cardinality two, so that the difference between the elements in the same block is $r-j$. Since $r-j\leq r-d$, then it follows that blocks from partition X_0 cannot have two elements in common with any block from classes $X_1,\ldots,X_m,X_{m+1}$. Partition X_0 will have the following form:

$$X_0=\left\{\overline{b},\left\{e^{i\frac{2\pi}{m}},e^{i\frac{2\pi}{m}\cdot\frac{m+2}{2}}\right\},\ldots,\left\{e^{i\frac{2\pi}{m}\cdot\frac{m-2}{2}},e^{i\frac{2\pi}{m}\cdot(m-1)}\right\},\left\{e^{i\pi},e^{i2\pi}\right\}\right\}.$$

Thus, $m+2=t-d+2$ partitions $X_0,X_1,\ldots,X_m,X_{m+1}$ were formed that satisfy conditions (1a) and (1b). If we put $\mathcal{B}=X_0\cup\cdots\cup X_{m+1}$, then we get that $\mathcal{J}_k=(\mathbb{P},\mathcal{B})$, i.e., $\mathcal{J}_k=(\mathbb{P},\{X_0,X_1,\ldots,X_m,X_{m+1}\})$ is a $k-$regular decomposable incidence structure of rank d.

Case (1)b): Let $d=2j,\ j\in\mathbb{N}$ and r is even. In this case, the partitions $X_1,\ldots,X_s,\ldots X_m$ will be formed according to the same methodology as in case (1)a). Now, the general class $X_s,\ s\in\mathbb{N}$ will have the following form:

$$\begin{aligned}X_s=\Bigg\{&\left\{e^{i\frac{2\pi}{m}\cdot s},2e^{i\frac{2\pi}{m}}\right\},\left\{e^{i\frac{2\pi}{m}\cdot(s+1)},2e^{i\frac{2\pi}{m}\cdot 2}\right\},\ldots,\left\{e^{i\frac{2\pi}{m}\cdot(d+s-2)},2e^{i\frac{2\pi}{m}\cdot(d-1)}\right\},\\&\left\{e^{i\frac{2\pi}{m}\cdot\frac{d+r}{2}+s-1},e^{i\frac{2\pi}{m}\cdot\frac{d+r}{2}}\right\},\left\{e^{i\frac{2\pi}{m}\cdot\frac{d+r-2}{2}+s-1},e^{i\frac{2\pi}{m}\cdot\frac{d+r+2}{2}+s-1}\right\},\\&\ldots,\left\{e^{i\frac{2\pi}{m}\cdot(d+s-1)},\ e^{i\frac{2\pi}{m}\cdot(r+s-1)}\right\},\ldots,\\&\left\{e^{i\frac{2\pi}{m}\cdot\frac{r+m}{2}+s-1},\ e^{i\frac{2\pi}{m}\cdot\frac{r+m+2}{2}+s-1}\right\},\ldots,\left\{e^{i\frac{2\pi}{m}(r+s)},\ e^{i\frac{2\pi}{m}(m+s-1)}\right\}\Bigg\}.\end{aligned}$$

The remaining two classes X_0,X_{m+1} will have the same form as in case (1)a). With the same conclusion as in case (1)a) it follows that $\mathcal{J}_k=(\mathbb{P},\{X_0,X_1\ldots,X_{m+1}\})$ is a $k-$regular decomposable incidence structure of rank d.

Case (1)c): Let $d=2j+1,\ j\in\mathbb{N}$, r is odd. In this case the partitions $X_1,\ldots,X_m$ will have exactly the same form as in case (1)b). However, in partition X_0, which contains block $\overline{b}=\left\{2e^{i\frac{2\pi}{m}},\ldots,2e^{i\frac{2\pi}{m}\cdot(d-1)},2e^{i\frac{2\pi}{m}\cdot\frac{d+r}{2}}\right\}$ with the maximum length, except the blocks with cardinality two, due to $m=2\,(r-j)-1$ which is odd, one block will be with cardinality one. We will form blocks with cardinality two so that the arc difference between the elements from the same block is $\alpha=\dfrac{2\pi}{m}\cdot\dfrac{m-1}{2}>\dfrac{2\pi}{m}\,(r-d)$.

The partition X_0 will have the following form:

$$X_0 = \left\{ \overline{b},\ \left\{ e^{i\frac{2\pi}{m}},\ e^{i\frac{2\pi}{m}\cdot\frac{m+1}{2}} \right\},\ \left\{ e^{i\frac{2\pi}{m}\cdot 2},\ e^{i\frac{2\pi}{m}\cdot\frac{m+3}{2}} \right\}, \right.$$
$$\left. \ldots,\ \left\{ e^{i\frac{2\pi}{m}\cdot\frac{m-1}{2}},\ e^{i\frac{2\pi}{m}\cdot(m-1)} \right\},\ \left\{ e^{i2\pi} \right\} \right\}.$$

We will form partition X_{m+1} out of blocks with cardinality one, except that we will form one block with cardinality two containing points $e^{i2\pi}$ and $e^{i\frac{2\pi}{m}\cdot\frac{m+1}{2}}$, because $m - \dfrac{m+1}{2} = \dfrac{m-1}{2} > r-d$, so class X_{m+1} will have the following form:

$$X_{m+1} = \left\{ \left\{ 2e^{i\frac{2\pi}{m}} \right\},\ \ldots,\ \left\{ e^{i\frac{2\pi}{m}} \right\},\ \ldots,\ \left\{ e^{i\frac{2\pi}{m}\cdot\frac{m-1}{2}} \right\},\ \left\{ e^{i\frac{2\pi}{m}\cdot\frac{m+3}{2}} \right\}, \right.$$
$$\left. \ldots,\ \left\{ e^{i\frac{2\pi}{m}\cdot(m-1)} \right\}, \left\{ e^{i2\pi},\ e^{i\frac{2\pi}{m}\cdot\frac{m+1}{2}} \right\} \right\}.$$

Similarly as in the previous cases, we come to a conclusion that in this case too $\mathcal{J}_k = (\mathbb{P}, \{X_0, X_1 \ldots, X_{m+1}\})$ is a $k-$regular decomposable incidence structure.

Case (1)d): Let $d = 2j+1,\ j \in \mathbb{N}$ *and* r is even. In this case the partitions $X_1, \ldots, X_m$ will have exactly the same form as in case (1)a). Partitions X_0 and X_{m+1} will have the same form as in case (2)a). It follows that in this case, the statement of the theorem holds.

Proof of case (2): The proof for case (2), i.e., when $t = 2r+1$, is carried out in an analogous way as for case (1). Therefore, we omit this proof. $\square$

Theorem 2. *Let $\mathbb{P}$ be the set of points such that $|\mathbb{P}| = t,\ t \in \{2, 3, \ldots\} \subset \mathbb{N}$. Then for*

$$2 \leq d < \frac{t+2}{2}$$

there is no regular decomposable incidence structure $\mathcal{J}_k = (\mathbb{P}, \{X_1 \ldots, X_k\})$ of rank d, so that $k > t-d+2$.

Proof. We will carry out the proof by determining the maximum number of blocks, that can be formed so that they contain an arbitrary point from the set $\mathbb{P}$, and which can be distributed in different partitions so that requirements (1a) and (1b) are met.

Without the loss of generality, we can carry out the consideration for an arbitrary point $\overline{r}$ from block $\overline{b} = \left\{ \overline{1}, \overline{2}, \ldots, \overline{r}, \overline{d-1}, \overline{d} \right\}$, whose rank is d. Out of points of the block $\overline{b}$, we can form only block $\{\overline{r}\}$ with cardinality one, because if we were to form a block with more than one point, then we would come into contradiction with the requirement (1b).

It remains to check the possibility of forming the maximum number of blocks that contain point $\overline{r}$ and one of the points from the set $\mathbb{P} \setminus \overline{b} = \overline{b}^c = \{1, 2, \ldots, m\}$, but in such a way that satisfies condition (1b).

The previous requirement is equivalent to the requirement that among the partitions of the set $\overline{b}^c$, the partition that has the most subsets is chosen, and then each of these subsets is expanded with the element $\overline{r}$. The required partition of the set $\overline{b}^c$ is the set of all subsets with cardinality one. This means that the maximum number of blocks that meet condition (1b), and contain point $\overline{r}$ and one of the points from set $\overline{b}^c$, has m. Such blocks have the form $\{\overline{r}, 1\}, \{\overline{r}, 2\}, \dots, \{\overline{r}, m\}$, which together with blocks $\overline{b} = \{\overline{1}, \overline{2}, \dots, \overline{d}\}$ and $\{\overline{r}\}$ give a total of $m+2$ blocks, satisfying condition (1b). Since in each partition, there must be a block containing the point $\overline{r}$, it follows that at most $m+2 = t-d+2$ partitions can be formed. Therefore, there is no regular decomposable incidence structure of rank d and degree $k > t-d+2$. $\square$

Now, after the proof of Theorem 1 and Theorem 2, it makes sense to state the following definition:

For a η-regular decomposable incidence structure $\mathcal{J}_\eta = (\mathbb{P}, \{X_1 \dots, X_\eta\})$ we say that it is a maximal regular decomposable incidence structure over the set $\mathbb{P}$ if and only if $\eta \geq k$ holds for every $k-$regular decomposable incidence structure over the set $\mathbb{P}$.

Based on the previous definition, and theorems 1 and 2, we directly obtain that the following statement is valid.

Corollary 1. *Let $\mathbb{P}$ be the set of points such that $|\mathbb{P}| = t \geq 2$. Then, for*

$$2 \leq d < \frac{t+2}{2}$$

there exists a maximal η-regular decomposable incidence structure of rank d over the set $\mathbb{P}$, such that $\eta = t-d+2$.

Example 1. Let $\mathbb{P}$ be a set of 6 points. According to Corollary 1, for $d = 2$, there exists a regular decomposable incidence structure of degree $k = t-d+2 = 6-2+2 = 6$. Let's construct that structure in accordance with the notation used in the paper. We will place the points of the longest block $\overline{b}$ on the outer of the two concentric circles, and the points from its complement on the inner circle. At the same time, let $\overline{b} = \{2e^{i\frac{\pi}{2}}, 2e^{i\pi}\}$, $\overline{b}^c = \{e^{i\frac{\pi}{2}}, e^{i\pi}, e^{i\frac{3\pi}{2}}, e^{i2\pi}\}$. So, set $\mathbb{P} = \{2e^{i\frac{\pi}{2}}, \dots, e^{i2\pi}\}$.

Figure 2 shows how we form the set

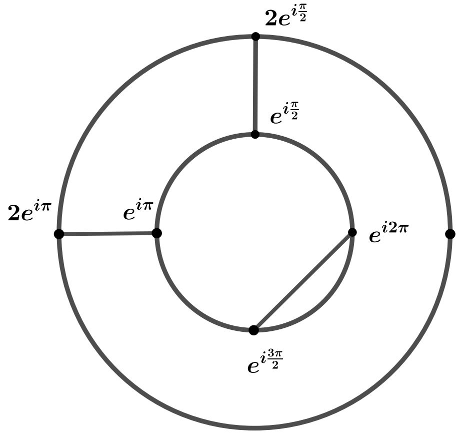
$$X_1 = \left\{\left\{2e^{i\frac{\pi}{2}}, e^{i\frac{\pi}{2}}\right\}, \left\{2e^{i\pi}, e^{i\pi}\right\}, \left\{e^{i\frac{3\pi}{2}}, e^{i2\pi}\right\}\right\}.$$

By rotating the inner circle three times by the angle $-\dfrac{\pi}{2}$, we get the following sets:

$$X_2 = \left\{\left\{2e^{i\frac{\pi}{2}}, e^{i\pi}\right\}, \left\{2e^{i\pi}, e^{i\frac{3\pi}{2}}\right\}, \left\{e^{i\frac{\pi}{2}}, e^{i2\pi}\right\}\right\},$$

$$X_3 = \left\{\left\{2e^{i\frac{\pi}{2}}, e^{i\frac{3\pi}{2}}\right\}, \left\{2e^{i\pi}, e^{i2\pi}\right\}, \left\{e^{i\frac{\pi}{2}}, e^{i\pi}\right\}\right\},$$

FIGURE 2.



$$X_4 = \left\{\left\{2e^{i\frac{\pi}{2}}, e^{i2\pi}\right\}, \left\{2e^{i\pi}, e^{i\frac{\pi}{2}}\right\}, \left\{e^{i\pi}, e^{i\frac{3\pi}{2}}\right\}\right\}.$$

According to Theorem 1, set X_5 consists of one-membered subsets:

$$X_5 = \left\{\left\{2e^{i\frac{\pi}{2}}\right\}, \left\{2e^{i\pi}\right\}, \left\{e^{i\frac{\pi}{2}}\right\}, \left\{e^{i\pi}\right\}, \left\{e^{i\frac{3\pi}{2}}\right\}, \left\{e^{i2\pi}\right\}\right\}.$$

Also,

$$X_0 = \left\{\left\{2e^{i\frac{\pi}{2}}, 2e^{i\pi}\right\}, \left\{e^{i\frac{\pi}{2}}, e^{i\pi}\right\}, \left\{e^{i\frac{3\pi}{2}}, e^{i2\pi}\right\}\right\}.$$

A set $\Pi = \{X_0, X_1, X_2, X_3, X_4, X_5\}$ is a partition of the set

$$\begin{aligned}\mathcal{B} = \Big\{&\left\{2e^{i\frac{\pi}{2}}\right\}, \left\{2e^{i\pi}\right\}, \left\{e^{i\frac{\pi}{2}}\right\}, \left\{e^{i\pi}\right\}, \left\{e^{i\frac{3\pi}{2}}\right\}, \left\{e^{i2\pi}\right\}, \left\{2e^{i\frac{\pi}{2}}, 2e^{i\pi}\right\},\\ &\left\{2e^{i\frac{\pi}{2}}, e^{i\frac{\pi}{2}}\right\}, \left\{2e^{i\frac{\pi}{2}}, e^{i\pi}\right\}, \left\{2e^{i\frac{\pi}{2}}, e^{i\frac{3\pi}{2}}\right\}, \left\{2e^{i\frac{\pi}{2}}, e^{i2\pi}\right\}, \left\{2e^{i\pi}, e^{i\frac{\pi}{2}}\right\},\\ &\left\{2e^{i\pi}, e^{i\pi}\right\}, \left\{2e^{i\pi}, e^{i\frac{3\pi}{2}}\right\}, \left\{2e^{i\pi}, e^{i2\pi}\right\}, \left\{e^{i\frac{\pi}{2}}, e^{i\pi}\right\}, \left\{e^{i\frac{\pi}{2}}, e^{i\frac{3\pi}{2}}\right\},\\ &\left\{e^{i\frac{\pi}{2}}, e^{i2\pi}\right\}, \left\{e^{i\pi}, e^{i\frac{3\pi}{2}}\right\}, \left\{e^{i\pi}, e^{i2\pi}\right\}, \left\{e^{i\frac{3\pi}{2}}, e^{i2\pi}\right\}\Big\}\end{aligned}$$

and satisfies the conditions of the Definition 1. Therefore $(\mathbb{P}, \Pi)$ is 6-regular decomposable incidence structure with rank 2.

Note. We notice that each element of the set $\mathbb{P}$ is located in exactly 6 blocks, which is a condition of the regularity of the structure $(\mathbb{P}, \Pi)$.

Example 2. Let $\mathbb{P}$ be a set with 34 points. A k-regular decomposable incidence structure of maximum rank $d = 6$ should be constructed. Since we have $r = 17$ due to $|\mathbb{P}| = 34 = 2r$, the requirement $2 \leq d \leq r$ is fulfilled.

Then, according to the theorem it follows that a $k-$regular decomposable incidence structure of maximum degree $k = t - d + 2 = 30$ can be constructed. Because d is even and r is odd, the computer program was created according to the scheme in the proof of the Theorem 1 for case (1)a), and the required incidence structure $\mathcal{J}_{30} = (\mathbb{P}, \{X_0, X_1 \dots, X_{29}\})$ will have the following partitions:

$$X_0 = \Big\{\{1,2,3,4,5,6\},\{7,21\},\{8,22\},\{9,23\},\{10,24\},\{19,33\},\{20,34\}\Big\}.$$

$$\begin{aligned} X_1 = \Big\{&\{1,7\},\{2,8\},\{3,9\},\{4,10\},\{5,11\},\{6,29\},\{12,23\},\\ &\{13,22\},\{14,21\},\{15,20\},\{16,19\},\{17,18\},\\ &\{24,34\},\{25,33\},\{26,32\},\{27,31\},\{28,30\}\Big\}. \end{aligned}$$

$$\begin{aligned} X_2 = \Big\{&\{1,8\},\{2,9\},\{3,10\},\{4,11\},\{5,12\},\{6,30\},\{13,24\},\\ &\{14,23\},\{15,22\},\{16,21\},\{17,20\},\{18,19\},\\ &\{25,7\},\{26,34\},\{27,33\},\{28,32\},\{29,31\}\Big\}. \end{aligned}$$

$$\vdots$$

$$\begin{aligned} X_{27} = \Big\{&\{1,33\},\{2,34\},\{3,7\},\{4,8\},\{5,9\},\{6,27\},\{10,21\},\\ &\{11,20\},\{12,19\},\{13,18\},\{14,17\},\{15,16\},\{22,32\},\\ &\{23,31\},\{24,30\},\{25,29\},\{26,28\}\Big\}. \end{aligned}$$

$$\begin{aligned} X_{28} = \Big\{&\{1,34\},\{2,7\},\{3,8\},\{4,9\},\{5,10\},\{6,28\},\{11,22\},\\ &\{12,21\},\{13,20\},\{14,19\},\{15,18\},\{16,17\},\\ &\{23,33\},\{24,32\},\{25,31\},\{26,30\},\{27,29\}\Big\}. \end{aligned}$$

$$\begin{aligned} X_{29} = \Big\{&\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\},\{11\},\{12\},\{13\},\\ &\{14\},\{15\},\{16\},\{17\},\{18\},\{19\},\{20\},\{21\},\{22\},\{23\},\{24\},\\ &\{25\},\{26\},\{27\},\{28\},\{29\},\{30\},\{31\},\{32\},\{33\},\{34\}\Big\}. \end{aligned}$$

3. Conclusion

Many researchers have worked on incidence structures in different branches of mathematics. In this paper, we investigate the degree of regularity of the incidence structure. The points are represented by the Euler's formula of complex number, which enables a more detailed representation of the construction of the partition of the incidence structure, considering which rank of the regular decomposable incidence structure will be maximal. We gave an algorithm for determining k-regular incidence structures of a given rank and showed the construction on two examples.

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