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Fixed point theorems of generalized multi-valued mappings in cone *b*-metric spaces

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Abstract. The aim of this paper is to establish fixed points for multi-valued mappings, by adapting the ideas in [1] to the cone b-metric space setting.

1. Introduction and preliminaries

The well-known Banach contraction principle and its several generalization in the setting of metric spaces play a central role for solving many problems of nonlinear analysis. For example, see [3, 10, 12, 20, 21]. Several authors introduced some interesting concept, see [28, 29, 30, 31, 32]. In [4], Bakhtin introduced b-metric spaces as a generalization of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalized the famous contraction principle in metric spaces. Since then, several papers have dealt with fixed point theory or the variational principle for single-valued and multi-valued operators in b-metric spaces(see [6, 7, 11] and reference therein). In recent investigations, the fixed point in non-convex analysis, especially in an ordered normed space, occupies a prominent place in many aspects (see [14, 15, 18, 22]). The authors define an ordering by using a cone, which naturally induces a partial ordering in Banach spaces. In 2007, Huang and Zhang [14] introduced the concept of cone metric spaces as a generalization of metric spaces and establish some fixed point theorems for contractive mappings in normal cone metric spaces. Subsequently, several other authors [2, 16, 23, 25] studied the existence of fixed points and common fixed points of mappings satisfying contractive type condition on a normal cone metric space. Recently, Rezapour and Hamlbarani [23] omitted the assumption of normality in cone metric space, which is a milestone in developing fixed point theory in cone metric space. In 2011, Hussain and

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Shah [15] introduced the concept of cone b-metric space as a generalization of b-metric space and cone metric spaces. They established some topological properties in such space and improved some recent results about KKM mappings in the setting of a cone b-metric space. In 2020, Wasfi Shatanawi, Zoran D. Mitrović, Nawab Hussain and Stojan Radenović [33] proved Generalized Hardy–Rogers Type α-Admissible Mappings in Cone b-Metric Spaces over Banach Algebras. Krishnakumar and Marudai [1] proved the following fixed point theorems of multi-valued mappings in cone metric spaces.

Theorem 1. *Let (X, d) be a complete cone metric space and the mapping $T\colon X \to CB(X)$ be multi-valued map satisfying for each $x, y \in X$,*

$$H(Tx, Ty) \le a[d(x, Tx) + d(y, Ty)] + b[d(x, Ty) + d(Tx, y)]$$

for all $x, y \in X$, and $a + b < \frac{1}{2}$, $a, b \in [0, \frac{1}{2})$. Then T has a fixed point in X.

Theorem 2. *Let (X, d) be a complete cone metric space and the mapping $T\colon X \to CB(X)$ be multi-valued map satisfy the condition,*

$$H(Tx, Ty) \le r \max\{d(x, y), d(x, Tx), d(y, Ty)\}$$

for all $x, y \in X$, and $r \in [0, 1)$. Then T has a fixed point in X.

Theorem 3. *Let (X, d) be a complete cone metric space and P a normal cone with normal constant K. Suppose the mapping $T\colon X \to CB(X)$ be multi valued mapping satisfying the condition*

$$H(Tx, Ty) \le r \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$, and $r \in [0, 1)$. Then T has a unique fixed point in X.

Definition 1 ([14])**.** Let E be a real Banach space. A subset P of E is called a cone whenever the following conditions hold:

(C_1) P is closed, nonempty and $P \ne \{0\}$;
(C_2) $a, b \in R$, $a, b \ge 0$ and $x, y \in P$ imply $ax + by \in P$;

(C_3) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\le$ with respect to P by $x \le y$ if and if $y - x \in P$. We shall write $x < y$ to indicate that $x \le y$ but $x \ne y$, while $x \ll y$ will stand for $y - x \in P^0$, where P^0 stands for the interior of P. If $P^0 \ne \emptyset$ then P is called a solid cone(see[23]).
There exist two kinds of cone-normal(with the normal constant K) and non-normal ones [12].
Let E be a real Banach space, $P \subset E$ a cone and $\le$ partial ordering defined by P. Then P is called normal if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \le x \le y \quad \text{imply} \quad \|x\| \le K\|y\|, \tag{1}$$

or equivalently, if $(\forall n) x_n \leq y_n \leq z_n$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \quad \text{imply} \quad \lim_{n\to\infty} y_n = x. \tag{2}$$

The least positive number K satisfying (1) is called the normal constant of P.
The cone P is called regular if every increasing sequence which is bounded above is convergent and every decreasing sequence which is bounded below is convergent.

Example 1 (see [24]). Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ on $P = \{x \in E \colon x(t) \geq 0\}$. This cone is not normal. Consider, for example, $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$. Then $0 \leq x_n \leq y_n$, and $\lim_{n\to\infty} y_n = 0$, but $\|x_n\| = \max_{t\in[0,1]} |\frac{t^n}{n}| + \max_{t\in[0,1]} |t^{n-1}| = \frac{1}{n} + 1 > 1$; hence x_n does not converge to zero. It follows by (2) that P is a non-normal cone.

Definition 2 ([14, 26]). Let X be a nonempty set. Suppose that the mapping $d \colon X \times X \to E$ satisfies:

- (d_1) $0 \leq d(x,y)$ for all $x, y \in X$ with $x \neq y$ and $d(x,y) = 0$ if and only if $x = y$;
- (d_2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
- (d_3) $d(x,y) \leq d(x,z) + d(z,y)$ $x, y, z \in X$.

Then d is called a cone metric [14] or$K$-metric [26] on X and (X,d) is called a cone metric [14] or K-metric space [26].

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, +\infty)$.

Example 2 (see [14]). Let $E = \mathbb{R}^2$, $P = \{(x,y) \in \mathbb{R}^2 \colon x \geq 0, y \geq 0\}$, $X = \mathbb{R}$ and $d \colon X \times X \to E$ defined by $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Then (X,d) is a cone metric space with normal cone P where $K = 1$.

Example 3 (see [22]). Let $E = \ell^2$, $P = \{\{x_n\}_{n\geq} \in E \colon x_n \geq 0 foralln\}$, (X,ρ) a metric space, and $d \colon X \times X \to E$ defined by $d(x,y) = \{\rho(x,y)/2^n\}_{n\geq 1}$. Then (X,d) is a cone metric space.

Clearly, the above examples show that class of cone metric spaces contains the class of metric spaces.

Definition 3 ([15]). Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d \colon X \times X \to E$ is said to be cone b-metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:

- (i) $0 \leq d(x,y)$ with $x \neq y$ and $d(x,y) = 0$ if and only if $x = y$;
- (ii) $d(x,y) = d(y,x)$;
- (iii) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

The pair (X,d) is called a cone b-metric space.

Remark 1. The class of cone b-metric spaces is larger than the class of cone metric space since any cone metric spaces must be a cone b-metric spaces. Therefore, it is obvious that cone b-metric spaces generalize b-metric spaces and cone metric spaces.

We give some examples, which show that introducing a cone b-metric space instead of a cone metric space is meaningful since there exist cone b-metric space which are not cone metric space.

Example 4 (see [13])**.** Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E \colon x \geq 0, y \geq 0\} \subset E$, $X = \mathbb{R}$ and $d \colon X \times X \to E$ defined by $d(x, y) = (|x-y|^p, \alpha|x-y|^p)$, where $\alpha \geq 0$ and $p > 1$ are two constants. Then (X, d) is a cone b-metric space with the coefficient $s = 2^p > 1$, but not a cone metric space.

Example 5 (see [13])**.** Let $X = \ell^p$ with $0 < p < 1$, where $\ell^p = \{\{x_n\} \subset \mathbb{R} \colon \sum\limits_{n=1}^{\infty} |x_n|^p < \infty\}$. Let $d \colon X \times X \to \mathbb{R}_+$ defined $d(x, y) = (\sum\limits_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}$, where $x = \{x_n\}, y = \{y_n\} \in \ell^p$. Then (X, d) is a cone b-metric space with the coefficient $s = 2^p > 1$, but not a cone metric space.

Example 6 (see [13])**.** Let $X = \{1, 2, 3, 4\}$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E \colon x \geq 0, y \geq 0\}$. Define $d \colon X \times X \to E$ by

$$d(x, y) = \begin{cases} (|x-y|^{-1}, |x-y|^{-1}), & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases} \tag{3}$$

Then (X, d) is a cone b-metric space with the coefficient $s = \frac{6}{5} > 1$. But it is not a cone metric space since the triangle inequality is not satisfied,

$$d(1, 2) > d(1, 4) + d(4, 2), \quad d(3, 4) > d(3, 1) + d(1, 4).$$

Definition 4 ([14])**.** Let (X, d) be a cone b-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then:

(i) $\{x_n\}$ is a Cauchy sequence whenever, if for every $c \in E$ with $0 \ll c$, then there is natural number N such that for all $n, m \geq N$, $d(x_n, x_m) \ll c$;

(ii) $\{x_n\}$ converges to x whenever, for every $c \in E$ with $0 \ll c$, then there is a natural number N such that for all $n \geq N$, $d(x_n, x) \ll c$. We denote this by $\lim\limits_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(iii) (X, d) is a complete cone b-metric space if every Cauchy sequence is convergent.

In the following (X, d) will stand for a cone b-metric space with respect to a cone P with $P^0 \neq \emptyset$ in a real Banach space E and $\leq$ is partial ordering in E with respect to P. The following lemmas are often used(in particular while dealing with cone metric spaces in which the cone need not be normal).

Lemma 1 ([18]). *Let P be a cone and $\{a_n\}$ be a sequence in E. If $c \in intP$ and $0 \leq a_n \to 0$ as $n \to \infty$, then there exists N such that for all $n > N$, we have $a_n \ll c$.*

Lemma 2 ([18]). *Let $x, y, z \in E$, if $x \leq y$ and $y \ll z$, then $x \ll z$.*

Lemma 3 ([15]). *Let P be a cone and if $0 \leq u \ll c$ for each $c \in intP$, then $u = 0$.*

Lemma 4 ([9]). *Let P be a cone, if $u \in P$ and $u \leq ku$ for some $0 \leq k < 1$, then $u = 0$.*

Lemma 5 ([18]). *Let P be a cone and $a \leq b + c$ for each $c \in intP$, $a \leq b$.*

Let (X, d) be a metric space. We denote by $CB(X)$ the family of nonempty closed bounded subset of X. Let H be the Hausdorff distance on $CB(X)$. That is, for $A, B \in CB(X)$,

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\},$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$ is the distance from the point a to the subset B. An element $x \in X$ is said to be a fixed point of a multi-valued mapping $T\colon X \to 2^X$ if $x \in T(X)$.

In this paper, we study the existence of fixed points for multi-valued mappings by adapting the ideas in [1] to the cone b-metric spaces setting.

2. Main results

Theorem 4. *Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$ and the mapping $T\colon X \to CB(X)$ be multi-valued map satisfying for each $x, y \in X$*

$$H(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)] + b[d(x, Ty) + d(Tx, y)]$$

for all $x, y \in X$, and $a, b \in [0, 1)$ are constants such that $2a + 2bs < 1$. Then T has a fixed point in X.

Proof. For every $x_0 \in X$ and $n \geq 1$, $x_1 \in Tx_0$ and $x_{n+1} \in Tx_n$. We have

$$\begin{aligned}
d(x_{n+1}, x_n) &\leq H(Tx_n, Tx_{n-1}) \\
&\leq a[d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})] \\
&\quad + b[d(x_n, Tx_{n-1}) + d(Tx_n, x_{n-1})] \\
&\leq a[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + b[d(x_n, x_n) + d(x_{n+1}, x_{n-1})] \\
&\leq a[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + b[d(x_{n+1}, x_{n-1})] \\
&\leq a[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \\
&\quad + bs[d(x_{n+1}, x_n) + d(x_n, x_{n-1})] \\
&\leq (a + bs)[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)],
\end{aligned}$$

$$d(x_{n+1}, x_n) \leq Ld(x_n, x_{n-1}), \tag{4}$$

where $L = \frac{a+bs}{1-(a+bs)}$. As $2a + 2bs < 1$, we obtain that $L < 1$. Similarly, we obtain

$$d(x_n, x_{n+1}) \leq Ld(x_{n-1}, x_{n-2}). \tag{5}$$

Using (5) in (4), we get

$$d(x_{n+1}, x_n) \leq L^2 d(x_n, x_{n-1}).$$

Continuing this process, we obtain

$$d(x_{n+1}, x_n) \leq L^n d(x_1, x_0).$$

For any $m \geq 1$, $p \geq 1$, we have

$$\begin{aligned}
d(x_m, x_{m+p}) &\leq s[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + sd(x_{m+1}, x_{m+p}) \\
&\leq sd(x_m, x_{m+1}) + s^2[d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^2 d(x_{m+2}, x_{m+p}) \\
&\leq sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^3 d(x_{m+2}, x_{m+3}) \\
&+ \cdots + s^{p-1} d(x_{m+p-2}, x_{m+p-1}) + s^{p-1} d(x_{m+p-1}, x_{m+p}) \\
&\leq sL^m d(x_1, x_0) + s^2 L^{m+1} d(x_1, x_0) + s^3 L^{m+2} d(x_1, x_0) \\
&+ \cdots + s^{p-1} L^{m+p-2} d(x_1, x_0) + s^p L^{m+p-1} d(x_1, x_0) \\
&= sL^m[1 + sL + s^2L^2 + s^3L^3 + \cdots + (sL)^{p-1}]d(x_1, x_0) \\
&\leq (\frac{sL^m}{1 - sL})d(x_1, x_0).
\end{aligned}$$

Let $0 \ll r$ be given. Note that $(\frac{sL^m}{1-sL})d(x_1, x_0) \to 0$ as $m \to \infty$ for any p. Making full use of ([13], Lemma 1.8), we find $m_0 \in \mathbb{N}$ such that

$$\left(\frac{sL^m}{1 - sL}\right) d(x_1, x_0) \ll r$$

for each $m > m_0$. Thus,

$$d(x_m, x_{m+p}) \leq \left(\frac{sL^m}{1 - sL}\right) d(x_1, x_0) \ll r$$

for all $m \geq 1$, $p \geq 1$. Therefore,$\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete cone b-metric space, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Take $n_0 \in \mathbb{N}$ such that $d(x_n, z) \ll r\frac{1-as-bs}{s(1+b)}$ for all $n > n_0$. Hence,

$$\begin{aligned}
d(z, Tz) &\leq s[d(z, x_{n+1}) + d(x_{n+1}, Tz)] \\
&\leq sd(z, Tx_n) + sH(Tx_n, Tz) \\
&\leq sd(z, x_{n+1}) + s[a(d(x_n, Tx_n) + d(z, Tz)) \\
&+ b(d(x_n, Tz) + d(Tx_n, z))]
\end{aligned}$$

$$\leq sd(z, x_{n+1}) + s[a(d(x_n, x_{n+1}) + d(z, T(z))) + b(d(x_n, Tz) + d(x_{n+1}, z))].$$

This implies that

$$d(z, Tz) \leq \left(\frac{s(1+b)}{1 - as - bs}\right) d(x_n, z) \ll r,$$

for $n > n_0$. Then, by Lemma (1.10), we deduce that $d(Tz, z) = 0$, that is $Tz = z$. □

Example 7. Let $X = [0, 1]$ endowed with the standard order and $E = C^1_R[0, 1]$ with $\|u\| = \|u\|_\infty + \|u'\|_\infty$, $u \in E$ and let $P = \{u \in E : u(t) \geq 0 \quad \text{on} \quad [0, 1]\}$. It is well known that this cone is solid, but it is not normal. Define a cone b-metric $d : X \times X \to E$ by $d(x, y)(t) = |x - y|^2 \exp^t$. Then (X, d) is a complete cone b-metric space with the coefficient $s = 2$. Define $T : X \to CB(X)$ by

$$T(x) = \begin{cases} \{\frac{1}{3}, \frac{2}{3}\}, & \text{if} \quad 0 \leq x < 1, \\ \{\frac{1}{3}\}, & \text{if} \quad x = 1. \end{cases} \tag{6}$$

Let $x, y \in X$. Without loss of generality, take $x \leq y$.
If $x = y$ or $x, y < 1$, then $Tx = Ty$. Hence $H(Tx, Ty) = 0$.
If $x < 1$ and $y = 1$, then

$$\begin{aligned} H(Tx, Ty) &= \frac{1}{9} \exp^t \\ &\leq \frac{4}{27} \exp^t \\ &= \frac{1}{3} . \frac{4}{9} \exp^t \\ &= \frac{1}{3}(d(x, Tx) + d(y, Ty)) \\ &\leq b(d(x, Tx) + d(y, Ty)) \end{aligned}$$

where $b = \frac{1}{3} \in [0, 1)$ and $a = 0$. So all the conditions of Theorem 2.1 are satisfied. Moreover, $\frac{1}{3}$ and $\frac{2}{3}$ are the two fixed points of T.

Corollary 1. *Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$ and the mapping $T \colon X \to CB(X)$ be multi valued map satisfies condition*

$$d(Tx, Ty) \leq b(d(x, Ty) + d(x, Ty))$$

for all $x, y \in X$, where $b \in [0, \frac{1}{2s})$ is a constant. Then T has a fixed point in X.

Proof. The proof of the corollary immediately follows by putting $a = 0$ in the previous theorem. □

Corollary 2. *Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$ and the mapping $T\colon X \to CB(X)$ be multi valued map satisfies condition*

$$d(Tx, Ty) \leq a(d(x, Tx) + d(y, Ty))$$

for all $x, y \in X$, where $a \in [0, \frac{1}{2s})$ is a constant. Then T has a fixed point in X.

Proof. The proof of the corollary immediately follows by putting $b = 0$ in the previous theorem. □

Theorem 5. *Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$ and the mapping $T\colon X \to CB(X)$ be multi valued map satisfy the condition, $H(Tx, Ty) \leq r \max\{d(x, y), d(x, Tx), d(y, Ty)\}$ for all $x, y \in X$, and $r \in [0, 1)$. Then T has a unique fixed point in X.*

Proof. For every $x_0 \in X$ and $n \geq 1$, $x_1 \in Tx_0$ and $x_{n+1} \in Tx_n$

$$\begin{aligned}
d(x_{n+1}, x_n) &\leq H(Tx_n, Tx_{n-1}) \\
&\leq r \max\{d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1})\} \\
&\leq r \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \\
&\leq rd(x_{n-1}, x_n) \\
&\leq r^n d(x_1, x_0)
\end{aligned}$$

For any $m \geq 1$, $p \geq 1$, we have

$$\begin{aligned}
d(x_m, x_{m+p}) &\leq s[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + sd(x_{m+1}, x_{m+p}) \\
&\leq sd(x_m, x_{m+1}) + s^2[d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^2 d(x_{m+2}, x_{m+p}) \\
&\leq sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^3 d(x_{m+2}, x_{m+3}) \\
&+ \cdots + s^{p-1} d(x_{m+p-2}, x_{m+p-1}) + s^{p-1} d(x_{m+p-1}, x_{m+p}) \\
&\leq sr^m d(x_1, x_0) + s^2 r^{m+1} d(x_1, x_0) + s^3 r^{m+2} d(x_1, x_0) \\
&+ \cdots + s^{p-1} r^{m+p-2} d(x_1, x_0) + s^p r^{m+p-1} d(x_1, x_0) \\
&= sr^m[1 + sr + s^2r^2 + s^3r^3 + \cdots + (sr)^{p-1}] d(x_1, x_0) \\
&\leq \left(\frac{sr^m}{1 - sr}\right) d(x_1, x_0).
\end{aligned}$$

Let $0 \ll r$ be given. Note that $(\frac{sr^m}{1-sr}) d(x_1, x_0) \to 0$ as $m \to \infty$ for any p. Making full use of ([13], Lemma 1.8), we find $m_0 \in \mathbb{N}$ such that

$$\left(\frac{sr^m}{1 - sr}\right) d(x_1, x_0) \ll c$$

for each $m > m_0$. Thus,

$$d(x_m, x_{m+p}) \leq \left(\frac{sr^m}{1-sr}\right) d(x_1, x_0) \ll c$$

for all $m \geq 1$, $p \geq 1$. Therefore, $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete cone b-metric space, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Take $n_0 \in \mathbb{N}$ such that $d(x_n, z) \ll c\frac{1-s}{s}$ for all $n > n_0$. Hence,

$$\begin{aligned}
d(z, Tz) &\leq s[d(z, x_{n+1}) + d(x_{n+1}, Tz)] \\
&\leq sd(z, Tx_n) + sH(Tx_n, Tz) \\
&\leq sd(z, x_{n+1}) + s[\max\{d(x_n, z), d(x_n, Tx_n), d(z, Tz)\}] \\
&\leq sd(z, x_{n+1}) + s[\max\{0, d(x_n, x_{n+1}), d(z, Tz)\}] \\
&\leq sd(z, x_{n+1}) + s[\max\{0, 0, d(z, Tz)\}] \\
&\leq sd(z, x_n) + sd(z, Tz).
\end{aligned}$$

This implies that

$$d(z, Tz) \leq \left(\frac{s}{1-s}\right) d(x_n, z) \ll c,$$

for $n > n_0$. Then, by Lemma (1.10), we deduce that $d(Tz, z) = 0$,
that is $Tz = z$.
Assume that there is another fixed point q in X such that $Tq = q$.

$$\begin{aligned}
\therefore d(z, q) &\leq H(Tz, Tq) \\
&\leq r \max\{d(z, q), d(z, Tz), d(q, Tq), d(z, Tq), d(q, Tz)\} \\
&\leq r \max\{d(z, q), d(z, z), d(q, q), d(z, q), d(q, z)\} \\
&\leq rd(z, q)
\end{aligned}$$

This is contradiction and hence T has a unique fixed point in X. □

Example 8. Let $X = [0, \infty)$ endowed with the standard order and $E = C_R^1[0, 1]$ with $\|u\| = \|u\|_\infty + \|u'\|_\infty$, $u \in E$ and let $P = \{u \in E : u(t) \geq 0 on[0, 1]\}$. It is well known that this cone is solid, but it is not normal. Define a cone metric $d : X \times X \to E$ by $d(x, y)(t) = |x - y|^2 \exp^t$. Then (X, d) is a complete cone b-metric space with the coefficient $s = 2$. Define $T : X \to CB(X)$ by

$$T(x) = \begin{cases} \{\frac{2}{3}\}, & \text{if } \ 0 \leq x < 1, \\ \{\frac{1}{3}\}, & \text{if } \ x > 1. \end{cases} \tag{7}$$

Let $x, y \in X$. Without loss of generality, take $x \leq y$.
If $x = y$ or $x, y < 1$, then $Tx = Ty$. Hence $H(Tx, Ty) = 0$.
If $x < 1$ and $y = 1$, then

$$\begin{aligned}
H(Tx, Ty) &= \frac{1}{9}\exp^t \\
&\leq \frac{4}{27}\exp^t \\
&= \frac{1}{3}\cdot\frac{4}{9}\exp^t \\
&= \frac{1}{3}d(y, Ty) \\
&\leq r\max\{d(x, y), d(x, Tx), d(y, Ty)\}
\end{aligned}$$

where $r = \frac{1}{3} \in [0, 1)$. So all the conditions of Theorem 2.5 are satisfied. Moreover, 0 is a unique fixed point of T.

Corollary 3. *Let(X, d) be a complete cone b-metric space with the coefficient $s \geq 1$ and the mapping $T\colon X \to CB(X)$ be multi valued mapping satisfy the condition*

$$H(Tx, Ty) \leq kd(x, y) \tag{8}$$

for all $x, y \in X$ where $k \in [0, \frac{1}{2s})$ is a constant. Then T has a unique fixed point in X.

Proof. The proof of the corollary immediately follows by taking $d(x, y)$ as maximum value in the previous theorem. □

We prove the above theorems in the setting of P is a normal cone with normal constant K.

Theorem 6. *Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$ and P a normal cone with normal constant K. Suppose the mapping $T\colon X \to CB(X)$ be multi valued mapping satisfying the condition, $H(Tx, Ty) \leq r\max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$ for all $x, y \in X$, and $r \in [0, 1), 2sr < 1$. Then T has a unique fixed point in X.*

Proof. For every $x_0 \in X$ and $n \geq 1$, $x_1 \in Tx_0$ and $x_{n+1} \in Tx_n$,

$$\begin{aligned}
d(x_{n+1}, x_n) &\leq H(Tx_n, Tx_{n-1}) \\
&\leq r\max\{d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1}), \\
&\qquad\qquad d(x_n, Tx_{n-1}), d(x_{n-1}, Tx_n)\} \\
&\leq r\max\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_n), \\
&\qquad\qquad d(x_{n+1}, x_{n-1})\} \\
&\leq r\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n+1}, x_{n-1})\}.
\end{aligned}$$

Case (i)

If $d(x_{n+1}, x_n) \leq rd(x_n, x_{n-1})$ then we get, $d(x_{n+1}, x_n) \leq r^n d(x_1, x_0)$.
For any $m \geq 1$, $p \geq 1$, we have

$$
\begin{aligned}
d(x_m, x_{m+p}) &\leq s[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + sd(x_{m+1}, x_{m+p}) \\
&\leq sd(x_m, x_{m+1}) + s^2[d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^2 d(x_{m+2}, x_{m+p}) \\
&\leq sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^3 d(x_{m+2}, x_{m+3}) \\
&\quad + \cdots + s^{p-1} d(x_{m+p-2}, x_{m+p-1}) + s^{p-1} d(x_{m+p-1}, x_{m+p}) \\
&\leq sr^m d(x_1, x_0) + s^2 r^{m+1} d(x_1, x_0) + s^3 r^{m+2} d(x_1, x_0) \\
&\quad + \cdots + s^{p-1} r^{m+p-2} d(x_1, x_0) + s^p r^{m+p-1} d(x_1, x_0) \\
&= sr^m[1 + sr + s^2 r^2 + s^3 r^3 + \cdots + (sr)^{p-1}] d(x_1, x_0) \\
&\leq \left(\frac{sr^m}{1 - sr}\right) d(x_1, x_0).
\end{aligned}
$$

We get $\|d(x_m, x_m + p)\| \leq K(\frac{sr^m}{1-sr})\|d(x_1, x_0)\|.d(x_m, x_m + p) \to 0$ as $p, m \to \infty$. Hence $\{x_m\}$ is a Cauchy sequence. By the completeness of X, there is $z \in X$ such that $x_m \to z$ as $m \to \infty$.

$$
\begin{aligned}
d(z, Tz) &\leq s[d(z, x_{n+1}) + d(x_{n+1}, Tz)] \\
&\leq sd(z, Tx_n) + sH(Tx_n, Tz) \\
&\leq sd(z, x_{n+1}) + s[r \max\{d(x_n, z), d(x_n, Tx_n), \\
&\qquad\qquad d(z, Tz), d(x_n, Tz), d(z, Tx_n)\}] \\
&\leq sd(z, x_{n+1}) + s[r \max\{0, d(x_n, x_{n+1}), d(z, Tz), \\
&\qquad\qquad d(x_n, Tz), d(z, x_{n+1})\}] \\
&\leq sd(z, x_{n+1}) + s[r \max\{0, 0, d(z, Tz)\}] \\
&\leq sd(z, x_n) + srd(z, Tz) \\
&\leq srd(z, Tz),
\end{aligned}
$$

which implies that $d(Tz, z) = 0$. Hence $z \in Tz$.

Case (ii)

If $d(x_{n+1}, x_n) \leq rd(x_{n+1}, x_{n-1})$ then we get

$$
\begin{aligned}
d(x_{n+1}, x_n) &\leq rs[d(x_{n+1}, x_n) + d(x_n, x_{n-1})] \\
&\leq \frac{sr}{1 - sr} d(x_n, x_{n-1}) \\
&\leq hd(x_n, x_{n-1}), \quad \text{where} \quad h = \frac{sr}{1 - sr} < 1.
\end{aligned}
$$

For any $m \geq 1$, $p \geq 1$, we have

$$
\begin{aligned}
d(x_m, x_{m+p}) &\leq s[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + sd(x_{m+1}, x_{m+p}) \\
&\leq sd(x_m, x_{m+1}) + s^2[d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+p})]
\end{aligned}
$$

$$\begin{aligned}
&= sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^2 d(x_{m+2}, x_{m+p}) \\
&\leq sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^3 d(x_{m+2}, x_{m+3}) \\
&\quad + \cdots + s^{p-1} d(x_{m+p-2}, x_{m+p-1}) + s^{p-1} d(x_{m+p-1}, x_{m+p}) \\
&\leq sh^m d(x_1, x_0) + s^2 h^{m+1} d(x_1, x_0) + s^3 h^{m+2} d(x_1, x_0) \\
&\quad + \cdots + s^{p-1} h^{m+p-2} d(x_1, x_0) + s^p h^{m+p-1} d(x_1, x_0) \\
&= sh^m [1 + sh + s^2h^2 + s^3h^3 + \cdots + (sh)^{p-1}] d(x_1, x_0) \\
&\leq \left(\frac{sh^m}{1 - sh} \right) d(x_1, x_0).
\end{aligned}$$

We get $\|d(x_m, x_m + p)\| \leq K(\frac{sh^m}{1-sh})\|d(x_1, x_0)\|$. $d(x_m, x_m + p) \to 0$ as $p, m \to \infty$. Hence $\{x_m\}$ is a Cauchy sequence. By the completeness of X, there is $z \in X$ such that $x_m \to z$ as $m \to \infty$.

$$\begin{aligned}
d(z, Tz) &\leq s[d(z, x_{n+1}) + d(x_{n+1}, Tz)] \\
&\leq sd(z, Tx_n) + sH(Tx_n, Tz) \\
&\leq sd(z, x_{n+1}) + s[r \max\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), \\
&\qquad\qquad d(x_n, Tz), d(z, Tx_n)\}] \\
&\leq sd(z, x_{n+1}) + s[r \max\{0, d(x_n, x_{n+1}), d(z, Tz), d(x_n, Tz), \\
&\qquad\qquad d(z, x_{n+1})\}] \\
&\leq sd(z, x_{n+1}) + s[r \max\{0, 0, d(z, Tz)\}] \\
&\leq sd(z, x_n) + srd(z, Tz) \\
&\leq srd(z, Tz) \\
d(Tz, z) &= 0.
\end{aligned}$$

Hence $z \in Tz$.
Assume that there is another fixed point q in X such that $Tq = q$.

$$\begin{aligned}
\therefore d(z, q) &\leq H(Tz, Tq) \\
&\leq r \max\{d(z, q), d(z, Tz), d(q, Tq), d(z, Tq), d(q, Tz)\} \\
&\leq r \max\{d(z, q), d(z, z), d(q, q), d(z, q), d(q, z)\} \\
&\leq rd(z, q)
\end{aligned}$$

This is contradiction and hence T has a unique fixed point in X. □

Example 9. Let $X = [0, 1]$, $E = \mathbb{R}^2$. Take $P = \{(x, y) \in E : x, y \geq 0\}$. We define $d\colon X \times X \to E$ as $d(x, y) = (|x - y|^2, |x - y|^2)$ for all $x, y \in X$. Then (X, d) is a complete cone b-metric. Let us define $T\colon X \to CB(X)$ as

$$T(x) = \begin{cases} \{\frac{2}{5}\}, & \text{if } \ 0 \leq x < 1, \\ \{\frac{1}{5}\}, & \text{if } \ x = 1. \end{cases} \tag{9}$$

Let $x, y \in X$. Without loss of generality, take $x \leq y$.

If $x = y$ or $x, y < 1$, then $Tx = Ty$. Hence $H(Tx, Ty) = 0$.
If $x < 1$ and $y = 1$, then

$$\begin{aligned} H(Tx,Ty) &= (\tfrac{1}{25}, \tfrac{1}{25}) \leq (\tfrac{16}{125}, \tfrac{16}{125}) = \tfrac{1}{5}(\tfrac{1}{25}, \tfrac{1}{25}) = \tfrac{1}{5} d(y, Ty) \\ &\leq r \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}, \end{aligned}$$

where $r = \frac{1}{5} \in [0, 1)$. So all the conditions of Theorem 2.8 are satisfied. Moreover, $\frac{2}{5}$ is a unique fixed point of T.

Corollary 4. *Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$ and P a normal cone with normal constant K. Suppose the mapping $T\colon X \to CB(X)$ be multi-valued mapping satisfies the condition, $H(Tx, Ty) \leq r \max\{d(x,y), d(x,Tx), d(y,Ty)\}$ for all $x, y \in X$, and $r \in [0, 1)$. Then T has a unique fixed point in X.*

Proof. The proof of the corollary immediately follows since

$$\max\{d(x,y), d(x,Tx), d(y,Ty)\} \leq \\ \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$ □

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