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The influence of θ-function to the class of MWP operators

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Abstract. In this work, taking into account the θ-function, we present a general class of multivalued weakly Picard operators on complete metric space. We also provide an example showing that it includes some earlier classes as properly.

1. Introduction

One of the most important concept of metric fixed point theory is Multivalued Weakly Picard (shortly MWP) operator introduced by Rus [21] in 1991. Let (X, d) be a metric space and $\mathcal{P}(X)$ be the family of all nonempty subsets of X. A multivalued mapping $T : X \to \mathcal{P}(X)$ is Multivalued Weakly Picard operator if there exists a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ for any initial point x_0, which converges to a fixed point of T. We shall denote the class of all MWP operators on X by $\mathcal{MWP}(X)$. There are a lot of papers and results about MWP operators in the literature (see [17, 18, 19, 20]).

For the sake of completeness we recall some important concepts and results about multivalued mappings.

Let (X, d) be a metric space. We denote by $\mathcal{CB}(X)$ the family of all nonempty closed and bounded subsets of X and by $\mathcal{K}(X)$ the family of all nonempty compact subsets of X. Let H be the Pompeiu-Hausdorff metric (see [2, 11]) with respect to d, that is,

$$H(A, B) = \max\left\{\sup_{x\in A} D(x, B), \sup_{y\in B} D(y, A)\right\},$$

for every $A, B \in \mathcal{CB}(X)$, where $D(x, A) = \inf\{d(x, y) : y \in A\}$. In 1969, Nadler [17] initiated the idea for multivalued contraction mapping and extended the Banach contraction principle to multivalued mappings and proved the following:

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Theorem 1. *Let (X,d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be multivalued mapping. If T is a multivalued contraction, that is, there exists $\lambda \in [0,1)$ such that*

$$H(Tx,Ty) \leq \lambda d(x,y),$$

for all $x, y \in X$, then there exists $z \in X$ such that $z \in Tz$.

Later on, several researches were conducted on a variety of generalizations, extensions and applications of this result of Nadler (see [3, 8, 9, 13, 14, 16]). Furthermore, Berinde and Berinde [1] introduced the concepts of multivalued almost contraction and multivalued nonlinear almost contraction as follows: Let (X,d) be a metric space and $T : X \to \mathcal{CB}(X)$ be a mapping. Then,

(i) T is said to be a multivalued almost contraction if there exist two constants $\lambda \in (0,1)$ and $L \geq 1$ such that

$$H(Tx,Ty) \leq \lambda d(x,y) + LD(y,Tx),$$

for all $x, y \in X$. We will denote the class of multivalued almost contractions on X by $\mathcal{MA}(X)$.

(ii) T is said to be a multivalued nonlinear almost contraction if there exists a constant $L \geq 0$ and a function $\varphi : [0,\infty) \to [0,1)$ satisfying

$$\limsup_{t \to s^+} \varphi(t) < 1, \ \forall s \geq 0, \tag{1}$$

such that

$$H(Tx,Ty) \leq \varphi(d(x,y))d(x,y) + LD(y,Tx), \tag{2}$$

for all $x, y \in X$. We denote the class of all multivalued nonlinear almost contractions on X by $\mathcal{MNA}(X)$.

A function $\varphi : [0,\infty) \to [0,1)$ satisfying (1) is called Mizoguchi-Takahashi function (as short $\mathcal{MT}$-function [7, 8, 22]) in the literature. Let's note, by the symmetry property of the metric, the above contractive conditions implicitly includes their dual ones. If $L = 0$, then (2) turns to the famous Mizoguchi-Takahashi [16] contractive condition, which includes the multivalued contraction in sense of Nadler [17]. If we examine the proofs of Theorem 3 and Theorem 4 of [1], we can infer the following:

Theorem 2. *If (X,d) is a complete metric space, then*

$$\mathcal{MA}(X) \subseteq \mathcal{MNA}(X) \subseteq \mathcal{MWP}(X).$$

On the other hand, Jleli and Samet [12] presented an interesting generalization of the Banach contraction principle. They introduced a new type of contractive condition, which we shall call it as θ-contraction. Now, we recall basic definitions, relevant notions and some related results concerning θ-contraction. Let $\theta : (0,\infty) \to (1,\infty)$ be a function. Next we will consider the following properties for θ:

(θ_1) θ is nondecreasing;

(θ_2) For each sequence $\{t_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ and $\lim_{n\to\infty} t_n = 0^+$ are equivalent;

(θ_3) There exist $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$;

(θ_4) $\theta(\inf A) = \inf \theta(A)$ for all $A \subset (0,\infty)$ with $\inf A > 0$.

We denote by Θ and Ω be the set of all functions θ satisfying (θ_1)-(θ_3) and (θ_1)-(θ_4), respectively. It is clear that $\Omega \subset \Theta$. Some examples of the functions belonging Ω are $\theta_1(t) = e^{\sqrt{t}}$ and $\theta_2(t) = e^{\sqrt{te^t}}$. If we define

$$\theta_3(t) = \begin{cases} e^{\sqrt{t}}, & t < 1, \\ 9, & t \geq 1, \end{cases}$$

then, we can see $\theta_3 \in \Theta \backslash \Omega$. Note that, if a function θ satisfies (θ_1), then it satisfies (θ_4) if and only if it is right continuous.

By considering the conditions (θ_1)-(θ_3), Jleli and Samet [12] introduced the concept of θ-contraction, which is more general than Banach contraction. Let (X,d) be a metric space and $\theta \in \Theta$. A mapping $T : X \to X$ is said to be a θ-contraction if there exists a constant $k \in [0,1)$ such that

$$\theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k, \tag{3}$$

for all $x,y \in X$ with $d(Tx,Ty) > 0$. As a real generalization of Banach contraction principle, Jleli and Samet proved that every θ-contraction on a complete metric space has a unique fixed point. In addition, from (θ_1) and (3), it is easy to concluded that every θ-contraction T is a contractive mapping, i.e., $d(Tx,Ty) < d(x,y)$ for all $x,y \in X$ with $Tx \neq Ty$. Thus, every θ-contraction mapping on a metric space is continuous.

Afterwards, many researches were conducted on a variety of generalizations, extensions and applications of the result of Jleli and Samet (See [4, 5, 6, 10, 15]). Hançer et al. [10] also extended the concept of θ-contraction to multivalued case. Moreover in these directions, Durmaz and Altun [5] and Mınak and Altun [15] presented the following concepts: Let (X,d) be a metric space and $T : X \to \mathcal{CB}(X)$ be given a mapping. Then,

(i) T is said to be a multivalued almost θ-contraction with $\theta \in \Theta$ [5] if there exist two constants $k \in (0,1)$ and $\lambda \geq 0$ such that

$$\theta(H(Tx,Ty)) \leq [\theta(d(x,y) + \lambda D(y,Tx))]^k,$$

for all $x,y \in X$ with $H(Tx,Ty) > 0$.

(ii) T is said to be a multivalued nonlinear θ-contraction with $\theta \in \Theta$ [15] if there exists a function $k : (0,\infty) \to [0,1)$ such that

$$\limsup_{t\to s^+} k(t) < 1, \ \forall s \geq 0,$$

satisfying

$$\theta(H(Tx,Ty)) \leq [\theta(d(x,y))]^{k(d(x,y))},$$

for all $x,y \in X$ with $H(Tx,Ty) > 0$.

We shall denote the class of all multivalued almost θ-contractions with $\theta \in \Theta$ (resp. $\theta \in \Omega$) on X by $\mathcal{MA}_\Theta(X)$ (resp. $\mathcal{MA}_\Omega(X)$) and the class of all multivalued nonlinear θ-contractions with $\theta \in \Theta$ (resp. $\theta \in \Omega$) on X by $\mathcal{MN}_\Theta(X)$ (resp. $\mathcal{MN}_\Omega(X)$). If we examine the proof of Theorem 2.1 in [5] and the proof of Theorem 8 in [15], we can infer the following theorems, respectively:

Theorem 3. *If (X, d) is a complete metric space, then*

$$\mathcal{MA}_\Omega(X) \subseteq \mathcal{MWP}(X).$$

Theorem 4. *If (X, d) is a complete metric space, then*

$$\mathcal{MN}_\Omega(X) \subseteq \mathcal{MWP}(X).$$

We can see from the above definitions and theorems that if (X, d) is a metric space, then

$$\mathcal{MA}(X) \subseteq \mathcal{MA}_\Omega(X) \subseteq \mathcal{MA}_\Theta(X)$$

and

$$\mathcal{MN}(X) \subseteq \mathcal{MN}_\Omega(X) \subseteq \mathcal{MN}_\Theta(X)$$

and further if (X, d) is complete metric space, then

$$\mathcal{MA}_\Omega(X) \cup \mathcal{MN}_\Omega(X) \subseteq \mathcal{MWP}(X).$$

However, Example 1 of [15] shows that, even if (X, d) is a complete metric space, then

$$\mathcal{MA}_\Theta(X) \nsubseteq \mathcal{MWP}(X) \text{ and } \mathcal{MN}_\Theta(X) \nsubseteq \mathcal{MWP}(X).$$

In this paper, we present a general class of MWP operators on a complete metric space (X, d) which includes the classes $\mathcal{MNA}(X)$, $\mathcal{MA}_\Omega(X)$ and $\mathcal{MN}_\Omega(X)$.

2. THE RESULTS

Definition 1. Let (X, d) be a metric space, $T : X \to \mathcal{CB}(X)$ be a mapping. We say that T is a multivalued nonlinear almost θ-contraction with $\theta \in \Theta$ if there exists a constant $\lambda \geq 0$ and a function $k : (0, \infty) \to [0, 1)$ such that

$$\limsup_{t \to s^+} k(t) < 1, \ \forall s \geq 0,$$

satisfying

$$\theta(H(Tx, Ty)) \leq [\theta(d(x, y) + \lambda D(y, Tx))]^{k(d(x,y))}, \tag{4}$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$.

We shall denote the class of all multivalued nonlinear almost θ-contractions with $\theta \in \Theta$ (resp. $\theta \in \Omega$) on X by $\mathcal{MNA}_\Theta(X)$ (resp. $\mathcal{MNA}_\Omega(X)$). It is clear that

$$\mathcal{MA}_\Theta(X) \cup \mathcal{MN}_\Theta(X) \cup \mathcal{MNA}(X) \subseteq \mathcal{MNA}_\Theta(X).$$

Now we give our main result, which presents a general class of MWP operators on complete metric space.

Theorem 5. *If* (X,d) *is a complete metric space, then* $\mathcal{MNA}_\Omega(X) \subseteq \mathcal{MWP}(X)$.

Proof. Let (X,d) be a complete metric space and $T \in \mathcal{MNA}_\Omega(X)$. Define a set $X^* = \{x \in X : D(x,Tx) > 0\}$. Let $x_0 \in X \backslash X^*$ be an arbitrary point, then x_0 is a fixed point of T and also the sequence $\{x_n\} = \{x_0, x_0, x_0, \dots\}$ converges to x_0 which satisfies $x_{n+1} \in Tx_n$. Now let $x_0 \in X^*$ and choose $x_1 \in Tx_0$. If $x_1 \in X \backslash X^*$, then x_1 is a fixed point of T and so we can construct a Picard sequence which converges to x_1. Suppose $x_1 \in X^*$, then we have $0 < D(x_1,Tx_1) \leq H(Tx_0,Tx_1)$ and so from (θ_1), we obtain

$$\theta(D(x_1,Tx_1)) \leq \theta(H(Tx_0,Tx_1)).$$

From (4), we can write that

$$\begin{aligned} \theta(D(x_1,Tx_1)) &\leq \theta(H(Tx_0,Tx_1)) \\ &\leq [\theta(d(x_0,x_1) + \lambda D(x_1,Tx_0))]^{k(d(x_0,x_1))} \\ &= [\theta(d(x_0,x_1))]^{k(d(x_0,x_1))} . \end{aligned} \tag{5}$$

From (θ_4), we can write

$$\theta(D(x_1,Tx_1)) = \inf_{y \in Tx_1} \theta(d(x_1,y))$$

and so from (5) we have

$$\begin{aligned} \inf_{y \in Tx_1} \theta(d(x_1,y)) &\leq [\theta(d(x_0,x_1))]^{k(d(x_0,x_1))} \\ &< [\theta(d(x_0,x_1))]^{\frac{k(d(x_0,x_1))}{2}} . \end{aligned} \tag{6}$$

Then, from (6) there exists $x_2 \in Tx_1$ such that

$$\theta(d(x_1,x_2)) \leq [\theta(d(x_0,x_1))]^{\frac{k(d(x_0,x_1))}{2}} .$$

If $x_2 \in X \backslash X^*$, then x_2 is a fixed point of T. Otherwise, by the same way, we can find $x_3 \in Tx_2$ such that

$$\theta(d(x_2,x_3)) \leq [\theta(d(x_1,x_2))]^{\frac{k(d(x_1,x_2))}{2}} .$$

Therefore, continuing recursively, we can obtain a sequence $\{x_n\}$ in X^* such that $x_{n+1} \in Tx_n$ and

$$\theta(d(x_n,x_{n+1})) \leq [\theta(d(x_{n-1},x_n))]^{\frac{kd(x_{n-1},x_n)}{2}} \tag{7}$$

for all $n \in \mathbb{N}$. Thus the sequence $\{d(x_n,x_{n+1})\}$ is decreasing and hence convergent. From (7), there exists $b \in (0,1)$ and $n_0 \in \mathbb{N}$ such that $k(d(x_n,x_{n+1})) < b$ for all $n \geq n_0$. Thus, we obtain for all $n \geq n_0$ the following inequalities:

$$\begin{aligned}
1 &< \theta(d(x_n, x_{n+1})) \\
&\leq [\theta(d(x_{n-1}, x_n))]^{k(d(x_{n-1},x_n))} \\
&\leq [\theta(d(x_{n-2}, x_{n-1}))]^{k(d(x_{n-1},x_n))k(d(x_{n-1},x_n))} \\
&\ \vdots \\
&\leq [\theta(d(x_0, x_1))]^{k(d(x_0,x_1))\cdots k(d(x_{n-1},x_n))k(d(x_{n-1},x_n))} \\
&= [\theta(d(x_0, x_1))]^{k(d(x_0,x_1))\cdots k(d(x_{n_0-1},x_{n_0}))k(d(x_{n_0},x_{n_0+1}))\cdots k(d(x_{n-1},x_n))k(d(x_{n-1},x_n))} \\
&\leq [\theta(d(x_0, x_1))]^{k(d(x_{n_0},x_{n_0+1}))\cdots k(d(x_{n-1},x_n))k(d(x_{n-1},x_n))} \\
&\leq [\theta(d(x_0, x_1))]^{b^{(n-n_0)}}.
\end{aligned}$$

Thus, we obtain

$$1 < \theta(d(x_n, x_{n+1})) \leq [\theta(d(x_0, x_1))]^{b^{(n-n_0)}} \tag{8}$$

for all $n \geq n_0$. Letting $n \to \infty$ in (8), we obtain

$$\lim_{n\to\infty} \theta(d(x_n, x_{n+1})) = 1. \tag{9}$$

From (θ_2), $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0^+$ and so from (θ_3) there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} = l.$$

Suppose that $l < \infty$. In this case, let $B = \frac{l}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\left| \frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} - l \right| \leq B.$$

This implies that, for all $n \geq n_0$,

$$\frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} \geq l - B = B.$$

Then, for all $n \geq n_0$,

$$n\,[d(x_n, x_{n+1})]^r \leq An\,[\theta(d(x_n, x_{n+1})) - 1]\,,$$

where $A = 1/B$.

Suppose now that $l = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} \geq B.$$

This implies that, for all $n \geq n_0$,

$$n\,[d(x_n, x_{n+1})]^r \leq An\,[\theta(d(x_n, x_{n+1})) - 1]\,,$$

where $A = 1/B$.

Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$n\left[d(x_n, x_{n+1})\right]^r \leq An\left[\theta(d(x_n, x_{n+1})) - 1\right].$$

Using (8), we obtain, for all $n \geq n_0$,

$$n\left[d(x_n, x_{n+1})\right]^r \leq An\left[\left[\theta(d(x_0, x_1))\right]^{b^{(n-n_0)}} - 1\right].$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} n\left[d(x_n, x_{n+1})\right]^r = 0.$$

Thus, there exits $n_1 \in \mathbb{N}$ such that $n\left[d(x_n, x_{n+1})\right]^r \leq 1$ for all $n \geq n_1$. So, we have, for all $n \geq n_1$

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{1/r}}. \tag{10}$$

In order to show that $\{x_n\}$ is a Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (10), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}. \end{aligned}$$

By the convergence of the series $\sum\limits_{i=1}^{\infty} \frac{1}{i^{1/r}}$, letting to limit $n \to \infty$, we get $d(x_n, x_m) \to 0$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$. From (θ_1) and (4), for all $x, y \in X$ with $H(Tx, Ty) > 0$, we get

$$H(Tx, Ty) < d(x, y) + \lambda D(y, Tx),$$

and so

$$H(Tx, Ty) \leq d(x, y) + \lambda D(y, Tx),$$

for all $x, y \in X$. Then

$$\begin{aligned} D(x_{n+1}, Tz) &\leq H(Tx_n, Tz) \\ &\leq d(x_n, z) + \lambda D(z, Tx_n) \\ &\leq d(x_n, z) + \lambda d(z, x_{n+1}). \end{aligned}$$

Passing to limit $n \to \infty$ in the above, we obtain $D(z, Tz) = 0$. Thus, we get $z \in Tz$. Therefore $T \in \mathcal{MWP}(X)$. □

Now, we give a significant example showing that $T \in \mathcal{MWP}(X)$, since $T \in \mathcal{MNA}_{\Omega}(X)$ when (X, d) is a complete metric space. However, $T \notin \mathcal{MNA}(X) \cup \mathcal{MN}_{\Omega}(X)$.

Example 1. Consider the complete metric space (X, d), where $X = \{\frac{1}{2^n} : n \in \mathbb{N}\} \cup \{0\}$ and

$$d(x,y) = \begin{cases} 0, & x = y, \\ \max\{x,y\}, & x \neq y. \end{cases}$$

Define a mapping $T : X \to \mathcal{CB}(X)$ by

$$Tx = \begin{cases} \{x\}, & x \in \{0,1\} \\ \{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}, \cdots\}, & x = \frac{1}{2^n}, \quad n \in \mathbb{N},\ n > 1. \end{cases}$$

We claim that $T \in \mathcal{MNA}_\Omega(X)$ with $\theta(t) = e^{\sqrt{te^t}}$, $\lambda = \frac{1}{2}$ and $k : (0,\infty) \to [0,1)$ defined by

$$k(t) = \begin{cases} e^{-\frac{1}{2^{n+2}}}, & \text{if } t = \frac{1}{2^n} \text{ for } n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $\limsup_{t \to s^+} k(t) = 0 < 1$ for all $s \in [0,\infty)$. Observe that taking $\theta(t) = e^{\sqrt{te^t}}$ and $\lambda = \frac{1}{2}$ the contractive condition (4) turns to

$$\text{(11)} \qquad \frac{H(Tx,Ty)e^{H(Tx,Ty)-d(x,y)-\lambda\min\{d(y,Tx),d(x,Ty)\}}}{d(x,y)+\lambda\min\{d(y,Tx),d(x,Ty)\}} \leq [k(d(x,y))]^2 .$$

for all $x, y \in X$ with $H(Tx,Ty) > 0$. Now we consider the following cases: for the brevity we will assign the left side of (11) as $A(x,y)$. Also without lost of generality we assume $x > y$ in all cases.

Case 1. Let $x = \frac{1}{2^n}$ and $y = \frac{1}{2^m}$ with $m > n > 1$, then

$$A(x,y) = \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} e^{-\frac{1}{2^{n+1}}} = \frac{1}{2} e^{-\frac{1}{2^{n+1}}} \leq k^2(\frac{1}{2^n}) = k^2(d(x,y)),$$

Case 2. Let $x = \frac{1}{n}$, $n > 1$ and $y = 0$, then

$$A(x,y) = \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} e^{-\frac{1}{2^{n+1}}} = \frac{1}{2} e^{-\frac{1}{2^{n+1}}} \leq k^2(\frac{1}{2^n}) = k^2(d(x,y)),$$

Case 3. Let $x = 1$ and $y = 0$, then

$$A(x,y) = \frac{1}{1+\frac{1}{2}} e^{-\frac{1}{2}} = \frac{2}{3} e^{-\frac{1}{2}} \leq e^{-\frac{1}{2}} = k^2(1) = k^2(d(x,y)).$$

Case 4. If $x = \frac{1}{n}$, $n > 1$ and $y = 1$, then

$$A(x,y) = \frac{1}{1+\frac{1}{2}} e^{-\frac{1}{2}} = \frac{2}{3} e^{-\frac{1}{2}} \leq e^{-\frac{1}{2}} = k^2(1) = k^2(d(x,y)).$$

This shows that $T \in \mathcal{MNA}_\Omega(X)$. Also since (X, d) is complete metric space, then by Theorem 5, $T \in \mathcal{MWP}(X)$.

On the other hand, since $H(T0,T1)=1=d(0,1)$, then for all $\theta\in\Omega$ and for all $k:(0,\infty)\to[0,1)$ satisfying inequality (1), we have

$$\theta(H(T0,T1))=\theta(1)>\theta(1)^{k(1)}=\theta(d(0,1))^{k(d(0,1))}.$$

Therefore, $T\notin\mathcal{MN}_{\Omega}(X)$.

The following result is interested in the mapping $T:X\to\mathcal{K}(X)$. Here, we can remove the condition (θ_4) on the function θ. For this, we will use that if A is compact subset of a metric space (X,d), then for every $x\in X$ there exists $a\in A$ such that $d(x,a)=d(x,A)$.

Theorem 6. *Let (X,d) be a complete metric space and $T:X\to\mathcal{K}(X)$ be a mapping. If $T\in\mathcal{MNA}_{\Theta}(X)$, then $T\in\mathcal{MWP}(X)$.*

Proof. As in proof of Theorem 5, we get

$$\begin{aligned}\theta(D(x_1,Tx_1)) &\le \theta(H(Tx_0,Tx_1))\\ &\le [\theta(d(x_0,x_1)+\lambda D(x_1,Tx_0))]^{k(d(x_0,x_1))}\\ &\le [\theta(d(x_1,x_0))]^{k(d(x_0,x_1))}.\end{aligned}\tag{12}$$

Since Tx_1 is compact, there exists $x_2\in Tx_1$ such that $d(x_1,x_2)=d(x_1,Tx_1)$. From (12),

$$\begin{aligned}\theta(d(x_1,x_2)) &\le \theta(H(Tx_0,Tx_1))\\ &\le [\theta(d(x_0,x_1)+\lambda D(x_1,Tx_0))]^{k(d(x_0,x_1))}\\ &< [\theta(d(x_1,x_0))]^{k(d(x_0,x_1))}.\end{aligned}\tag{13}$$

By induction, we obtain a sequence $\{x_n\}$ in X^* with the property that $x_{n+1}\in Tx_n$, and

$$\theta(d(x_n,x_{n+1}))\le[\theta(d(x_{n-1},x_n))]^{k(d(x_{n-1},x_n))}<\theta(d(x_{n-1},x_n)),$$

for all $n\in\mathbb{N}$.

The rest of the proof can be completed as in the proof of Theorem 5. □

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