

## New fixed figure results with the notion of $k$ -ellipse

HÜLYA AYTİMUR, ŞABAN GÜVENÇ, NİHAL TAŞ

ABSTRACT. In this paper, as a geometric approach to the fixed-point theory, we prove new fixed-figure results using the notion of  $k$ -ellipse on a metric space. For this purpose, we are inspired by the Caristi type mapping, Kannan type contraction, Chatterjea type contraction and Ćirić type contraction. After that, we give some existence and uniqueness theorems of a fixed  $k$ -ellipse. We also support our obtained results with illustrative examples. Finally, we present a new application to the  $S$ -Shaped Rectified Linear Activation Unit ( $SReLU$ ) to show the importance of our theoretical results.

### 1. INTRODUCTION AND BACKGROUND

Recently, the geometry of the fixed-point set  $Fix(T) = \{x \in X : Tx = x\}$  of a self-mapping  $T : X \rightarrow X$  has been studied as a new approach to the fixed-point theory. This approach began with the “fixed-circle problem” in [13]. This problem gains the importance since the self-mapping has non-unique fixed points and the set of non-unique fixed points includes some geometric figures such as circle, disc, ellipse etc. For example, let us consider the usual metric space  $(\mathbb{R}, d)$  and the self-mapping  $T : \mathbb{R} \rightarrow \mathbb{R}$  defined as

$$Tx = \begin{cases} x, & x \geq -1, \\ -x, & x < -1, \end{cases}$$

for all  $x \in \mathbb{R}$ . Then we have

$$C_{0,1} = \{x \in \mathbb{R} : |x| = 1\} = \{-1, 1\} \subset Fix(T) = [-1, \infty),$$

that is, the fixed-point set  $Fix(T)$  includes the unit circle. Similarly, we get

$$D_{0,1} = \{x \in \mathbb{R} : |x| \leq 1\} = [-1, 1] \subset Fix(T) = [-1, \infty),$$

that is, the fixed-point set  $Fix(T)$  includes the unit disc.

The notion of a fixed figure was defined as a generalization of the notions of a fixed circle as follows:

---

2020 *Mathematics Subject Classification*. Primary: 54H25; Secondary: 47H09; 47H10.

*Key words and phrases*. Fixed figure, fixed  $k$ -ellipse, metric space, activation function.

*Full paper*. Received 4 November 2022, accepted 26 December 2022, available online 24 January 2022.

A geometric figure  $\mathcal{F}$  (a circle, an ellipse, a hyperbola, etc.) contained in the fixed point set  $Fix(T)$  is called a *fixed figure* (a fixed circle, a fixed ellipse, a fixed hyperbola, etc.) of the self-mapping  $T$  (see [16] and the references therein). For example, some fixed-figure theorems were obtained using different techniques (see [5, 8, 16, 17] for more details).

A  $k$ -ellipse is the locus of points of the plane whose sum of distances to the  $k$  foci is a constant  $d$ . The 1-ellipse is the circle, and the 2-ellipse is the classic ellipse.  $k$ -ellipses can be considered as generalizations of ellipses. These special curves allow more than two foci [11].  $k$ -ellipses have many names such as:  $n$ -ellipse [15], multifocal ellipse [6], polyellipse [10] and eglipse [14]. It is noteworthy to mention that these curves were first studied by Scottish mathematician and scientist James Clerk Maxwell in 1846.

From the above reasons, in this paper, we investigate new solutions to the fixed-figure problem. Therefore, we use some known contractive conditions such as Caristi type mapping, Kannan type contraction, Chatterjea type contraction and Ćirić type contraction to obtain some existence and uniqueness theorems of a fixed  $k$ -ellipse. That is, to increase the number of fixed points of a self-mapping, we modify the known contraction mappings, which are used for the existence of unique fixed point, with the radius and the foci of a  $k$ -ellipse. Also, we give necessary examples to support our obtained results. Finally, we present a new application to the  $S$ -Shaped Rectified Linear Activation Unit ( $SReLU$ ) to show the importance of our theoretical results.

## 2. MAIN RESULTS

In this section, we give the definition of a  $k$ -ellipse with some examples and prove new fixed-figure theorems on metric spaces. Also, we investigate some properties of the obtained theorems with necessary examples.

**Definition 1.** Let  $(X, d)$  be a metric space. The  $k$ -ellipse is defined by

$$E[x_1, \dots, x_k; r] = \left\{ x \in X : \sum_{i=1}^k d(x, x_i) = r \right\}.$$

Clearly, if  $k = 2$  then we get an ellipse and if  $k = 1$  then we get a circle on a metric space with the same radius (see [15] for more details).

Now, we give the following examples of  $k$ -ellipses.

**Example 1.** Let  $(X = \mathbb{R}^2, d)$  be a metric space with the metric  $d : X \times X \rightarrow \mathbb{R}$  defined as

$$d(a, b) = |u_1 - u_2| + |v_1 - v_2|,$$

such that  $a = (u_1, v_1), b = (u_2, v_2) \in X$ . Let us define 3-ellipse for  $x_1 = (1, 0), x_2 = (0, 0), x_3 = (0, 1)$  as follows (see Figure 1):

$$E[x_1, x_2, x_3; r] = \{p(x, y) \in X : |x - 1| + |y| + |x| + |y| + |x| + |y - 1| = r\}.$$

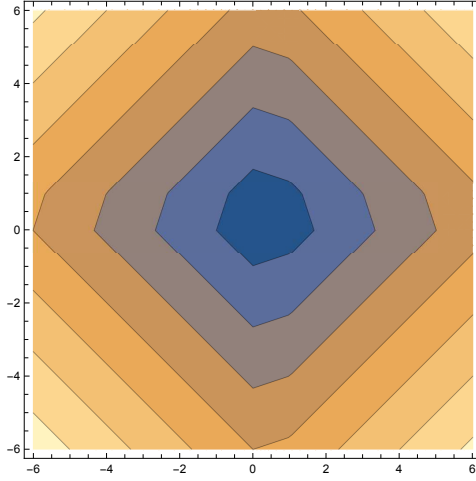


FIGURE 1. The 3-ellipse for  $x_1 = (1, 0)$ ,  $x_2 = (0, 0)$ ,  $x_3 = (0, 1)$ .

**Example 2.** Let  $(X = \mathbb{R}^2, d)$  be a metric space with the metric  $d : X \times X \rightarrow \mathbb{R}$  defined as

$$d(a, b) = \sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2},$$

such that  $a = (u_1, v_1)$ ,  $b = (u_2, v_2) \in X$ . Let us define 3-ellipse for  $x_1 = (3, 0)$ ,  $x_2 = (0, 0)$ ,  $x_3 = (0, 4)$  as follows (see Figure 2):

$$E[x_1, x_2, x_3; r] = \left\{ p(x, y) \in X : \sqrt{(x-3)^2 + y^2} + \sqrt{x^2 + y^2} + \sqrt{x^2 + (y-4)^2} = r \right\}.$$

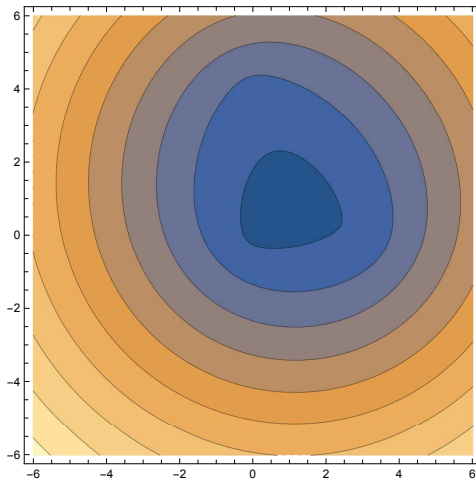


FIGURE 2. The 3-ellipse for  $x_1 = (3, 0)$ ,  $x_2 = (0, 0)$ ,  $x_3 = (0, 4)$ .

**Example 3.** Let  $(X = \mathbb{R}^2, d)$  be a metric space with the metric  $d : X \times X \rightarrow \mathbb{R}$  defined as

$$d(a, b) = \max \{|u_1 - u_2|, |v_1 - v_2|\},$$

such that  $a = (u_1, v_1), b = (u_2, v_2) \in X$ . Let us define 3-ellipse for  $x_1 = (1, 0), x_2 = (0, 0), x_3 = (0, 1)$  as follows (see Figure 3):

$$E[x_1, x_2, x_3; r] = \{p(x, y) \in X : \max\{|x-1|, |y|\} + \max\{|x|, |y|\} + \max\{|x|, |y-1|\} = r\}.$$

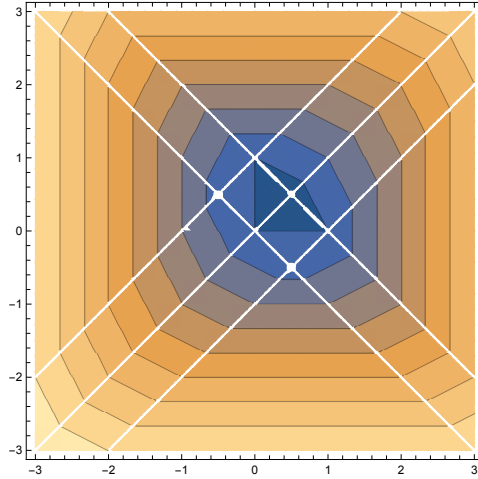


FIGURE 3. The 3-ellipse for  $x_1 = (1, 0), x_2 = (0, 0), x_3 = (0, 1)$ .

**Example 4.** Let  $(X = \mathbb{R}^3, d)$  be a metric space with the metric  $d : X \times X \rightarrow \mathbb{R}$  defined as

$$d(a, b) = \sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2 + (z_1 - z_2)^2},$$

such that  $a = (u_1, v_1, z_1), b = (u_2, v_2, z_2) \in X$ . Let us define 3-ellipse for  $x_1 = (5, 0, 0), x_2 = (0, 2, 0), x_3 = (0, 0, 1)$  as follows (see Figure 4):

$$E[x_1, x_2, x_3; r] = \left\{ p(x, y, z) \in X : \sqrt{(x-5)^2 + y^2 + z^2} + \sqrt{x^2 + (y-2)^2 + z^2} + \sqrt{x^2 + y^2 + (z-1)^2} = r \right\}.$$

**Example 5.** Let  $(X = \mathbb{R}^3, d)$  be a metric space with the metric  $d : X \times X \rightarrow \mathbb{R}$  defined as

$$d(a, b) = \sqrt[4]{(u_1 - u_2)^4 + (v_1 - v_2)^4 + (z_1 - z_2)^4},$$

such that  $a = (u_1, v_1, z_1), b = (u_2, v_2, z_2) \in X$ . Let us define 3-ellipse for  $x_1 = (-1, 0, 0), x_2 = (1, 0, 0), x_3 = (0, 1, 0)$  as follows (see Figure 5):

$$E[x_1, x_2, x_3; r] = \left\{ p(x, y, z) \in X : \sqrt[4]{(x+1)^4 + y^4 + z^4} + \sqrt[4]{(x-1)^4 + y^4 + z^4} + \sqrt[4]{x^4 + (y-1)^4 + z^4} = r \right\}.$$

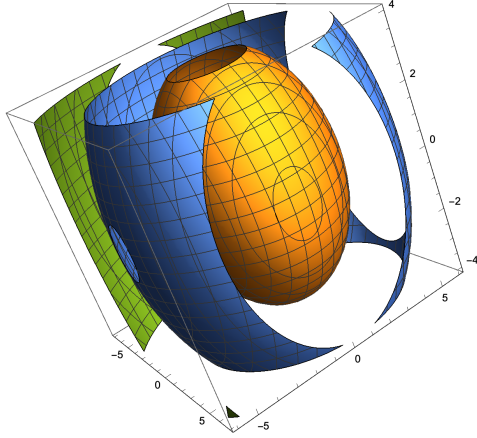


FIGURE 4. The 3-ellipse for  $x_1 = (5, 0, 0)$ ,  $x_2 = (0, 2, 0)$ ,  $x_3 = (0, 0, 1)$ .

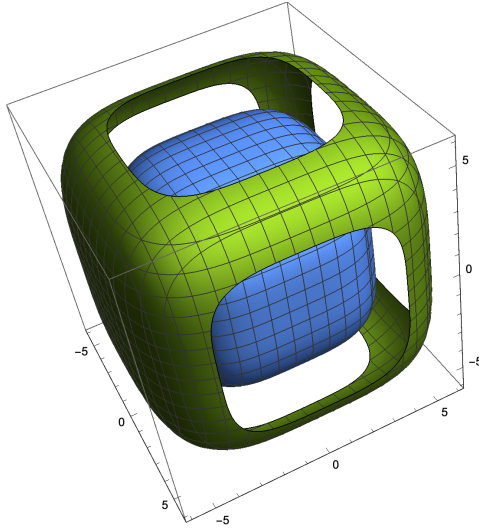


FIGURE 5. The 3-ellipse for  $x_1 = (-1, 0, 0)$ ,  $x_2 = (1, 0, 0)$ ,  $x_3 = (0, 1, 0)$ .

**Example 6.** Let  $(X = \mathbb{R}^2, d)$  be a metric space with the metric  $d : X \times X \rightarrow \mathbb{R}$  defined as

$$d(a, b) = \sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2},$$

such that  $a = (u_1, v_1), b = (u_2, v_2) \in X$ . Let us define 4-ellipse for  $x_1 = (2, 0)$ ,  $x_2 = (0, 0)$ ,  $x_3 = (0, 3)$ ,  $x_4 = (-2, 0)$  as follows (see Figure 6):

$$E[x_1, x_2, x_3, x_4; r] = \left\{ p(x, y) \in X : \sqrt{(x-2)^2 + y^2} + \sqrt{x^2 + y^2} + \sqrt{x^2 + (y-3)^2} + \sqrt{(x+2)^2 + y^2} = r \right\}.$$

We begin with the following proposition.

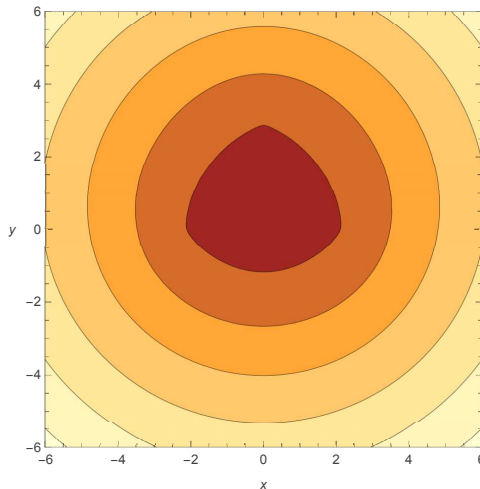


FIGURE 6. The 4-ellipse for  $x_1 = (2, 0)$ ,  $x_2 = (0, 0)$ ,  $x_3 = (0, 3)$ ,  $x_4 = (-2, 0)$ .

**Proposition 1.** *Let  $(X, d)$  be a metric space and  $E[x_1, \dots, x_k; r]$ ,  $E[x'_1, \dots, x'_k; r']$  two  $k$ -ellipses. Then there exists at least one of self-mapping  $T : X \rightarrow X$  such that  $T$  fixes the  $k$ -ellipses  $E[x_1, \dots, x_k; r]$  and  $E[x'_1, \dots, x'_k; r']$ .*

*Proof.* Let  $E[x_1, \dots, x_k; r]$  and  $E[x'_1, \dots, x'_k; r']$  be any  $k$ -ellipses on  $X$ . Let us define the self-mapping  $T : X \rightarrow X$  as

$$Tx = \begin{cases} x, & x \in E \cup E', \\ \alpha, & \text{otherwise,} \end{cases}$$

for all  $x \in X$ , where  $\alpha$  is a constant such that  $\sum_{i=1}^k d(x, x_i) \neq r$  and  $\sum_{i=1}^k d(x, x'_i) \neq r'$ . It can be easily seen that  $x \in \text{Fix}(T)$  for all  $x \in E \cup E'$ , that is,  $T$  fixes the  $k$ -ellipses both  $E$  and  $E'$ .  $\square$

**Remark 1.**

- 1) Proposition 1 generalizes Proposition 3.1 given in [13] and Proposition 4 given in [8].
- 2) Proposition 1 can be extended as follows: "Let  $(X, d)$  be a metric space and  $E[x_1, \dots, x_k; r], \dots, E[x_1^n, \dots, x_k^n; r^n]$  any  $k$ -ellipses. Then there exists at least one self-mapping  $T : X \rightarrow X$  such that  $T$  fixes the  $k$ -ellipses  $E[x_1, \dots, x_k; r], \dots, E[x_1^n, \dots, x_k^n; r^n]$ ."

From the above reasons, it is important to investigate the existence and uniqueness conditions of the fixed  $k$ -ellipses.

Now, we give some theorems about the existence and uniqueness conditions of the fixed  $k$ -ellipses.

**Theorem 1.** Let  $(X, d)$  be a metric space and  $E[x_1, \dots, x_k; r]$  any  $k$ -ellipse on  $X$ . Let us define the mapping  $\xi : X \rightarrow [0, \infty)$  as

$$\xi(x) = \sum_{i=1}^k d(x, x_i),$$

for all  $x \in X$ . If there exists a self-mapping  $T : X \rightarrow X$  such that

$$(E_k1) \quad d(x, Tx) \leq \xi(x) - \xi(Tx), \text{ for each } x \in E[x_1, \dots, x_k; r],$$

$$(E_k2) \quad \sum_{i=1}^k d(Tx, x_i) \geq r, \text{ for each } x \in E[x_1, \dots, x_k; r],$$

$$(E_k3) \quad d(Tx, Ty) \leq h[d(Tx, x) + d(Ty, y)], \text{ for each } x \in E[x_1, \dots, x_k; r], \\ y \in X - E[x_1, \dots, x_k; r] \text{ and some } h \in (0, \frac{1}{2}),$$

then  $E[x_1, \dots, x_k; r]$  is a unique fixed  $k$ -ellipse of  $T$ .

*Proof.* At first, we prove the existence of a fixed  $k$ -ellipse of  $T$ . Let  $x \in E[x_1, \dots, x_k; r]$  be any point. Using the conditions  $(E_k1)$ ,  $(E_k2)$  and the definition of  $\xi$ , we get

$$\begin{aligned} d(x, Tx) &\leq \xi(x) - \xi(Tx) = \sum_{i=1}^k d(x, x_i) - \sum_{i=1}^k d(Tx, x_i) \\ &= r - \sum_{i=1}^k d(Tx, x_i) \leq r - r = 0, \end{aligned}$$

that is,  $x \in \text{Fix}(T) = \{x \in X : x = Tx\}$ . Hence,  $E[x_1, \dots, x_k; r]$  is a fixed  $k$ -ellipse of  $T$ .

Now, we show that  $E[x_1, \dots, x_k; r]$  is a unique fixed  $k$ -ellipse of  $T$ . To do this, on the contrary, we suppose that  $E[x'_1, \dots, x'_k; r']$  is another fixed  $k$ -ellipse of  $T$ . Let  $x \in E[x_1, \dots, x_k; r]$  and  $y \in E[x'_1, \dots, x'_k; r']$  such that  $x \neq y$ . Then using the condition  $(E_k3)$ , we obtain

$$d(Tx, Ty) = d(x, y) \leq h[d(Tx, x) + d(Ty, y)] = h[d(x, x) + d(y, y)] = 0,$$

a contradiction with  $x \neq y$ . It should be  $x = y$ . Consequently  $E[x_1, \dots, x_k; r]$  is a unique fixed  $k$ -ellipse of  $T$ .  $\square$

If we consider the above theorem, we obtain the following remark.

**Remark 2.**

- 1) If  $k = 1$ , then we have  $E[x_1; r] = C_{x_1, r}$  and so, by Theorem 2.1 given in [13],  $E[x_1; r]$  is a fixed 1-ellipse of  $T$  or  $C_{x_1, r}$  is a fixed circle of  $T$ .
- 2) If  $k = 2$ , then we have  $E[x_1, x_2; r] = E_r(x_1, x_2)$  and so, by Theorem 1 given in [8],  $E[x_1, x_2; r]$  is a fixed 2-ellipse of  $T$  or  $E_r(x_1, x_2)$  is a fixed ellipse of  $T$ .
- 3) The condition  $(E_k3)$  can be changed with a proper contractive condition such as

$$d(Tx, Ty) \leq hd(x, y),$$

where  $h \in (0, 1)$ . This contraction can be considered as Banach type contractive condition.

- 4) If we pay attention, the condition  $(E_k1)$  can be considered as Caristi type condition [1] and the condition  $(E_k3)$  can be considered as Kannan type contractive condition [9].
- 5) The condition  $(E_k1)$  guarantees that  $Tx$  is not in the exterior of the  $k$ -ellipse  $E[x_1, \dots, x_k; r]$  for each  $x \in E[x_1, \dots, x_k; r]$  and the condition  $(E_k2)$  guarantees that  $Tx$  is not in the interior of the  $k$ -ellipse  $E[x_1, \dots, x_k; r]$  for each  $x \in E[x_1, \dots, x_k; r]$ . Therefore, we say  $T(E[x_1, \dots, x_k; r]) \subset E[x_1, \dots, x_k; r]$ .

Now, we give the examples that satisfy the aforementioned theorem.

**Example 7.** Let  $(X, d)$  be a metric space,  $E = E[x_1, \dots, x_k; r]$  any  $k$ -ellipse and  $z$  a constant such that

$$2d(x, z) < d(y, z),$$

for all  $x \in E$  and  $y \in X - E$ .

Let us define the self-mapping  $T : X \rightarrow X$  as

$$Tx = \begin{cases} x, & x \in E, \\ z, & \text{otherwise,} \end{cases}$$

for all  $x \in X$ . Then it is clear that  $T$  satisfies the conditions  $(E_k1)$  and  $(E_k2)$  and so  $E[x_1, \dots, x_k; r]$  is a fixed  $k$ -ellipse of  $T$ . Now we show that  $T$  satisfies the condition  $(E_k3)$ . Let  $x \in E[x_1, \dots, x_k; r]$  and  $y \in X - E[x_1, \dots, x_k; r]$ . Then we have

$$d(Tx, Ty) = d(x, z) \leq h[d(Tx, x) + d(Ty, y)] = h[d(x, x) + d(z, y)] = hd(z, y),$$

with  $h \in (0, \frac{1}{2})$ . Therefore,  $E[x_1, \dots, x_k; r]$  is a unique fixed  $k$ -ellipse of  $T$ .

In the following example, we see that the fixed  $k$ -ellipse of  $T$  does not have to be unique.

**Example 8.** Let  $(X, d)$  be a metric space,  $E_1 = E[x_1, \dots, x_k; r]$ ,  $E_2 = E[x'_1, \dots, x'_k; r']$  any two  $k$ -ellipses and  $z$  a constant such that

$$\sum_{i=1}^k d(x, x_i) \neq r \text{ and } \sum_{i=1}^k d(x, x'_i) \neq r'.$$

Let us define the self-mapping  $T : X \rightarrow X$  as

$$Tx = \begin{cases} x, & x \in E_1 \cup E_2, \\ z, & \text{otherwise,} \end{cases}$$

for all  $x \in X$ . Then  $T$  satisfies the conditions  $(E_k1)$  and  $(E_k2)$  for both  $x \in E_1$  and  $x \in E_2$ . Hence  $T$  fixes both  $E_1$  and  $E_2$ . But  $T$  does not satisfy the condition  $(E_k3)$ . Indeed, for  $x \in E_1$  and  $y \in E_2$  with  $x \neq y$ , we have

$$d(Tx, Ty) = d(x, y) \leq h[d(Tx, x) + d(Ty, y)] = 0,$$



a contradiction with  $x \neq y$ .

Consequently, the fixed  $k$ -ellipse of  $T$  is not unique.

**Example 9.** Let  $X = \mathbb{R}$  be the usual metric space with the usual metric  $d$  defined as

$$d(x, y) = |x - y| \quad (x, y \in \mathbb{R}).$$

Let us take a 3-ellipse  $E[-1, 0, 1; 9] = \{x \in \mathbb{R} : d(x, -1) + d(x, 0) + d(x, 1) = 9\}$  and define the self-mapping  $T : \mathbb{R} \rightarrow \mathbb{R}$  as

$$Tx = \begin{cases} 0, & x \in \{-3, 3\}, \\ -3, & x \in \mathbb{R} - \{-3, 3\}, \end{cases}$$

for all  $x \in \mathbb{R}$ . Then  $T$  satisfies the condition  $(E_k1)$  but does not satisfy the condition  $(E_k2)$ . Therefore,  $E[-1, 0, 1; 9]$  is not a fixed 3-ellipse of  $T$ .

On the other hand, let us define the self-mapping  $S : \mathbb{R} \rightarrow \mathbb{R}$  as

$$Sx = \begin{cases} 5, & x \in \{-3, 3\}, \\ 0, & x \in \mathbb{R} - \{-3, 3\}, \end{cases}$$

for all  $x \in \mathbb{R}$ . Then  $S$  satisfies the condition  $(E_k2)$  but does not satisfy the condition  $(E_k1)$ . Hence  $E[-1, 0, 1; 9]$  is not a fixed 3-ellipse of  $S$ .

**Theorem 2.** Let  $(X, d)$  be a metric space and  $E[x_1, \dots, x_k; r]$  any  $k$ -ellipse on  $X$  and the mapping  $\xi : X \rightarrow [0, \infty)$  defined as in Theorem 1. If there exists a self-mapping  $T : X \rightarrow X$  such that

- $(E'_k1)$   $d(x, Tx) \leq \xi(x) + \xi(Tx) - 2r$ , for each  $x \in E[x_1, \dots, x_k; r]$ ,
- $(E'_k2)$   $\sum_{i=1}^k d(Tx, x_i) \leq r$ , for each  $x \in E[x_1, \dots, x_k; r]$ ,
- $(E'_k3)$   $d(Tx, Ty) \leq h[d(Tx, y) + d(Ty, x)]$ , for each  $x \in E[x_1, \dots, x_k; r]$ ,  
 $y \in X - E[x_1, \dots, x_k; r]$  and some  $h \in [0, \frac{1}{2})$ ,

then  $E[x_1, \dots, x_k; r]$  is a unique fixed  $k$ -ellipse of  $T$ .

*Proof.* At first, we prove the existence of a fixed  $k$ -ellipse of  $T$ . Let  $x \in E[x_1, \dots, x_k; r]$  be any point. Using the conditions  $(E'_k1)$ ,  $(E'_k2)$  and the definition of  $\xi$ , we get

$$\begin{aligned} d(x, Tx) &\leq \xi(x) + \xi(Tx) - 2r \\ &= \sum_{i=1}^k d(x, x_i) + \sum_{i=1}^k d(Tx, x_i) - 2r \\ &= r + \sum_{i=1}^k d(Tx, x_i) - 2r \\ &= \sum_{i=1}^k d(Tx, x_i) - r \leq r - r = 0, \end{aligned}$$

that is,  $x \in \text{Fix}(T)$ . So  $E[x_1, \dots, x_k; r]$  is a fixed  $k$ -ellipse of  $T$ .

Now, we show that  $E[x_1, \dots, x_k; r]$  is a unique fixed  $k$ -ellipse of  $T$ . To do this, on the contrary, we assume that  $E[x'_1, \dots, x'_k; r']$  is another fixed

$k$ -ellipse of  $T$ . Let  $x \in E[x_1, \dots, x_k; r]$  and  $y \in E[x'_1, \dots, x'_k; r']$  such that  $x \neq y$ . Then using the condition  $(E'_k3)$ , we find

$$d(Tx, Ty) = d(x, y) \leq h[d(Tx, y) + d(Ty, x)] = h[d(x, y) + d(y, x)] = 2hd(x, y),$$

a contradiction with  $h \in [0, \frac{1}{2})$ . It should be  $x = y$ .

Consequently,  $E[x_1, \dots, x_k; r]$  is a unique fixed  $k$ -ellipse of  $T$ .  $\square$

**Remark 3.**

- 1) If  $k = 1$ , then we have  $E[x_1; r] = C_{x_1, r}$  and so by Theorem 2.2 given in [13],  $E[x_1; r]$  is a fixed 1-ellipse of  $T$  or  $C_{x_1, r}$  is a fixed circle of  $T$ .
- 2) If  $k = 2$ , then we consider Theorem 2 as a fixed-ellipse theorem.
- 3) The determination of the condition  $(E'_k3)$  is not unique.
- 4) If we pay close attention, the condition  $(E'_k3)$  can be considered as a Chatterjea type contractive condition [3].
- 5) The condition  $(E'_k1)$  guarantees that  $Tx$  is not in the interior of the  $k$ -ellipse  $E[x_1, \dots, x_k; r]$  for each  $x \in E[x_1, \dots, x_k; r]$  and the condition  $(E'_k2)$  guarantees that  $Tx$  is not in the exterior of the  $k$ -ellipse  $E[x_1, \dots, x_k; r]$  for each  $x \in E[x_1, \dots, x_k; r]$ . Hence, we say  $T(E[x_1, \dots, x_k; r]) \subset E[x_1, \dots, x_k; r]$ .

**Example 10.** Let  $X = \{-4, -1, 0, 1, 2, 18\}$  be the usual metric space. Let us take a 4-ellipse  $E[-1, 0, 1, 2; 18]$  such as

$$E[-1, 0, 1, 2; 18] = \{x \in X : |x + 1| + |x| + |x - 1| + |x - 2| = 18\} = \{-4\}.$$

Let us define the self-mapping  $T : X \rightarrow X$  as

$$Tx = \begin{cases} -4, & x \in X - \{-1\}, \\ 0, & x = -1, \end{cases}$$

for all  $x \in X$ . Then  $T$  satisfies the conditions  $(E'_k1)$  and  $(E'_k2)$  and so  $E[-1, 0, 1, 2; 18]$  is a fixed 4-ellipse of  $T$ . Also  $T$  satisfies the condition  $(E'_k3)$  with  $h = \frac{4}{9}$ . Consequently,  $E[-1, 0, 1, 2; 18]$  is a unique fixed 4-ellipse of  $T$ .

**Theorem 3.** Let  $(X, d)$  be a metric space and  $E[x_1, \dots, x_k; r]$  any  $k$ -ellipse on  $X$  and the mapping  $\xi : X \rightarrow [0, \infty)$  defined as in Theorem 1. If there exists a self-mapping  $T : X \rightarrow X$  satisfying the conditions  $(E_k1)$ ,  $(E_k3)$  and  $(E''_k2)$   $\mu d(x, Tx) + \sum_{i=1}^k d(Tx, x_i) \geq r$ , for each  $x \in E[x_1, \dots, x_k; r]$  and some  $\mu \in [0, 1)$ ,

then  $E[x_1, \dots, x_k; r]$  is a unique fixed  $k$ -ellipse of  $T$ .

*Proof.* Using the similar approaches given in the proof of Theorem 1, we can easily prove that  $E[x_1, \dots, x_k; r]$  is a unique fixed  $k$ -ellipse of  $T$ .  $\square$

**Remark 4.**

- 1) If  $k = 1$ , then by Theorem 2.3 given in [13],  $E[x_1; r]$  is a fixed 1-ellipse of  $T$  or  $C_{x_1, r}$  is a fixed circle of  $T$ .

- 2) If  $k = 2$ , then we consider Theorem 2 given in [8],  $E[x_1, x_2; r]$  as a fixed 2-ellipse of  $T$  or  $E_r(x_1, x_2)$  is a fixed ellipse of  $T$ .
- 3) The condition  $(E''_k2)$  implies that  $Tx$  can be lies on or exterior or interior of the  $k$ -ellipse  $E[x_1, \dots, x_k; r]$ .

**Remark 5.** If we consider Example 8, then  $T$  satisfies the conditions  $(E'_k1)$  and  $(E'_k2)$  but  $T$  does not satisfy the condition  $(E'_k3)$ . Also, if we consider Example 9, then  $T$  satisfies the condition  $(E'_k2)$  but  $T$  does not satisfy the condition  $(E'_k1)$ . Similarly, in the same example, if we define the self-mapping  $H : \mathbb{R} \rightarrow \mathbb{R}$  as

$$Hx = \begin{cases} 10, & x \in \{-3, 3\}, \\ 0, & x \in \mathbb{R} - \{-3, 3\}, \end{cases}$$

for all  $x \in \mathbb{R}$ . Then  $H$  satisfies the condition  $(E'_k1)$  but does not satisfy the condition  $(E'_k2)$ .

Finally, if we consider Example 7 and Example 8, then  $T$  satisfies the conditions of  $(E''_k2)$ .

The selection of the auxiliary function is not unique. For example, we give the following fixed  $k$ -ellipse theorem.

**Theorem 4.** Let  $(X, d)$  be a metric space and  $E[x_1, \dots, x_k; r]$  any  $k$ -ellipse on  $X$ . Let us define the mapping  $\psi : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$  as

$$\psi(x) = \begin{cases} x - r, & x > 0, \\ 0, & x = 0, \end{cases}$$

for all  $x \in \mathbb{R}^+ \cup \{0\}$ . If there exists a self-mapping  $T : X \rightarrow X$  satisfying

- $(E'''_k1)$   $\sum_{i=1}^k d(Tx, x_i) = r$ , for each  $x \in E[x_1, \dots, x_k; r]$ ,
- $(E'''_k2)$   $d(Tx, Ty) > r$ , for each  $x, y \in E[x_1, \dots, x_k; r]$  with  $x \neq y$ ,
- $(E'''_k3)$   $d(Tx, Ty) \leq d(x, y) - \psi(d(x, Tx))$ , for each  $x, y \in E[x_1, \dots, x_k; r]$ ,
- $(E'''_k4)$   $d(Tx, Ty) \leq h \max \{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(x, y)\}$ , for each  $x \in E[x_1, \dots, x_k; r]$ ,  $y \in X - E[x_1, \dots, x_k; r]$  and some  $h \in (0, 1)$ ,

then  $E[x_1, \dots, x_k; r]$  is a unique fixed  $k$ -ellipse of  $T$ .

*Proof.* To show the existence of a fixed  $k$ -ellipse of  $T$ , we assume that  $x \in E[x_1, \dots, x_k; r]$  is any point. By the condition  $(E'''_k1)$ , we say that  $Tx \in E[x_1, \dots, x_k; r]$  for each  $x \in E[x_1, \dots, x_k; r]$ . Now we prove  $x \in Fix(T)$ . On the contrary, let  $x \notin Fix(T)$ , that is,  $x \neq Tx$ . Using the condition  $(E'''_k2)$ , we get

$$(1) \quad d(Tx, T^2x) > r$$

and using the condition  $(E'''_k3)$ , we obtain

$$\begin{aligned} d(Tx, T^2x) &\leq d(x, Tx) - \psi(d(x, Tx)) \\ &= d(x, Tx) - d(x, Tx) + r = r, \end{aligned}$$

a contradiction with the inequality (1). Thereby, it should be  $x \in \text{Fix}(T)$  and  $E[x_1, \dots, x_k; r]$  is a fixed  $k$ -ellipse of  $T$ .

Finally, we prove the uniqueness of the fixed  $k$ -ellipse  $E[x_1, \dots, x_k; r]$  of  $T$ . On the contrary, let  $E[x'_1, \dots, x'_k; r']$  be another fixed  $k$ -ellipse of  $T$ ,  $x \in E[x_1, \dots, x_k; r]$  and  $y \in E[x'_1, \dots, x'_k; r']$  such that  $x \neq y$ . Using the condition  $(E_k'''4)$ , we find

$$\begin{aligned} d(Tx, Ty) &\leq h \max \{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(x, y)\}, \\ d(x, y) &= h \max \{0, 0, d(x, y), d(y, x), d(x, y)\} \\ &= hd(x, y), \end{aligned}$$

a contradiction with  $h \in (0, 1)$ . It should be  $x = y$ . Consequently,  $E[x_1, \dots, x_k; r]$  is a unique fixed  $k$ -ellipse of  $T$ .  $\square$

**Remark 6.**

- 1) If  $k = 1$ , then by Theorem 3 given in [12],  $E[x_1; r]$  is a fixed 1-ellipse of  $T$  or  $C_{x_1, r}$  is a fixed circle of  $T$ .
- 2) If  $k = 2$ , then we consider Theorem 4 as a new fixed-ellipse result.
- 3) The condition  $(E_k'''4)$  can be considered as Ćirić type contractive condition [4] and the selection is not unique.
- 4) The converse statements of Theorem 1, Theorem 2 and Theorem 3 are also true but the converse statement of Theorem 4 is not always true.

**Example 11.** Let  $X = \{-1, 0, 1, 4, 12\}$  be the usual metric space. Let us take a 3-ellipse  $E[-1, 0, 1; 12]$  such as

$$E[-1, 0, 1; 12] = \{x \in \mathbb{R} : |x + 1| + |x| + |x - 1| = 12\} = \{4\}.$$

Let us define the self-mapping  $T : X \rightarrow X$  as

$$Tx = 4,$$

for all  $x \in X$ . Then satisfies the conditions  $(E_k'''1)$ ,  $(E_k'''2)$  and  $(E_k'''3)$ . So  $E[-1, 0, 1; 12]$  is a fixed 3-ellipse of  $T$ . Also  $T$  satisfy the condition  $(E_k'''4)$  and we say that  $E[-1, 0, 1; 12]$  is a unique fixed 3-ellipse of  $T$ .

**Example 12.** Let  $X = \{-2, -1\} \cup [0, \infty)$  and  $(X, d)$  be a usual metric space. Let us take a 3-ellipse  $E[-2, 0, 2; 21]$  such as

$$E[-2, 0, 2; 21] = \{x \in \mathbb{R} : |x + 2| + |x| + |x - 2| = 21\} = \{7\}.$$

Let us define the self-mapping  $T : X \rightarrow X$  as

$$Tx = \begin{cases} x, & x \in [0, \infty), \\ 0, & x \in \{-2, -1\}, \end{cases}$$

for all  $x \in X$ . Then  $T$  satisfies the conditions  $(E_k'''1)$ ,  $(E_k'''2)$  and  $(E_k'''3)$ . Hence  $E[-2, 0, 2; 21]$  is a fixed 3-ellipse of  $T$ . But  $T$  does not satisfy the condition  $(E_k'''4)$  for  $x \in [0, \infty) - \{7\}$ . Indeed, we have

$$d(7, x) \leq h \max \{0, 0, d(7, x), d(x, 7), d(x, 7)\}$$

$$= hd(7, x),$$

a contradiction with  $h \in (0, 1)$ . So  $E[-2, 0, 2; 21]$  is not a unique fixed 3-ellipse of  $T$ . For example,  $E[-1, 0, 1; 21]$  is another fixed 3-ellipse of  $T$ .

**Example 13.** Let  $X = \mathbb{R}$  be the usual metric space. Let us take a 3-ellipse  $E[-1, 0, 1; 27]$  such as

$$E[-2, 0, 2; 27] = \{x \in \mathbb{R} : |x + 2| + |x| + |x - 2| = 27\} = \{-9, 9\}.$$

Let us define the self-mapping  $T : \mathbb{R} \rightarrow \mathbb{R}$  as

$$Tx = \begin{cases} x, & x \in \{-9, 9\}, \\ \frac{1}{x+1}, & x \in \mathbb{R} - \{-9, 9\}, \end{cases}$$

for all  $x \in \mathbb{R}$ . Then  $T$  fixes the 3-ellipse  $E[-1, 0, 1; 27]$  but  $T$  does not satisfy the condition  $(E_k''')$ .

In the following theorem, we investigate a contractive condition excludes the identity map  $I_x : X \rightarrow X$ , defined by  $I_x(x) = x$  for all  $x \in X$ , in Theorems 1, 2, 3 and 4.

**Theorem 5.** Let  $(X, d)$  be a metric space,  $E[x_1, \dots, x_k; r]$  any  $k$ -ellipse on  $X$  and the mapping  $\xi : X \rightarrow [0, \infty)$  defined as in Theorem 1.  $T$  satisfies the condition  $(I_k)$

$$(I_k) \quad d(x, Tx) \leq \frac{[\xi(x) - \xi(Tx)]}{k + 1},$$

for all  $x \in X$  if and only if  $T = I_x$ .

*Proof.* Assume that  $T$  satisfies the condition  $(I_k)$  and  $x \notin \text{Fix}(T)$  for  $x \in X$ . Then we get

$$\begin{aligned} d(x, Tx) &\leq \frac{[\xi(x) - \xi(Tx)]}{k + 1} \\ &= \frac{1}{k + 1} \left[ \sum_{i=1}^k d(x, x_i) - \sum_{i=1}^k d(Tx, x_i) \right] \\ &\leq \frac{1}{k + 1} \left[ kd(x, Tx) + \sum_{i=1}^k d(Tx, x_i) - \sum_{i=1}^k d(Tx, x_i) \right] \\ &= \frac{k}{k + 1} d(x, Tx) < d(x, Tx), \end{aligned}$$

a contradiction. Hence it should be  $Tx = x$  for each  $x \in X$  and  $T = I_x$ . The converse statement is clearly proved.  $\square$

**Remark 7.** If a self-mapping  $T : X \rightarrow X$  satisfying the conditions of Theorem 1 (resp. Theorem 2, Theorem 3 and Theorem 4) does not satisfy the condition  $(I_k)$  then we exclude the identity map.

### 3. AN APPLICATION TO ( $SReLU$ )

In this section, we investigate a new application to the activation functions using the notion of fixed  $k$ -ellipse. Why do we choose an activation function? Activation functions are extensively used and important part in neural networks since they decide whether a neuron should be activated or not. In the literature, there are a lot of examples of activation functions. For example, one of them is  $S$ -Shaped Rectified Linear Activation Unit ( $SReLU$ ) defined by

$$SReLU(x) = \begin{cases} t_l + a_l(x - t_l), & x \leq t_l, \\ x, & t_l < x < t_r, \\ t_r + a_r(x - t_r), & x \geq t_r, \end{cases}$$

where  $t_l, a_l, t_r, a_r$  are parameters [7].

Let us take  $X = \mathbb{R}, t_l = -6, t_r = 6, a_l = 2$  and  $a_r = 3$ . Then we get

$$SReLU(x) = \begin{cases} 2x + 6, & x \leq -6, \\ x, & -6 < x < 6, \\ 3x - 12, & x \geq 6 \end{cases}$$

for all  $x \in \mathbb{R}$  as seen in Figure 7.

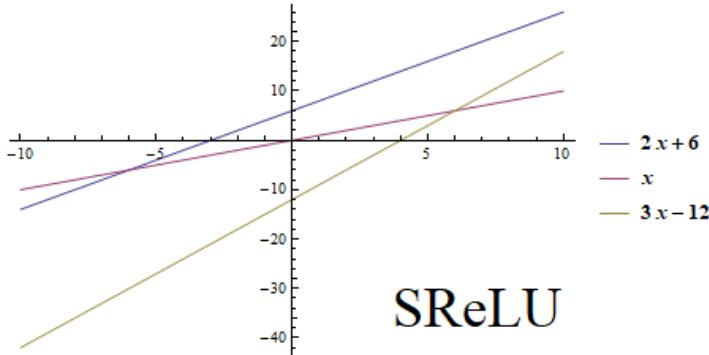


FIGURE 7. The activation function  $SReLU(x)$ .

Then we can easily say that the activation function  $SReLU(x)$  fixes at least one  $k$ -ellipse on  $\mathbb{R}$ . For example, let us consider the following 3-ellipses:

$$E[-1, 0, 1; 15] = \{x \in \mathbb{R} : |x + 1| + |x| + |x - 1| = 15\} = \{-5, 5\},$$

$$E[-2, 0, 2; 6] = \{x \in \mathbb{R} : |x + 2| + |x| + |x - 2| = 6\} = \{-2, 2\}$$

and

$$E[-\alpha, 0, \alpha; 9] = \{x \in \mathbb{R} : |x + \alpha| + |x| + |x - \alpha| = 9, \alpha \in \mathbb{R}\} = \{-3, 3\}.$$

Then  $E[-1, 0, 1; 15]$ ,  $E[-2, 0, 2; 6]$  and  $E[-\alpha, 0, \alpha; 9]$  are fixed 3-ellipse of  $SReLU$ . Also, it can be easily seen that the activation function  $SReLU$

satisfies the conditions  $(E_k1)$  and  $(E_k2)$  (resp.  $(E'_k1)$  and  $(E'_k2)$ ,  $(E''_k2)$ ). Therefore, we say that there exists at least one fixed  $k$ -ellipse of  $SReLU$ . But  $SReLU$  does not satisfy the conditions  $(E_k3)$  and  $(E'_k3)$ . Hence the fixed  $k$ -ellipse of  $SReLU$  is not unique as seen in the above examples. If we pay attention, this activation function  $SReLU$  fixes infinite number of  $k$ -ellipses. This case is important in view of increasing the number of fixed points in neural networks.

#### 4. CONCLUSION

In this paper, we obtain new fixed  $k$ -ellipse results on metric spaces. The proved results are new solutions to the fixed-figure problem. This problem can be investigated using different contractions on metric and generalized metric spaces (for example, [2] and the references therein).

#### REFERENCES

- [1] J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Transactions of the American Mathematical Society, 215 (1976), 241–251.
- [2] N. Chandra, B. Joshi, M. C. Choshi, *Generalized fixed point theorems on metric spaces*, Mathematica Moravica, 26 (2) (2022), 85–101.
- [3] S. K. Chatterjea, *Fixed point theorems*, Comptes rendus de l'Academie bulgare des Sciences, 25 (1972), 727–730.
- [4] Lj. B. Ćirić, *Generalized contractions and fixed-point theorems*, Publications de l'Institut Mathematique, 12 (26) (1971), 19–26.
- [5] G. Z. Erçınar, *Some geometric properties of fixed points*, Ph.D. Thesis, Eskişehir Osmangazi University, 2020.
- [6] P. Erdos, I. Vincze, István, *On the Approximation of Convex, Closed Plane Curves by Multifocal Ellipses*, Journal of Applied Probability, 19 (1982) 89–96.
- [7] X. Jin, C. Xu, J. Feng, Y. Wei, J. Xiong, S. Yan, *Deep learning with s-shaped rectified linear activation units*, In Proceedings of the AAAI Conference on Artificial Intelligence, 30 (1) (2016).
- [8] M. Joshi, A. Tomar, S. K. Padaliya, *Fixed point to fixed ellipse in metric spaces and discontinuous activation function*, Applied Mathematics. E-Notes, 21 (2021), 225–237.
- [9] R. Kannan, *Some results on fixed points*, Bulletin of the Calcutta Mathematical Society, 62 (1968), 71–76.
- [10] Z. A. Melzak, J.S. Forsyth, *Polyconics 1. polyellipses and optimization*, Quarterly of Applied Mathematics, (1977), 239–255, .
- [11] J. Nie, P.A. Parrilo, B. Sturmfels, *Semidefinite representation of the  $k$ -ellipse*, Algorithms in Algebraic Geometry, I.M.A. Volumes in Mathematics and its Applications, Springer, New York, 146 (2008), 117–132.
- [12] N. Y. Özgür, N. Taş, *Some fixed-circle theorems and discontinuity at fixed circle*, AIP Conference Proceedings 1926, 020048 (2018).

- [13] N. Y. Özgür, N. Taş, *Some fixed-circle theorems on metric spaces*, Bulletin of the Malaysian Mathematical Sciences Society, 42 (4) (2019), 1433–1449.
- [14] P.V. Sahadevan, *The theory of egglipe—a new curve with three focal points*, International Journal of Mathematical Education in Science and Technology, 18 (1987), 29–39.
- [15] J. Sekino,  *$n$ -ellipses and the minimum distance sum problem*, American Mathematical Monthly, 106 (3) (1999), 193–202.
- [16] N. Taş, N. Özgür, *New fixed-figure results on metric spaces*, In: Debnath, P., Srivastava, H.M., Kumam, P., Hazarika, B. (eds) Fixed Point Theory and Fractional Calculus, Forum for Interdisciplinary Mathematics, Springer, Singapore, 2022.
- [17] N. Taş, *A geometric approach to the Proinov type contractions*, Mathematica Moravica, 26 (1) (2022), 123–132.

**HÜLYA AYTİMUR**

BALIKESİR UNIVERSITY

DEPARTMENT OF MATHEMATICS

10145 BALIKESİR

TURKEY

*E-mail address:* hulya.aytimur@balikesir.edu.tr

**ŞABAN GÜVENÇ**

BALIKESİR UNIVERSITY

DEPARTMENT OF MATHEMATICS

10145 BALIKESİR

TURKEY

*E-mail address:* sguvenc@balikesir.edu.tr

**NİHAL TAŞ**

BALIKESİR UNIVERSITY

DEPARTMENT OF MATHEMATICS

10145 BALIKESİR

TURKEY

*E-mail address:* nihaltas@balikesir.edu.tr