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Common fixed points under strict conditions

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Abstract. In this contribution, three new concepts called reciprocally continuous, strictly subweakly compatible and strictly subreciprocally continuous single and multivalued mappings are given for obtention some common fixed point theorems in a metric space. Our results improve and complement the results of Aliouche and Popa, Azam and Beg, Deshpande and Pathak, Kaneko and Sessa, Popa and others.

1. Introduction and preliminaries

Let $(\mathcal{X}, d)$ be a metric space. We denote by $CL(\mathcal{X})$ (resp., $CB(\mathcal{X})$) the nonempty closed (resp., closed and bounded) subsets of $\mathcal{X}$ and H the Hausdorff metric on $CL(\mathcal{X})$ (resp., $CB(\mathcal{X})$)

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup x \in Bd(A, x)\},$$

where A, $B \in CL(\mathcal{X})$ or $CB(\mathcal{X})$, and

$$d(x, A) = \inf_{y\in A}\{d(x, y)\}.$$

Now, let f and g be two self-mappings of a metric space $(\mathcal{X}, d)$. In 1982, Sessa [11] gave the weaker concept of the commutativity, namely the weakly commuting notion. f and g are weakly commuting if

$$d(fgx, gfx) \leq d(gx, fx),$$

for all $x \in \mathcal{X}$.

In 1986, Jungck [5] gave a generalization of the weak commutativity by giving the notion of compatible mappings. He defined f and g to be compatible if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$.

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Weakly commuting mappings are compatible. However, compatible mappings need not be weakly commuting (see example 2.2 of [13]).

In 1996, Jungck [6] generalized the above notion by introducing the concept of weakly compatible mappings. He defined f and g to be weakly compatible if they commute at their coincidence points; i.e., if $fu = gu$ for some $u \in \mathcal{X}$, then $fgu = gfu$.

If f and g are compatible then they are obviously weakly compatible but as shown in example 2.52 of [1] the converse is not true.

In their paper [8], Kaneko and Sessa extended the definition of compatibility to include multivalued mappings in the following way: mappings $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to CB(\mathcal{X})$ are compatible if $fFx \in CB(\mathcal{X})$ for all $x \in \mathcal{X}$ and

$$\lim_{n\to\infty} H(fFx_n, Ffx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} Fx_n = M \in CB(\mathcal{X})$ and $\lim_{n\to\infty} fx_n = t \in M$.

To generalize the above notion, Jungck and Rhoades [7] gave the concept of weakly compatible mappings. $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to CB(\mathcal{X})$ are said to be weakly compatible if they commute at their coincidence points; i.e., if $fFx = Ffx$ whenever $fx \in Fx$.

Recall that a point $t \in \mathcal{X}$ is called a **strict coincidence point** (resp. **strict common fixed point**) of mappings $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to CB(\mathcal{X})$ if $Ft = \{ft\}$ (resp. $Ft = \{ft\} = \{t\}$).

2. Main results

Our first objective in this contribution is to generalize the above definition by introducing the concept of strictly subweakly compatible single and multivalued mappings.

Definition 1. Mappings $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to CB(\mathcal{X})$ are **strictly subweakly compatible (shortly sswc)** if and only if $fFx \in CB(\mathcal{X})$ and there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = t$, $\lim_{n\to\infty} Fx_n = \{t\}$ for some $t \in \mathcal{X}$ and $\lim_{n\to\infty} (fFx_n, Ffx_n) = 0$.

The example below shows that there exist sswc mappings which are not weakly compatible.

Example 1. Let $\mathcal{X} = [0, \infty)$ and $d(x, y) = |x - y|$. Define $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to CB(\mathcal{X})$ by

$$fx = x^2 \text{ and } Fx = \begin{cases} [4, x+2], & \text{if } x \in [2,4] \cup (9,\infty), \\ \{x+12\}, & \text{if } x \in [0,2) \cup (4,9]. \end{cases}$$

We have $fFx \in CB(\mathcal{X})$. Consider the sequence $\{x_n\}$ in $\mathcal{X}$ defined by $x_n = 2 + \frac{1}{n}$ for $n = 1, 2, \ldots$, we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_n^2 = 4 = t,$$
$$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} [4, x_n + 2] = \{4\} = \{t\},$$
$$\lim_{n\to\infty} H(fFx_n, Ffx_n) = \lim_{n\to\infty} H([16, (x_n + 2)^2], \{x_n^2 + 12\}) = 0,$$

therefore f and F are sswc.

On the other hand, we have $fx \in Fx$ if and only if $x \in [4, x + 2]$, but $Ff(x) \neq fF(x)$, therefore f and F are not weakly compatible.

In 1999, Pant [9] introduced the concept of reciprocally continuous single-valued mappings as a generalization of continuous mappings: f and g are reciprocally continuous if and only if $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$ whenever $\{x_n\} \subset \mathcal{X}$ is such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in \mathcal{X}$.

In 2002, Singh and Mishra [12] introduced the concept of reciprocal continuity for single and multivalued mappings as follows.

Definition 2. The mappings $F : \mathcal{X} \to CL(\mathcal{X})$ and $f : \mathcal{X} \to \mathcal{X}$ are reciprocally continuous on $\mathcal{X}$ (resp., at $t \in \mathcal{X}$) if and only if $fFx \in CL(\mathcal{X})$ for each $x \in \mathcal{X}$ (resp., $fFt \in CL(\mathcal{X})$) and $\lim_{n\to\infty} fFx_n = fM$, $\lim_{n\to\infty} Ffx_n = Ft$ whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} Fx_n = M \in CL(\mathcal{X})$, $\lim_{n\to\infty} fx_n = t \in M$.

Motivated by Pant, Singh and Mishra, we give the following notion of reciprocally continuous single and multivalued mappings which is different from the above definition and represents our second objective.

Definition 3. Mappings $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to CB(\mathcal{X})$ are **reciprocally continuous** if and only if $\lim_{n\to\infty} fFx_n = \{ft\}$ and $\lim_{n\to\infty} Ffx_n = Ft$ whenever $\{x_n\} \subset \mathcal{X}$ is a sequence such that $\lim_{n\to\infty} fx_n = t$, $\lim_{n\to\infty} Fx_n = \{t\}$ for some $t \in \mathcal{X}$.

Our third objective here is to extend the concept of reciprocally continuous mappings of Pant and the above one to the setting of single and multivalued mappings.

Definition 4. Mappings $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to CB(\mathcal{X})$ are **strictly subreciprocally continuous (shortly ssrc)** if and only if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = t$, $\lim_{n\to\infty} Fx_n = \{t\}$ for some $t \in \mathcal{X}$, $\lim_{n\to\infty} fFx_n = \{ft\}$ and $\lim_{n\to\infty} Ffx_n = Ft$.

The next example shows that there exist ssrc mappings which are not continuous.

Example 2. Let $\mathcal{X} = \mathbb{R}$. Define $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to CB(\mathcal{X})$ by

$$fx = \begin{cases} x-1, & \text{if } x < 0, \\ x, & \text{if } x \geq 0, \end{cases} \qquad Fx = \begin{cases} [x-1, -1], & \text{if } x < 0, \\ [0, x], & \text{if } x \geq 0. \end{cases}$$

It is clear to see that f and F are discontinuous at $x = 0$.

Consider the sequence $\{x_n\}$ in $\mathcal{X}$ defined by $x_n = \frac{1}{n}$ for $n = 1, 2, \ldots$ We have

$$\begin{aligned} \lim_{n\to\infty} fx_n &= \lim_{n\to\infty} x_n = 0 = t, \\ \lim_{n\to\infty} Fx_n &= \lim_{n\to\infty} [0, x_n] = \{0\} = \{t\}, \\ \lim_{n\to\infty} fFx_n &= \lim_{n\to\infty} [0, x_n] = \{0\} = \{f(0)\} = \{f(t)\}, \\ \lim_{n\to\infty} Ffx_n &= \lim_{n\to\infty} [0, x_n] = \{0\} = F(0) = F(t), \end{aligned}$$

therefore f and F are ssrc.

Now, we are ready to present and prove our main result.

Theorem 1. *Let $(\mathcal{X}, d)$ be a metric space. Let f, $g : \mathcal{X} \to \mathcal{X}$ and F, $G : \mathcal{X} \to CB(\mathcal{X})$ be single and multivalued mappings respectively such that f and F as well as g and G are reciprocally continuous and sswc or ssrc and compatible. Let $\varphi : \mathbb{R}_+^6 \to \mathbb{R}$ be a lower semi continuous function satisfying:*

(φ_1): *φ is nonincreasing in variables t_5 and t_6,*

(φ_2): *$\varphi(u, u, 0, 0, u, u) > 0$ for all $u > 0$ and the inequality*

$$\varphi\Big(H(Fx, Gy), d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)\Big) \leq 0, \tag{1}$$

for all x and y in $\mathcal{X}$, then, f, g, F and G have a strict common fixed point in $\mathcal{X}$.

Proof. Since f and F as well as g and G are reciprocally continuous and sswc or ssrc and compatible then, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = t$, $\lim_{n\to\infty} Fx_n = \{t\}$ for some $t \in \mathcal{X}$, $\lim_{n\to\infty} fFx_n = \{ft\}$ and $\lim_{n\to\infty} Ffx_n = Ft$; $\lim_{n\to\infty} gy_n = z$, $\lim_{n\to\infty} Gy_n = \{z\}$ for some $z \in \mathcal{X}$, $\lim_{n\to\infty} gGy_n = \{gz\}$ and $\lim_{n\to\infty} Ggy_n = Gz$.

First we prove that $ft = gz$. In fact, by (1) we have

$$\varphi\Big(H(Ft, Gz), d(ft, gz), d(ft, Ft), d(gz, Gz), d(ft, Gz), d(gz, Ft)\Big) \leq 0.$$

Since $Ft = \{ft\}$, $Gz = \{gz\}$ and φ is nonincreasing in t_5 and t_6, we get

$$\varphi\Big(d(ft, gz), d(ft, gz), 0, 0, d(ft, gz), d(gz, ft)\Big) \leq 0,$$

which is a contradiction with φ_2. Then $ft = gz$.

Now, we claim that $t = ft$, by (1) we have

$$\varphi\Big(H(Fx_n, Gz), d(fx_n, gz), d(fx_n, Fx_n), \\ d(gz, Gz), d(fx_n, Gz), d(gz, Fx_n)\Big) \leq 0.$$

Letting n tends to infinity and taking in account that φ is lower semi continuous, we get

$$\varphi(d(t, ft), d(t, ft), 0, 0, d(t, ft), d(ft, t)) \leq 0,$$

which contradicts φ_2. Then $ft = t$.

Next, we prove that $z = t$. Indeed, by (1) we have

$$\varphi\Big(H(Fx_n, Gy_n), d(fx_n, gy_n), d(fx_n, Fx_n), \\ d(gy_n, Gy_n), d(fx_n, Gy_n), d(gy_n, Fx_n)\Big) \leq 0.$$

When n tends to infinity, we get

$$\varphi\Big(d(t, z), d(t, z), 0, 0, d(t, z), d(z, t)\Big) \leq 0,$$

which contradicts (φ_2), therefore $z = t$. Consequently, t is a strict common fixed point of f, g, F and G. $\square$

Corollary 1. *Let $(\mathcal{X}, d)$ be a metric space. Let f, $g : \mathcal{X} \to \mathcal{X}$ and F, $G : \mathcal{X} \to CB(\mathcal{X})$ be single and multivalued mappings respectively such that f and F as well as g and G are reciprocally continuous and sswc or ssrc and compatible. Assume that*

$$H(Fx, Gy) \leq m \max\Big\{d(fx, gy), d(fx, Fx), d(gy, Gy), \\ \tfrac{1}{2}\big((d(fx, Gy) + d(gy, Fx)\big)\Big\}, \tag{2}$$

where $m \in (0, 1)$, or

$$H^2(Fx, Gy) \leq m^2 \max\Big\{d^2(fx, gy), d(fx, Fx)d(gy, Gy), d(fx, Gy)d(gy, Fx), \\ d(fx, Gy)d(fx, Fx), d(gy, Fx)d(gy, Gy)\Big\}, \tag{3}$$

where $m^2 \in (0, 1)$, or

$$H^2(Fx, Gy) + \frac{H(Fx, Gy)}{1 + d(fx, Gy)d(gy, Fx)} - \\ -[ad^2(fx, gy) + bd^2(fx, Fx) + cd^2(gy, Gy)] \leq 0, \tag{4}$$

where a, b, $c > 0$ and $a + b + c < 1$.

Then f, g, F and G have a strict common fixed point.

Remark 1. (1) Our main result improves the main result of Popa [10].

(2) By the above corollary and (2) for $f = g$ and $F = G$, we obtain an extension of the main result of Kaneko and Sessa [8].

(3) The main result of Azam and Beg [3] follows from the above corollary and (2), because

$$d(fx, gy) \leq \max\Big\{d(fx, gy), d(fx, Fx), d(gy, Gy), \\ \tfrac{1}{2}[d(fx, Gy) + d(gy, Fx)]\Big\}.$$

(4) Also, our main result improves the main results of Aliouche and Popa [2], Deshpande and Pathak [4] because, in our work, we have not continuity, neither completeness nor inclusion, and we did not impose a lot of conditions on the four mappings.

Example 3. Let $\mathcal{X} = [0, 2]$ endowed with the Euclidean metric, define mappings f, g, F and G as follows:

$$fx = \begin{cases} \frac{1}{2}(x+1), & 0 \leq x \leq 1, \\ 0, & 1 < x \leq 2, \end{cases} \qquad gx = \begin{cases} x + \frac{1}{2}, & 0 \leq x < 1, \\ 1, & x = 1, \\ 2, & 1 < x \leq 2, \end{cases}$$

$$Fx = \begin{cases} [1, 2-x], & 0 \leq x \leq 1, \\ \{\frac{3}{2}\}, & 1 < x \leq 2, \end{cases} \qquad Gx = \begin{cases} \{1\}, & 0 \leq x \leq 1 \\ [0, \frac{1}{2}], & 1 < x \leq 2. \end{cases}$$

We consider a sequence $\{x_n\}$ defined for each $n \geq 1$ by $x_n = 1 - \frac{1}{n}$, clearly $\lim_{n\to\infty} Fx_n = \{1\}$ and $\lim_{n\to\infty} fx_n = 1$, also we have

$$\lim_{n\to\infty} fFx_n = \{1\} = f\{1\},$$
$$\lim_{n\to\infty} Ffx_n = F1 = \{1\}.$$

Also, $\lim_{n\to\infty} H(fFx_n, Ffx_n) = 0$. Hence f and F are ssrc and compatible.

For g and G, consider a sequence $\{y_n\}$ defined by $y_n = \frac{1}{2}(1 - e^{-n})$, for all $n \geq 1$. It is clear that

$$\lim_{n\to\infty} gy_n = 1, \quad \lim_{n\to\infty} Gy_n = \{1\},$$
$$\lim_{n\to\infty} gGy_n = g\{1\} = \{1\}, \quad \lim_{n\to\infty} Ggy_n = G1 = \{1\},$$
$$\lim_{n\to\infty} H(gGy_n, Ggy_n) = 0,$$

i.e., g and G are ssrc and compatible.

By taking $m = \frac{4}{5}$ in corollary 1, we show that inequality (2) is satisfied. We have the following cases:

(1) For $x, y \in [0, 1]$, we have $H(Fx, Gy) = 0$, obviously inequality (2) is satisfied.

(2) For $x \in [0, 1]$ and $y \in (1, 2]$, we have

$$H(Fx, Gy) = 1 \leq \frac{6}{5} = \frac{4}{5}d(gy, Gy).$$

(3) For $x \in (1,2]$ and $y \in [0,1]$, we have

$$H(Fx,Gy) = \frac{1}{2} \leq \frac{6}{5} = \frac{4}{5}d(fx,Fx).$$

(4) For $x, y \in (1,2]$, we have

$$H(Fx,Gy) = 1 \leq \frac{8}{5} = \frac{4}{5}d(fx,gy),$$

then all hypotheses of corollary 1 are satisfied and the point 1 is a strict common fixed point for f, g, F and G.

Corollary 2. *Let f, $g : \mathcal{X} \to \mathcal{X}$ be single valued mappings and let F, $G : \mathcal{X} \to CB(\mathcal{X})$ be multivalued mappings on a metric space $(\mathcal{X}, d)$ such that the pairs f and F as well as g and G are ssrc and compatible for all x, $y \in \mathcal{X}$. Then the pair of mappings (f,F) and (g,G) has a strict coincidence point. Moreover f, g, F and G have a strict common fixed point in $\mathcal{X}$ provided that mappings satisfy*

$$H(Fx,Gy) \leq \phi\Big(d(fx,gy), d(fx,Fx), d(gy,Gy), d(fx,Gy), d(gy,Fx)\Big),$$

where $\phi : \mathbb{R}_+^5 \to \mathbb{R}_+$ is an upper semi continuous function such that $\phi(0) = 0$ and $\phi(t,0,0,t,t) < t$ for each $t > 0$.

Proof. The proof follows immediately on taking

$$\varphi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \phi(t_2,t_3,t_4,t_5,t_6)$$

in Theorem 1, where $\phi : \mathbb{R}_+^5 \to \mathbb{R}_+$ is an upper semi continuous function such that $\phi(0) = 0$ and $\phi(t,0,0,t,t) < t$ for each $t > 0$. $\square$

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