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On a family of bi-univalent functions related to the Fibonacci numbers

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Abstract. In this study, we construct a new family of holomorphic bi-univalent functions in the open unit disc by the help of $q-$analogue of Noor integral operator, principle of subordination and Fibonacci polynomials. Also we obtain coefficient bounds and Fekete Szegö inequality for functions belonging this family. We have illustrated relevant families and consequences.

1. Introduction

Let $\mathcal{A}$ indicate the collection of functions h having the form

$$h(z) = z + \sum_{\vartheta=2}^{\infty} d_{\vartheta} z^{\vartheta}, \tag{1}$$

also holomorphic in the open unit disc $\mathfrak{D} = \{z : |z| < 1\}$. Let

$$S = \{h \in \mathcal{A} : h \ \text{ is univalent in } \mathfrak{D}\}.$$

According to the Koebe one-quarter theorem ([9]), the range of every function $h \in S$ contains the disc of radius $\left\{w : |w| < \frac{1}{4}\right\}$. Thus every such function $h \in S$ has an inverse h^{-1} which satisfies

$$h^{-1}(h(z)) = z \quad (z \in \mathfrak{D})$$

and

$$h\left(h^{-1}(w)\right) = w \quad \left(|w| < r_0(h) \ , \ r_0(h) \geq \frac{1}{4}\right), \quad (w \in \mathfrak{D})$$

where

$$\begin{aligned} k(w) &= h^{-1}(w) = \\ &= w - d_2 w^2 + \left(2d_2^2 - d_3\right) w^3 - \left(5d_2^3 - 5d_2 d_3 + d_4\right) w^4 + \cdots . \end{aligned} \tag{2}$$

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Definition 1. If both h and h^{-1} are univalent in $\mathfrak{D}$, then a function $h \in \mathcal{A}$ is called to be bi-univalent in $\mathfrak{D}$. We say that h is in the class Σ for such functions.

Definition 2. For analytic functions h and k, h is called to be subordinate to k, denoted by

$$h(z) \prec k(z),$$

if there is a function w, analytic, such that

$$w(0) = 0 \ \ , \ |w(z)| < 1 \ \text{ and } \ h(z) = k(w(z)).$$

Definition 3. For $q \in (0,1)$, the q-derivative of function $h \in \mathcal{A}$ is defined by (see [23])

$$\partial_q h(z) = \frac{h(qz) - h(z)}{(q-1)z}, \quad z \neq 0$$

and

$$\partial_q h(0) = h'(0).$$

Thus we have

$$\partial_q h(z) = 1 + \sum_{\vartheta=2}^{\infty} [\vartheta, q]\, d_\vartheta z^{\vartheta-1},$$

where $[\vartheta, q]$ is given by

$$[\vartheta, q] = \frac{1-q^\vartheta}{1-q}, \qquad [0,q] = 0$$

and define the q-fractional by

$$[\vartheta, q]! = \begin{cases} \prod_{n=1}^{\vartheta} [n,q]\,, & \vartheta \in \mathbb{N}, \\ \quad 1, & \vartheta = 0. \end{cases}$$

On the other hand, the $q-$generalized Pochhammer symbol for $\mathfrak{p} \geq 0$ is defined by

$$[\mathfrak{p}, q]_\vartheta = \begin{cases} \prod_{n=1}^{\vartheta} [\mathfrak{p} + n - 1, q]\,, & \vartheta \in \mathbb{N}, \\ \quad 1, & \vartheta = 0. \end{cases}$$

In addition, as $q \to 1, [\vartheta, q] \to 1$, taking $k(z) = z^\vartheta$, then we get

$$\partial_q k(z) = \partial_q z^\vartheta = [\vartheta, q]\, z^{\vartheta-1} = k'(z),$$

where k' is the ordinary derivative. Recently, the function $F_{q,\mu+1}^{-1}(z)$, given with the following relation, was defined by Arif et al. (see [8])

$$F_{q,\mu+1}^{-1}(z) * F_{q,\mu+1}(z) = z\partial_q h(z), \quad (\mu > -1)$$

where

$$F_{q,\mu+1}(z) = z + \sum_{\vartheta=2}^{\infty} \frac{[\mu+1, q]_{\vartheta-1}}{[\vartheta-1, q]!} z^\vartheta, \quad z \in \mathfrak{D}.$$

Due to the fact that series defined in last equation is convergent absolutely in $z \in \mathfrak{D}$, by taking advantage of the characterization of q-derivative via convolution, one can define the integral operator $\zeta_q^\mu : \mathfrak{D} \to \mathfrak{D}$ by

$$\zeta_q^\mu h(z) = F_{q,\mu+1}^{-1}(z) * h(z) = z + \sum_{\vartheta=2}^{\infty} \phi_{\vartheta-1} d_\vartheta z^\vartheta, \quad (z \in \mathfrak{D})$$

where

$$\phi_{\vartheta-1} = \frac{[\vartheta, q]!}{[\mu+1, q]_{\vartheta-1}}.$$

We note that

$$\zeta_q^0 h(z) = z\partial_q h(z), \ \zeta_q' h(z) = h(z),$$

also

$$\lim_{q\to 1} \zeta_q^\mu h(z) = z + \sum_{\vartheta=2}^{\infty} \frac{\vartheta!}{(\mu+1)_{\vartheta-1}} d_\vartheta z^\vartheta.$$

Last equation means that by letting $q \to 1$, the operator $\zeta_q^\mu h$, defined as above, reduces to the Noor integral operator introduced in [19,20]. One can find more details on the q-analogue of differential and integral operators, in the study of Aldweby and Darus (see [21]).

Such polynomials as the Fibonacci polynomials, the Lucas polynomials, the Chebyshev polynomials, the Pell polynomials, Lucas–Lehmer polynomials, and the families of orthogonal polynomials also other special polynomials and their generalizations are very important in different disciplines of sciences (also, see [2,3,11,12,14,15,17,26,28]).

Up to now, coefficient estimates for bi-univalent holomorphic functions was studied by many authors. This subject goes back the studies in [1,5,9, 16,23–25]. In the literature, there are limited number of studies (by help of the Faber polynomial expansions) determining the general coefficient bounds $|d_\vartheta|$ for bi-univalent functions ([6,7,13,18,22,27]). Thus, identification of the bounds for each of $|d_\vartheta|$ ($\vartheta = 3, 4, 5, \cdots, \vartheta \in \mathbb{N}$) is still an open problem for functions in the collection Σ.

In this study, we introduce the family $\mathfrak{B}_\Sigma^{\mu,q}(\beta, \gamma; \widetilde{\wp})$ associated with the Fibonacci numbers and q-analogue of Noor integral operators. For functions in $\mathfrak{B}_\Sigma^{\mu,q}(\beta, \gamma; \widetilde{\wp})$, we have obtained coefficient inequalities.

2. The class $\mathfrak{B}_\Sigma^{\mu,q}(\beta, \gamma; \widetilde{\wp})$

Definition 4. A function $h \in \Sigma$, given by (1), is called to be in the class $\mathfrak{B}_\Sigma^{\mu,q}(\beta, \gamma; \widetilde{\wp})$ if it satisfies the conditions:

$$1 + \frac{1}{\gamma}\left[(1-\beta)\frac{\zeta_q^\mu h(z)}{z} + \beta\partial_q(\zeta_q^\mu h(z)) - 1\right] \prec \widetilde{\wp}(z) = \frac{1+\tau^2 z^2}{1-\tau^2 z^2 - \tau z} \tag{3}$$

and

$$(4)\qquad \begin{aligned}1+\frac{1}{\gamma}\left[(1-\beta)\frac{\zeta_q^{\mu}k(w)}{w}+\beta\partial_q(\zeta_q^{\mu}k(w))-1\right]\\ \prec\widetilde{\wp}(w)=\frac{1+\tau^2w^2}{1-\tau^2w^2-\tau w},\end{aligned}$$

where $(0<\mu\leq 1,\ 0<q<1,\gamma>0,\beta\geq 0)$, the k is given by (2) and $\tau=\frac{1-\sqrt{5}}{2}\approx -0,618$.

Remark 1. The function $\widetilde{\wp}(z)$ is not univalent in $\mathfrak{D}$, $\widetilde{\wp}(z)$ is univalent in the disc $|z|<\frac{3-\sqrt{5}}{2}\approx 0,38$. Assume that $\widetilde{\wp}(0)=\widetilde{\wp}(-\frac{1}{2\tau})$ and $\widetilde{\wp}(e^{\mp i\arccos(\frac{1}{4})})=\frac{\sqrt{5}}{5}$. Additionally, we can write it as

$$\frac{1}{|\tau|}=\frac{|\tau|}{1-|\tau|},$$

which indicates the number $|\tau|$ divides $[0,1]$ such that it fulfills the golden section (see for details [10, 27]).

Also, a relation between the function $\widetilde{\wp}(z)$ and the Fibonacci numbers was found in [10].

Assume that $\mathcal{F}_n$ is the sequence of Fibonacci numbers.

$$\mathcal{F}_{n+2}=\mathcal{F}_n+\mathcal{F}_{n+1},\quad n\in\mathbb{N}_0=0,1,2,\dots$$

with $\mathcal{F}_0=0,\mathcal{F}_1=1$, then

$$\mathcal{F}_n=\frac{(1-\tau)^n-\tau^n}{\sqrt{5}},\quad \tau=\frac{1-\sqrt{5}}{2}.$$

If we let

$$\begin{aligned}\widetilde{\wp}(z)&=1+\sum_{n=1}\widetilde{\wp}_nz^n=1+(F_0+F_2)\,\tau z+(F_1+F_3)\,\tau^2z^2\\&+\sum_{n=3}(F_{n-3}+F_{n-2}+F_n)\,\tau^nz^n,\end{aligned}$$

then we arrive at

$$\widetilde{\wp}_n=\begin{cases}\tau, & n=1,\\ 3\tau^2, & n=2,\\ \tau\widetilde{\wp}_{n-1}+\tau^2\widetilde{\wp}_{n-2}, & n\geq 3.\end{cases}$$

It should be noted that the special values of β,γ,μ,q give us different subfamilies.

Remark 2. For $q\to 1^-$, easy to see that $h\in\mathfrak{B}_{\Sigma}^{\mu,1}(\beta,\gamma;\widetilde{\wp})$ if it satisfies the conditions

$$1+\frac{1}{\gamma}\left[(1-\beta)\frac{\zeta^{\mu}h(z)}{z}+\beta(\zeta^{\mu}h(z))'-1\right]\prec\widetilde{\wp}(z)=\frac{1+\tau^2z^2}{1-\tau^2z^2-\tau z}$$

and

$$1+\frac{1}{\gamma}\left[(1-\beta)\frac{\zeta^{\mu}k(w)}{w}+\beta(\zeta^{\mu}k(w))'-1\right]\prec\widetilde{\wp}(w)=\frac{1+\tau^2w^2}{1-\tau^2w^2-\tau w},$$

where $k=h^{-1}$ is given by (2) .

Remark 3. For $q\to 1^-,\gamma=1$ and $\beta=1$, easy to see that $h\in\Sigma$ is in

$$\mathfrak{B}_{\Sigma}^{\mu,1}(1,1;\widetilde{\wp})=\mathfrak{B}_{\Sigma}^{\mu,q}(\widetilde{\wp})$$

if it satisfies the conditions

$$(\zeta^{\mu}h(z))'\prec\frac{1+\tau^2z^2}{1-\tau^2z^2-\tau z}$$

and

$$(\zeta^{\mu}k(w))'\prec\frac{1+\tau^2w^2}{1-\tau^2w^2-\tau w},$$

where $k=h^{-1}$ is given by (2) .

Remark 4. For $q\to 1^-,\gamma=1,\beta=1$ and $\mu=1$, easy to see that $h\in\Sigma$ is in

$$\mathfrak{B}_{\Sigma}^{1,1}(1,1;\widetilde{\wp})=\mathfrak{B}_{\Sigma}(\widetilde{\wp})=\Sigma'(\widetilde{\wp})$$

if it satisfies the conditions

$$h'(z)\prec\frac{1+\tau^2z^2}{1-\tau^2z^2-\tau z}$$

and

$$k'(w)\prec\frac{1+\tau^2w^2}{1-\tau^2w^2-\tau w},$$

where $k=h^{-1}$ is given by (2) . The class $\Sigma'(\wp)$ was investigated and studied by Alamous [4].

Remark 5. For $q\to 1^-$,$\gamma=1$,for $\beta=1$ and $\mu=0$, easy to see that $h\in\Sigma$ is in

$$\mathfrak{B}_{\Sigma}^{0,1}(1,1;\widetilde{\wp})=\mathfrak{B}_{\Sigma}(\widetilde{\wp})$$

if it satisfies the conditions

$$(z\partial h(z))'\prec\frac{1+\tau^2z^2}{1-\tau^2z^2-\tau z}$$

and

$$(w\partial k(w))'\prec\frac{1+\tau^2w^2}{1-\tau^2w^2-\tau w},$$

where $k=h^{-1}$ is given by (2).

3. Initial coefficient estimates

We first state and prove the following result.

Theorem 1. *Let* $h \in \mathfrak{B}_{\Sigma}^{\mu,q}(\beta,\gamma;\widetilde{\wp})$ *given by* (1). Then,

$$|d_2| \leq \gamma \frac{|\tau|}{\sqrt{\left|(1+\beta q)^2\phi_1^2 + \left[\gamma(1+\beta q+\beta q^2)\phi_2 - 3(1+\beta q)^2\phi_1^2\right]\tau\right|}}$$

and

$$|d_3| \leq \gamma \frac{\tau^2}{(1+\beta q)^2\phi_1^2} + \gamma^2 \frac{|\tau|}{(1+\beta q+\beta q^2)\phi_2},$$

where

$$0 < \mu \leq 1, 0 < q < 1, \gamma > 0, \beta \geq 0.$$

Proof. Let $h \in \mathfrak{B}_{\Sigma}^{\mu,q}(\beta,\gamma;\widetilde{\wp}), 0 < \mu \leq 1, 0 < q < 1, \gamma > 0, \beta \geq 0$. From the subordination relations given in (3) and (4), there exists two analytic functions $f, g : \mathfrak{D} \to \mathfrak{D}$,with $f(0) = g(0)$ such that

$$1 + \frac{1}{\gamma}\left[(1-\beta)\frac{\zeta_q^{\mu}h(z)}{z} + \beta\partial_q(\zeta_q^{\mu}h(z)) - 1\right] = \widetilde{\wp}(f(z))$$

and

$$1 + \frac{1}{\gamma}\left[(1-\beta)\frac{\zeta_q^{\mu}k(w)}{w} + \beta\partial_q(\zeta_q^{\mu}k(w)) - 1\right] = \widetilde{\wp}(g(w)).$$

If we determine the functions p_1 and p_2 as

$$p_1(z) = \frac{1+f(z)}{1-f(z)} = 1 + x_1 z + x_2 z^2 + \cdots$$

and

$$p_2(w) = \frac{1+g(w)}{1-g(w)} = 1 + y_1 w + y_2 w^2 + \cdots,$$

then p_1 and p_2 are analytic in $\mathfrak{D}$ with $p_1(0) = p_2(0) = 1$. Thus,

$$f(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[x_1 z + \left(x_2 - \frac{x_1^2}{2}\right)z^2 + \cdots\right]$$

and

$$g(w) = \frac{p_2(w)-1}{p_2(w)+1} = \frac{1}{2}\left[y_1 w + \left(y_2 - \frac{y_1^2}{2}\right)w^2 + \cdots\right]$$

lead to

$$\widetilde{\wp}(f(z)) = 1 + \frac{\widetilde{\wp}_1 x_1}{2}z + \left\{\left(x_2 - \frac{x_1^2}{2}\right)\frac{\widetilde{\wp}_1}{2} + \frac{x_1^2}{4}\widetilde{\wp}_2\right\}z^2 + \cdots$$

and

$$\widetilde{\wp}(g(w)) = 1 + \frac{\widetilde{\wp}_1 y_1}{2}w + \left\{\left(y_2 - \frac{y_1^2}{2}\right)\frac{\widetilde{\wp}_1}{2} + \frac{y_1^2}{4}\widetilde{\wp}_2\right\}w^2 + \cdots.$$

Hence,

$$1+\frac{1}{\gamma}\left[(1-\beta)\frac{\zeta_q^\mu h(z)}{z}+\beta\partial_q(\zeta_q^\mu h(z))-1\right]=$$
$$1+\frac{\widetilde{\wp}_1 x_1}{2}z+\left\{\left(x_2-\frac{x_1^2}{2}\right)\frac{\widetilde{\wp}_1}{2}+\frac{x_1^2}{4}\widetilde{\wp}_2\right\}z^2+\cdots$$

and

$$1+\frac{1}{\gamma}\left[(1-\beta)\frac{\zeta_q^\mu k(w)}{w}+\beta\partial_q(\zeta_q^\mu k(w))-1\right]=$$
$$1+\frac{\widetilde{\wp}_1 y_1}{2}w+\left\{\left(y_2-\frac{y_1^2}{2}\right)\frac{\widetilde{\wp}_1}{2}+\frac{y_1^2}{4}\widetilde{\wp}_2\right\}w^2+\cdots.$$

If we compare the corresponding coefficients in last two equations, then we can obtain

$$\frac{1}{\gamma}\left(1+\beta q\right)\phi_1 d_2=\frac{\widetilde{\wp}_1 x_1}{2}, \tag{5}$$

$$\frac{1}{\gamma}\left[(1+\beta q+\beta q^2)\phi_2 d_3\right]=\frac{1}{2}\left(x_2-\frac{x_1^2}{2}\right)\widetilde{\wp}_1+\frac{x_1^2}{4}\widetilde{\wp}_2 \tag{6}$$

and

$$-\frac{1}{\gamma}\left(1+\beta q\right)\phi_1 d_2=\frac{\widetilde{\wp}_1 y_1}{2}, \tag{7}$$

$$\frac{1}{\gamma}\left[(1+\beta q+\beta q^2)\phi_2\left(2d_2^2-d_3\right)\right]=\frac{1}{2}\left(y_2-\frac{y_1^2}{2}\right)\widetilde{\wp}_1+\frac{y_1^2}{4}\widetilde{\wp}_2. \tag{8}$$

From equalities (5) and (6), we find that

$$x_1=-y_1,$$

and

$$2\left[\frac{1}{\gamma}\left(1+\beta q\right)\phi_1\right]^2 d_2^2=\frac{\widetilde{\wp}_1^2}{4}\left(x_1^2+y_1^2\right). \tag{9}$$

Also, by using (6) and (8), we obtain

$$\frac{2}{\gamma}(1+\beta q+\beta q^2)\phi_2 d_2^2=\frac{\widetilde{\wp}_1}{2}\left(x_2+y_2\right)+\frac{\widetilde{\wp}_2-\widetilde{\wp}_1}{4}\left(x_1^2+y_1^2\right). \tag{10}$$

By substituting $x_1^2+y_1^2$ from (9) and putting in (10), we reduce that

$$d_2^2=\frac{\gamma^2\widetilde{\wp}_1^3\left(x_2+y_2\right)}{4\left[\gamma\widetilde{\wp}_1^2(1+\beta q+\beta q^2)\phi_2-(1+\beta q)^2\phi_1^2(\widetilde{\wp}_2-\widetilde{\wp}_1)\right]},$$

which yields

$$|d_2| \leq \gamma \frac{|\tau|}{\sqrt{\left|(1+\beta q)^2\phi_1^2 + \left[\gamma(1+\beta q+\beta q^2)\phi_2 - 3(1+\beta q)^2\phi_1^2\right]\tau\right|}}.$$

Moreover, if we subtract (6) from (8), we have

$$\frac{2}{\gamma}(1+\beta q+\beta q^2)\phi_2(d_3 - d_2^2) = \frac{\widetilde{\wp}_1}{2}\left(x_2 - y_2\right).$$

Then, in view of (9), the last equation becomes

$$d_3 = \gamma^2 \frac{\widetilde{\wp}_1^2}{8(1+\beta q)^2\phi_1^2}\left(x_1^2 + y_1^2\right) + \gamma \frac{\widetilde{\wp}_1\left(x_2 - y_2\right)}{4(1+\beta q+\beta q^2)\phi_2}.$$

Applying $h_2(x)$ and taking modulus, we deduce that

$$|d_3| \leq \gamma^2 \frac{\tau^2}{(1+\beta q)^2\phi_1^2} + \gamma \frac{|\tau|}{(1+\beta q+\beta q^2)\phi_2}.$$

$\square$

4. Consequences

In this study, we studied the analytic bi-univalent function class $\mathfrak{B}_\Sigma^{\mu,q}(\beta,\gamma;\widetilde{\wp})$ associated with the Fibonacci numbers. For functions belonging to this class, we have derived Taylor-Maclaurin coefficient inequalities. The geometric properties this new class varies to the values according to the parameters included. This approach has been extended to find more examples of bi-univalent functions with the Fibonacci numbers.

Upon setting $q \to 1^-$in Theorem 1, we get the following corollaries.

Corollary 1. *Let $h \in \mathfrak{B}_\Sigma^{\mu,1}(\beta,\gamma;\widetilde{\wp})$, given by* (1). Then,

$$|d_2| \leq |\gamma| \frac{|\tau|}{\sqrt{\left|(1+\beta)^2\phi_1^2 + \left[\gamma(1+2\beta)\phi_2 - 3(1+\beta)^2\phi_1^2\right]\tau\right|}}$$

and

$$|d_3| \leq \gamma^2 \frac{\tau^2}{(1+\beta)^2\phi_1^2} + |\gamma| \frac{|\tau|}{(1+2\beta)\phi_2},$$

where

$$0 < \mu \leq 1, \gamma > 0, \beta \geq 0.$$

Upon setting $q \to 1^-, \gamma = 1$ and $\beta = 1$ in Theorem 1, we get next corollary.

Corollary 2. *Let $h \in \mathfrak{B}_\Sigma^{\mu,1}(1,1;\widetilde{\wp})$, given by* (1). Then,

$$|d_2| \leq \frac{|\tau|}{\sqrt{\left|4\phi_1^2 + 3(\phi_2 - 4\phi_1^2)\tau\right|}}$$

and

$$|d_3| \leq \frac{\tau^2}{4\phi_1^2} + \frac{|\tau|}{3\phi_2},$$

where

$$0 < \mu \leq 1.$$

Restricting our assumptions for a choosen univalent function $\widetilde{\wp}(z)$ in $\mathfrak{D}$, we can examine mapping problems for other regions of the complex $z-$plane. Thus, one can define different subclasses of the function class which we have studied in this work.

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