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# New inequalities for $F$-convex functions pertaining generalized fractional integrals

Hüseyİn Budak, Pinar Kösem, Artion Kashuri

Abstract. In this paper, the authors, utilizing $F$-convex functions which are defined by B. Samet, establish some new Hermite-Hadamard type inequalities via generalized fractional integrals. Some special cases of our main results recaptured the well-known earlier works.

## 1. Introduction

Let $f : I \subseteq R \to R$ be a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. If $f$ is a convex function then the following double inequality, which is well known in the literature as the Hermite–Hadamard inequality, holds [17]:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities in (1) hold in the reversed direction if $f$ is concave.

Over the last decade, this classical double inequality has been improved and generalized in a number of ways, see [5, 7, 8, 13, 18], [23]–[25] and the references therein. Also, many types of convexities have been defined, such as quasi–convex in [6], pseudo–convex in [14], strongly convex in [20], $\varepsilon$–convex in [11], $s$–convex in [10], $h$–convex in [28], etc. Recently, Samet in [21], has defined a new concept of convexity that depends on a certain function satisfying some axioms, that generalizes different types of convexity.

Recall the family $\mathcal{F}$ of mappings $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \to \mathbb{R}$ satisfying the following axioms:

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(A1) If $e_i \in L^1(0,1)$, $i = 1,2,3$, then for every $\lambda \in [0,1]$, we have

$$\int_0^1 F(e_1(t), e_2(t), e_3(t), \lambda)\mathrm{d}t = F\left(\int_0^1 e_1(t)\mathrm{d}t, \int_0^1 e_2(t)\mathrm{d}t, \int_0^1 e_3(t)\mathrm{d}t, \lambda\right);$$

(A2) For every $u \in L^1(0,1)$, $w \in L^\infty(0,1)$ and $(z_1, z_2) \in \mathbb{R}^2$, we have

$$\int_0^1 F(w(t)u(t), w(t)z_1, w(t)z_2, t)\mathrm{d}t = T_{F,w}\left(\int_0^1 w(t)u(t)\mathrm{d}t, z_1, z_2\right),$$

where $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function that depends on $(F, w)$, and it is nondecreasing with respect to the first variable;

(A3) For any $(w, e_1, e_2, e_3) \in \mathbb{R}^4$, $e_4 \in [0,1]$, we have

$$wF(e_1, e_2, e_3, e_4) = F(we_1, we_2, we_3, e_4) + L_w,$$

where $L_w \in \mathbb{R}$ is a constant that depends only on $w$.

**Definition 1.** Let $f : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, be a given function. We say that $f$ is a convex function with respect to some $F \in \mathcal{F}$ (or $F$-convex function), if and only if:

$$F(f(tx + (1-t)y), f(x), f(y), t) \le 0, \quad (x,y,t) \in [a,b] \times [a,b] \times [0,1].$$

**Remark 1.** 1) Let $\varepsilon \ge 0$, and let $f : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, be an $\varepsilon$-convex function, see [11], that is

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \varepsilon, \ \ (x,y,t) \in [a,b] \times [a,b] \times [0,1].$$

Define the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}$ by

$$F(e_1, e_2, e_3, e_4) = e_1 - e_4 e_2 - (1 - e_4)e_3 - \varepsilon \tag{2}$$

and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$T_{F,w}(e_1, e_2, e_3) = e_1 - \left(\int_0^1 tw(t)\mathrm{d}t\right) e_2 - \left(\int_0^1 (1-t)w(t)\mathrm{d}t\right) e_3 - \varepsilon. \tag{3}$$

For

$$L_w = (1 - w)\varepsilon, \tag{4}$$

it is clear that $F \in \mathcal{F}$ and

$$F(f(tx+(1-t)y), f(x), f(y), t) = f(tx+(1-t)y) - tf(x) - (1-t)f(y) - \varepsilon \le 0,$$

that is $f$ is an $F$–convex function. Particularly, taking $\varepsilon = 0$, we show that if $f$ is a convex function then $f$ is an $F$–convex function with respect to $F$ defined above.

2) Let $h : J \to [0, +\infty)$ be a given function which is not identical to 0, where $J$ is an interval in $\mathbb{R}$ such that $(0, 1) \subseteq J$. Let $f : [a, b] \to [0, +\infty)$, $(a, b) \in \mathbb{R}^2$, $a < b$, be an $h$-convex function, see [28], that is

$$f(tx + (1 - t)y) \leq h(t)f(x) + h(1 - t)f(y), \quad (x, y, t) \in [a, b] \times [a, b] \times [0, 1] .$$

Define the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R}\times [0, 1] \to \mathbb{R}$ by

$$F(e_1, e_2, e_3, e_4) = e_1 - h(e_4)e_2 - h(1 - e_4)e_3 \tag{5}$$

and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$T_{F,w}(e_1, e_2, e_3) = e_1 - \left(\int_0^1 h(t)w(t)\mathrm{d}t\right) e_2 - \left(\int_0^1 h(1 - t)w(t)\mathrm{d}t\right) e_3. \tag{6}$$

For $L_w = 0$, it is clear that $F \in \mathcal{F}$ and

$$F(f(tx+(1-t)y), f(x), f(y), t) = f(tx+(1-t)y)-h(t)f(x)-h(1-t)f(y) \leq 0,$$

that is, $f$ is an $F$–convex function.

Samet in [21], established the following Hermite–Hadamard type inequalities using the new convexity concept:

**Theorem 1.** *Let $f : [a, b] \to \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, be an $F$-convex function, for some $F \in \mathcal{F}$. Suppose that $f \in L^1[a, b]$. Then*

$$F\left(f\left(\frac{a + b}{2}\right), \frac{1}{b - a}\int_a^b f(x)\mathrm{d}x, \frac{1}{b - a}\int_a^b f(x)\mathrm{d}x, \frac{1}{2}\right) \leq 0,$$

$$T_{F,1}\left(\frac{1}{b - a}\int_a^b f(x)\mathrm{d}x, \ f(a), \ f(b)\right) \leq 0.$$

**Definition 2.** Let $f \in L^1[a, b]$. The Riemann–Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x - t)^{\alpha-1} f(t)\mathrm{d}t, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t - x)^{\alpha-1} f(t)\mathrm{d}t, \quad x < b,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and

$$J^0_{a+}f(x) = J^0_{b-}f(x) = f(x).$$

**Definition 3.** Let $f \in L^1[a, b]$. Then $k$–fractional integrals of order $\alpha, k > 0$ are defined by

$$I^{\alpha}_{a^+,k}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x - t)^{\frac{\alpha}{k}-1}f(t)\mathrm{d}t, \quad x > a,$$

and

$$I^{\alpha}_{b^-,k}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^b (t-x)^{\frac{\alpha}{k}-1}f(t)\mathrm{d}t, \quad b > x, \tag{7}$$

where $\Gamma_k(\cdot)$ stands for the $k$-gamma function. For $k = 1$, the $k$–fractional integrals yield Riemann–Liouville integrals. For $\alpha = k = 1$, the $k$–fractional integrals yield classical integrals. For more details, see [9, 12, 15, 19].

It is remarkable that Sarikaya et al. in [26], first give the following interesting integral inequalities of Hermite–Hadamard type involving Riemann–Liouville fractional integrals.

**Theorem 2.** *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L^1[a, b]$. If $f$ is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right] \le \frac{f(a)+f(b)}{2}, \tag{8}$$

*with $\alpha > 0$.*

Budak et al. in [1], prove the following Hermite-Hadaamrd type inequalities for $F$-convex functions via fractional integrals:

**Theorem 3.** *Let $I \subseteq R$ be an interval, $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping on $I^{\circ}$, $a, b \in I^{\circ}$, $a < b$. If $f$ is $F$-convex on $[a, b]$ for some $F \in \mathcal{F}$, then we have*

$$\begin{aligned} &F\left(f\left(\frac{a+b}{2}\right), \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J^{\alpha}_{a+}f(b), \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J^{\alpha}_{b-}f(a), \frac{1}{2}\right) \\ &+ \int_0^1 L_{w(t)}\mathrm{d}t \le 0, \end{aligned} \tag{9}$$

*and*

$$\begin{aligned} &T_{F,w}\left(\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right], f(a)+f(b), f(a)+f(b)\right) \\ &+ \int_0^1 L_{w(t)}\mathrm{d}t \le 0, \end{aligned} \tag{10}$$

*where $w(t) = \alpha t^{\alpha-1}$.*

For other papers involving $F$-convex functions, see [1]-[4], [16, 27].

Now we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [22].

Let's define a function $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

$$\int_0^1 \frac{\varphi(t)}{t}\mathrm{d}t < +\infty, \tag{11}$$

(12) $$\frac{1}{A_1} \le \frac{\varphi(v)}{\phi(u)} \le A_1 \text{ for } \frac{1}{2} \le \frac{v}{u} \le 2,$$

(13) $$\frac{\varphi(u)}{u^2} \le A_2 \frac{\varphi(v)}{v^2} \text{ for } v \le u,$$

(14) $$\left| \frac{\varphi(u)}{u^2} - \frac{\varphi(v)}{v^2} \right| \le A_3 |u-v| \frac{\varphi(u)}{u^2} \text{ for } \frac{1}{2} \le \frac{v}{u} \le 2,$$

where $A_1, A_2, A_3 > 0$ are independent of $u, v > 0$. If $\varphi(u)u^\alpha$ is increasing for some $\alpha \ge 0$ and $\frac{\varphi(u)}{u^\beta}$ is decreasing for some $\beta \ge 0$, then $\varphi$ satisfies the above conditions.

The following left-sided and right–sided generalized fractional integral operators are defined respectively, as follows:

(15) $${}_{a^+}I_\varphi f(x) = \int_a^x \frac{\varphi(x-t)}{x-t} f(t) \mathrm{d}t, \quad x > a,$$

(16) $${}_{b^-}I_\varphi f(x) = \int_x^b \frac{\varphi(t-x)}{t-x} f(t) \mathrm{d}t, \quad x < b.$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann–Liouville fractional integral, $k$–Riemann–Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc.

Sarikaya and Ertuğral in [22], establish the following Hermite–Hadamard inequality and lemmas for the generalized fractional integral operators:

**Theorem 4.** *Let $f : [a, b] \to \mathbb{R}$ be a convex function on $[a, b]$ with $a < b$, then the following inequalities for fractional integral operators hold:*

(17) $$f\left(\frac{a+b}{2}\right) \le \frac{1}{2\Psi(1)} \left[{}_{a+}I_\varphi f(b) + {}_{b-} I_\varphi f(a)\right] \le \frac{f(a)+f(b)}{2},$$

*where the mapping $\Lambda : [0, 1] \to \mathbb{R}$ is defined by*

$$\Psi(x) = \int_0^x \frac{\varphi((b-a)t)}{t} \mathrm{d}t.$$

Budak et al. prove the following Hermite Hadamard type inequalities for $F$-convex functions.

**Theorem 5** ([4])**.** *Let $I \subseteq \mathbb{R}$ be an interval, $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping on $I^\circ$, $a, b \in I^\circ$, $a < b$. If $f$ is $F$-convex on $[a, b]$ for some $F \in \mathcal{F}$, then we have*

$$F\left(f\left(\frac{a+b}{2}\right), \frac{1}{\Psi(1)}\ {}_{a+}I_\varphi f(b), \frac{1}{\Psi(1)}\ {}_{b-}I_\varphi f(a), \frac{1}{2}\right) + \int_0^1 L_{w(t)} \mathrm{d}t \le 0,$$

*and*

$$T_{F,w}\left(\frac{1}{\Psi(1)}\left[{}_{a+}I_{\varphi}f(b)+{}_{b-}I_{\varphi}f(a)\right],f(a)+f(b),f(a)+f(b)\right)+\int_{0}^{1}L_{w(t)}\mathrm{d}t\leq 0,$$

*where* $w(t)=\frac{\varphi((b-a)t)}{t\Psi(1)}$.

Motivated by the above literatures, the main objective of this article is to establish some new Hermite–Hadamard type inequalities via generalized fractional integrals utilizing $F$–convex functions. Some special cases of our main results recaptured the well–known earlier works. At the end, a briefly conclusion will be given as well.

## 2. Main results

In this section, we establish some inequalities of Hermite–Hadamard type including generalized fractional integrals via $F$–convex functions.

**Theorem 6.** *Let $I\subseteq\mathbb{R}$ be an interval, $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a mapping on $I^{\circ}$, $a,b\in I^{\circ}$, $a<b$ and let $F$ be linear with respect to the first three variables. If $f$ is $F$-convex on $[a,b]$ for some $F\in\mathcal{F}$, then we have*

$$F\left(f\left(\frac{a+b}{2}\right),\frac{1}{\Lambda(1)}\,{}_{\left(\frac{a+b}{2}\right)^{+}}I_{\varphi}f\left(b\right),\frac{1}{\Lambda(1)}\,{}_{\left(\frac{a+b}{2}\right)^{-}}I_{\varphi}f\left(a\right),\frac{1}{2}\right)+\int_{0}^{1}L_{w(t)}\mathrm{d}t\leq 0, \tag{18}$$

*and*

$$T_{F,w}\left(\frac{1}{\Lambda(1)}\left[{}_{\left(\frac{a+b}{2}\right)^{+}}I_{\varphi}f\left(b\right)+{}_{\left(\frac{a+b}{2}\right)^{-}}I_{\varphi}f\left(a\right)\right],f(a)+f(b),f(a)+f(b)\right)+\int_{0}^{1}L_{w(t)}\mathrm{d}t\leq 0, \tag{19}$$

*where* $w(t)=\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}$ *and the function* $\Lambda:[0,1]\to\mathbb{R}$ *is defined by*

$$\Lambda(x)=\int_{0}^{x}\frac{\varphi\left(\frac{b-a}{2}t\right)}{t}\mathrm{d}t.$$

*Proof.* Since $f$ is $F$–convex, we have

$$F\left(f\left(\frac{x+y}{2}\right),f(x),f(y),\frac{1}{2}\right)\leq 0,\quad\forall x,y\in\left[a,b\right].$$

For
$$x=\frac{t}{2}a+\left(\frac{2-t}{2}\right)b \text{ and } y=\left(\frac{2-t}{2}\right)a+\frac{t}{2}b,$$
we have
$$F\left(f\left(\frac{a+b}{2}\right),f\left(\frac{t}{2}a+\left(\frac{2-t}{2}\right)b\right),f\left(\left(\frac{2-t}{2}\right)a+\frac{t}{2}b\right),\frac{1}{2}\right)\leq 0.$$
for all $t\in[0,1]$. Multiplying this inequality by $w(t)=\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}$ and using axiom (A3), we get
$$F\left(\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}f\left(\frac{a+b}{2}\right),\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}f\left(\frac{t}{2}a+\left(\frac{2-t}{2}\right)b\right),\right.$$
$$\left.\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}f\left(\left(\frac{2-t}{2}\right)a+\frac{t}{2}b\right),\frac{1}{2}\right)+L_{w(t)}\leq 0$$
for all $t\in(0,1)$. Integrating over $(0,1)$ with respect to the variable $t$ and using axiom (A1), we obtain
$$F\left(\frac{f\left(\frac{a+b}{2}\right)}{\Lambda(1)}\int_0^1\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t}\mathrm{d}t,\frac{1}{\Lambda(1)}\int_0^1\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t}f\left(\frac{t}{2}a+\left(\frac{2-t}{2}\right)b\right)\mathrm{d}t,\right.$$
$$\left.\frac{1}{\Lambda(1)}\int_0^1\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t}f\left(\left(\frac{2-t}{2}\right)a+\frac{t}{2}b\right)\mathrm{d}t,\frac{1}{2}\right)+\int_0^1L_{w(t)}\mathrm{d}t\leq 0.$$
Using the facts that
$$\int_0^1\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t}f\left(\frac{t}{2}a+\left(\frac{2-t}{2}\right)b\right)\mathrm{d}t$$
$$=\int_{\frac{a+b}{2}}^b\frac{\varphi(b-x)}{b-x}f(x)\mathrm{d}x={}_{\left(\frac{a+b}{2}\right)^+}I_\varphi f(b)$$
and
$$\int_0^1\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t}f\left(\left(\frac{2-t}{2}\right)a+\frac{t}{2}b\right)\mathrm{d}t$$
$$=\int_a^{\frac{a+b}{2}}\frac{\varphi(x-a)}{x-a}f(x)\mathrm{d}x={}_{\left(\frac{a+b}{2}\right)^-}I_\varphi f(a),$$
we obtain
$$F\left(f\left(\frac{a+b}{2}\right),\frac{1}{\Lambda(1)}\,{}_{\left(\frac{a+b}{2}\right)^+}I_\varphi f(b),\frac{1}{\Lambda(1)}\,{}_{\left(\frac{a+b}{2}\right)^-}I_\varphi f(a),\frac{1}{2}\right)$$
$$+\int_0^1L_{w(t)}\mathrm{d}t\leq 0,$$

which gives (18).

On the other hand, since $f$ is $F$–convex, we have

$$F\left(f\left(\frac{t}{2}a+\left(\frac{2-t}{2}\right)b\right),f(a),f(b),t\right)\leq 0,\quad \forall t\in[0,1],$$

and

$$F\left(f\left(\left(\frac{2-t}{2}\right)a+\frac{t}{2}b\right),f(a),f(b),t\right)\leq 0,\quad \forall t\in[0,1].$$

Using the linearity of $F$, we get

$$\begin{aligned}F\Bigg(&f\left(\frac{t}{2}a+\left(\frac{2-t}{2}\right)b\right)+f\left(\left(\frac{2-t}{2}\right)a+\frac{t}{2}b\right),\\&f(a)+f(b),f(a)+f(b),t\Bigg)\leq 0,\end{aligned}$$

for all $t\in[0,1]$. Applying the axiom (A3) for $w(t)=\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}$, we obtain

$$\begin{aligned}F\Bigg(&\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}\left[f\left(\frac{t}{2}a+\left(\frac{2-t}{2}\right)b\right)+f\left(\left(\frac{2-t}{2}\right)a+\frac{t}{2}b\right)\right],\\&\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}\left[f(a)+f(b)\right],\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}\left[f(a)+f(b)\right],t\Bigg)+L_{w(t)}\leq 0,\end{aligned}$$

for all $t\in(0,1)$. Integrating over $(0,1)$ and using axiom (A2), we have

$$\begin{aligned}T_{F,w}\Bigg(&\int_0^1\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}\left[f\left(\frac{t}{2}a+\left(\frac{2-t}{2}\right)b\right)+f\left(\left(\frac{2-t}{2}\right)a+\frac{t}{2}b\right)\right]\mathrm{d}t,\\&f(a)+f(b),f(a)+f(b)\Bigg)+\int_0^1 L_{w(t)}\mathrm{d}t\leq 0,\end{aligned}$$

that is

$$\begin{aligned}T_{F,w}&\left(\frac{1}{\Lambda(1)}\left[{}_{\left(\frac{a+b}{2}\right)^+}I_{\varphi}f(b)+{}_{\left(\frac{a+b}{2}\right)^-}I_{\varphi}f(a)\right],f(a)+f(b),f(a)+f(b)\right)\\&+\int_0^1 L_{w(t)}\mathrm{d}t\leq 0.\end{aligned}$$

The proof of Theorem 6 is completed. □

**Remark 2.** If we choose $\varphi(t)=t$ in Theorem 6, then we have the following inequalities

$$F\left(f\left(\frac{a+b}{2}\right),\frac{2}{b-a}\int\limits_{\frac{a+b}{2}}^{b}f(t)\mathrm{d}t,\frac{2}{b-a}\int\limits_{a}^{\frac{a+b}{2}}f(t)\mathrm{d}t,\frac{1}{2}\right)+\int_0^1 L_{w(t)}\mathrm{d}t\le 0, \tag{20}$$

and

$$T_{F,w}\left(\frac{2}{b-a}\int\limits_{a}^{b}f(t)\mathrm{d}t,f(a)+f(b),f(a)+f(b)\right)+\int\limits_0^1 L_{w(t)}\mathrm{d}t\le 0, \tag{21}$$

where $w(t)=1$.

**Remark 3.** If we choose $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 6, then we have the following inequalities for Riemann-Liouville fractional integrals

$$F\left(f\left(\frac{a+b}{2}\right),\frac{2^\alpha\Gamma(\alpha+1)}{(b-a)^\alpha}J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b),\frac{2^\alpha\Gamma(\alpha+1)}{(b-a)^\alpha}J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a),\frac{1}{2}\right)+\int_0^1 L_{w(t)}\mathrm{d}t\le 0,$$

and

$$T_{F,w}\Bigg(\frac{2^\alpha\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)+J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)\right], f(a)+f(b),f(a)+f(b)\Bigg)+\int\limits_0^1 L_{w(t)}\mathrm{d}t\le 0,$$

where $w(t)=\alpha t^{\alpha-1}$ which is given by Budak et al. in [5].

**Corollary 1.** *If we take $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 6, then we have the following inequalities for $k$–Riemann–Liouville fractional integrals*

$$F\Bigg(f\left(\frac{a+b}{2}\right),\frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}I^\alpha_{\left(\frac{a+b}{2}\right)+,\ k}f(b), \frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}I^\alpha_{\left(\frac{a+b}{2}\right)-,\ k}f(a),\frac{1}{2}\Bigg)+\int_0^1 L_{w(t)}\mathrm{d}t\le 0,$$

*and*

$$T_{F,w}\Bigg(\frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I^\alpha_{\left(\frac{a+b}{2}\right)+,\ k}f(b)+I^\alpha_{\left(\frac{a+b}{2}\right)-,\ k}f(a)\right],$$

$$f(a)+f(b),f(a)+f(b)\Big)+\int_0^1 L_{w(t)}\mathrm{d}t\le 0,$$

*where* $w(t)=\frac{\alpha}{k}t^{\frac{\alpha}{k}-1}$.

**Theorem 7.** *Let $I\subseteq\mathbb{R}$ be an interval, $f:I^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be a mapping on $I^\circ$, $a,b\in I^\circ$, $a<b$ and let $F$ be linear with respect to the first three variables. If $f$ is $F$-convex on $[a,b]$ for some $F\in\mathcal{F}$, then we have*

$$\begin{aligned}&F\left(f\left(\frac{a+b}{2}\right),\frac{1}{\Lambda(1)}\ {}_{b-}I_\varphi f\left(\frac{a+b}{2}\right),\right.\\ &\left.\frac{1}{\Lambda(1)}\ {}_{a+}I_\varphi f\left(\frac{a+b}{2}\right),\frac{1}{2}\right)+\int_0^1 L_{w(t)}\mathrm{d}t\le 0,\end{aligned}\tag{22}$$

*and*

$$\begin{aligned}&T_{F,w}\Bigg(\frac{1}{\Lambda(1)}\left[{}_{a+}I_\varphi f\left(\frac{a+b}{2}\right)+\ {}_{b-}I_\varphi f\left(\frac{a+b}{2}\right)\right],\\ &f(a)+f(b),f(a)+f(b)\Bigg)+\int_0^1 L_{w(t)}\mathrm{d}t\le 0,\end{aligned}\tag{23}$$

*where* $w(t)=\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}$.

*Proof.* Since $f$ is $F$-convex, we have

$$F\left(f\left(\frac{x+y}{2}\right),f(x),f(y),\frac{1}{2}\right)\le 0,\quad\forall\,x,y\in[a,b]\,.$$

For

$$x=\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\ \text{ and }\ y=\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b,$$

we have

$$\begin{aligned}&F\left(f\left(\frac{a+b}{2}\right),f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right),\right.\\ &\left.f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right),\frac{1}{2}\right)\le 0,\end{aligned}$$

for all $t\in[0,1]$. Multiplying this inequality by $w(t)=\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}$ and using axiom (A3), we get

$$\begin{aligned}&F\Bigg(\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}f\left(\frac{a+b}{2}\right),\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right),\\ &\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right),\frac{1}{2}\Bigg)+L_{w(t)}\le 0,\end{aligned}$$

for all $t \in (0,1)$. Integrating over $(0,1)$ with respect to the variable $t$ and using axiom (A1), we obtain

$$F\left(\frac{f\left(\frac{a+b}{2}\right)}{\Lambda(1)}\int_0^1 \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t}\mathrm{d}t,\right.$$

$$\frac{1}{\Lambda(1)}\int_0^1 \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t}f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\mathrm{d}t,$$

$$\left.\frac{1}{\Lambda(1)}\int_0^1 \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t}f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\mathrm{d}t, \frac{1}{2}\right)$$

$$+\int_0^1 L_{w(t)}\mathrm{d}t \leq 0.$$

Using the facts that

$$\begin{aligned}
&\int_0^1 \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t}f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\mathrm{d}t \\
&= \int_{\frac{a+b}{2}}^b \frac{\varphi\left(x-\frac{a+b}{2}\right)}{x-\frac{a+b}{2}}f(x)\mathrm{d}x \\
&= {}_{b-}I_\varphi f\left(\frac{a+b}{2}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t}f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\mathrm{d}t \\
&= \int_a^{\frac{a+b}{2}} \frac{\varphi\left(\frac{a+b}{2}-x\right)}{\frac{a+b}{2}-x}f(x)\mathrm{d}x \\
&= {}_{a+}I_\varphi f\left(\frac{a+b}{2}\right),
\end{aligned}$$

we obtain

$$F\left(f\left(\frac{a+b}{2}\right), \frac{1}{\Lambda(1)}\,{}_{b-}I_\varphi f\left(\frac{a+b}{2}\right), \frac{1}{\Lambda(1)}\,{}_{a+}I_\varphi f\left(\frac{a+b}{2}\right), \frac{1}{2}\right)$$

$$+\int_0^1 L_{w(t)}\mathrm{d}t \leq 0,$$

which gives (22).

On the other hand, since $f$ is $F$–convex, we have

$$F\left(f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right), f(a), f(b), t\right) \leq 0, \quad \forall t \in [0,1],$$

and

$$F\left(f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right), f(a), f(b), t\right) \leq 0, \quad \forall t \in [0,1].$$

Using the linearity of $F$, we get

$$F\Bigg(f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)+f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right), \\ f(a)+f(b), f(a)+f(b), t\Bigg) \leq 0, \quad \forall t \in [0,1].$$

Applying the axiom (A3) for $w(t)=\dfrac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}$, we obtain

$$F\Bigg(\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)} \times \Bigg[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)+ \\ f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\Bigg], \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}\left[f(a)+f(b)\right], \\ \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}\left[f(a)+f(b)\right], t\Bigg) + L_{w(t)} \leq 0,$$

for all $t \in (0,1)$. Integrating over $(0,1)$ and using axiom (A2), we have

$$T_{F,w}\Bigg(\int_0^1 \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)} \times \Bigg[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)+ \\ f\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\Bigg]\mathrm{d}t, \\ f(a)+f(b), f(a)+f(b)\Bigg) + \int_0^1 L_{w(t)}\mathrm{d}t \leq 0,$$

that is

$$T_{F,w}\Bigg(\frac{1}{\Lambda(1)}\left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right], \\ f(a)+f(b), f(a)+f(b)\Bigg) + \int_0^1 L_{w(t)}\mathrm{d}t \leq 0.$$

The proof of Theorem 7 is completed. □

**Remark 4.** If we take $\varphi(t)=t$ in Theorem 7, then the inequalities (22) and (23) reduce to the inequalities (20) and (21)

**Remark 5.** If we take $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$ in Theorem 7, then we have the following inequalities for Riemann-Liouville fractional integrals

$$F\Bigg(f\left(\frac{a+b}{2}\right),\frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right),$$
$$\frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right),\frac{1}{2}\Bigg)+\int_0^1 L_{w(t)}\mathrm{d}t\le 0,$$

and

$$T_{F,w}\Bigg(\frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right],$$
$$f(a)+f(b),f(a)+f(b)\Bigg)+\int_0^1 L_{w(t)}\mathrm{d}t\le 0,$$

where $w(t)=\alpha t^{\alpha-1}$ which is given by Budak et al. in [5].

**Corollary 2.** *If we take* $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *in Theorem 7, then we have the following inequalities for $k$–Riemann–Liouville fractional integrals:*

$$F\left(f\left(\frac{a+b}{2}\right),\frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}I_{b-,\ k}^{\alpha}f\left(\frac{a+b}{2}\right),\right.$$
$$\left.\frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}I_{a+,\ k}^{\alpha}f\left(\frac{a+b}{2}\right),\frac{1}{2}\right)+\int_0^1 L_{w(t)}\mathrm{d}t\le 0,$$

*and*

$$T_{F,w}\left(\frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I_{a+,\ k}^{\alpha}f\left(\frac{a+b}{2}\right)+I_{b-,\ k}^{\alpha}f\left(\frac{a+b}{2}\right)\right],\right.$$
$$f(a)+f(b),f(a)+f(b))+\int_0^1 L_{w(t)}\mathrm{d}t\le 0,$$

*where* $w(t)=\frac{\alpha}{k}t^{\frac{\alpha}{k}-1}$.

**Remark 6.** One can obtain several results for convexity, $\varepsilon$-convexity, $h$-convexity, etc by special choice of the function $F$ in Theorems 6 and 7.

## 3. Conclusion

In the development of this work, using the definition of $F$-convex functions some new Hermite-Hadamard type inequalities via generalized fractional integrals have been deduced. We also give several results capturing Riemann-Liouville fractional integrals and $k$-Riemann-Liouville fractional integrals as special cases. The authors hope that these results will serve as a motivation for future work in this fascinating area.

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**Hüseyin Budak**
Department of Mathematics
Faculty of Science and Arts
Düzce University
Düzce
Turkey
*E-mail address*: hsyn.budak@gmail.com

**Pinar Kösem**
Department of Mathematics
Faculty of Science and Arts
Düzce University
Düzce
Turkey
*E-mail address*: pinarksm18@gmail.com

**Artion Kashuri**
Department of Mathematics
Faculty of Science and Arts
Düzce University
Düzce
Turkey
*E-mail address*: artionkashuri@gmail.com