

# Existence and Ulam stability results for nonlinear hybrid implicit Caputo fractional differential equations

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ABSTRACT. In this paper, we study the existence, uniqueness and estimate of solutions for nonlinear hybrid implicit Caputo fractional differential equations by using the contraction mapping principle and the generalization of Gronwall's inequality. After that, we also establish the Ulam stability for the problem at hand. Finally, an example is given to illustrate this work.

## 1. INTRODUCTION

The concept of fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non integer order. Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1]–[28], [30]–[36], [38]–[40] and the references therein. The study of Ulam stability for fractional differential equations was initiated by Wang et al. [39]. An overview on the development of theory of the Ulam-Hyers and the Ulam-Hyers-Rassias stability for fractional differential equations can be found in [39, 40] and the references given therein. Subsequently, many authors discussed various Ulam-Hyers stability problem for different kinds of fractional integral and fractional differential equations by using different techniques, see [10, 14, 32, 39, 40] and the references therein.

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Hybrid differential equations arise from a variety of different areas of applied mathematics and physics, e.g., in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [4, 20, 22, 21, 38].

Dhage and Lakshmikantham [21] discussed the existence of solutions for the following first-order hybrid differential equation

$$\begin{cases} \frac{d}{dt} \left( \frac{x(t)}{g(t, x(t))} \right) = f(t, x(t)) \text{ a.e. } t \in [t_0, t_0 + T], \\ x(t_0) = x_0, x_0 \in \mathbb{R}, \end{cases}$$

where  $t_0, T \in \mathbb{R}$  with  $T > 0$ ,  $g : [t_0, t_0 + T] \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$  and  $f : [t_0, t_0 + T] \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous functions. By using the fixed point theorem in Banach algebra, the authors obtained the existence results.

The hybrid fractional differential equation

$$\begin{cases} D^\alpha \left( \frac{x(t)}{g(t, x(t))} \right) = f(t, x(t)) \text{ a.e. } t \in [0, T], \\ x(0) = 0, \end{cases}$$

has been investigated in [38], where  $D^\alpha$  is the Riemann-Liouville fractional derivative of order  $0 < \alpha < 1$ ,  $g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$  and  $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous functions. By employing the fixed point theorem in Banach algebra, the authors obtained the existence of a solution.

In [14], Benchohra and Lazreg established four types of Ulam stability for the following implicit fractional differential equation

$$\begin{cases} {}^C D^\alpha x(t) = f(t, x(t), {}^C D^\alpha x(t)), t \in [0, T], 0 < \alpha \leq 1, \\ x(0) = x_0, x_0 \in \mathbb{R}, \end{cases}$$

where  ${}^C D^\alpha$  is the Caputo fractional derivative,  $f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is a given continuous function.

In [10], Ardjouni and Djoudi studied the existence, interval of existence and uniqueness of solutions for the hybrid implicit Caputo fractional differential equation

$$\begin{cases} {}^C D^\alpha \left( \frac{x(t)}{g(t, x(t))} \right) = f \left( t, x(t), {}^C D^\alpha \left( \frac{x(t)}{g(t, x(t))} \right) \right), t \in [0, T], \\ x(0) = \eta g(0, x(0)), \eta \in \mathbb{R}, \end{cases}$$

where  $f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$  are nonlinear continuous functions and  ${}^C D^\alpha$  denotes the Caputo fractional derivative of order  $0 < \alpha < 1$ .

Motivated by the above works, we study the existence and uniqueness of solution and the Ulam stability for the following nonlinear hybrid implicit Caputo fractional differential equation

$$(1) \begin{cases} {}^C D^\alpha \left( \frac{x(t) - f(t, x(t))}{g(t, x(t))} \right) = h \left( t, x(t), {}^C D^\alpha \left( \frac{x(t) - f(t, x(t))}{g(t, x(t))} \right) \right), t \in [0, T], \\ x(0) = \theta g(0, x(0)) + f(0, x(0)), \theta \in \mathbb{R}, \end{cases}$$

where  $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$  and  $h : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  are nonlinear continuous functions and  ${}^C D^\alpha$  denotes the Caputo fractional derivative of order  $0 < \alpha < 1$ . To show the existence, uniqueness and estimate of solutions of (1), we transform (1) into an integral equation and then use the contraction mapping principle and the generalization of Gronwall's inequality. Further, we obtain Ulam-Hyers and Ulam-Hyers-Rassias stability results of (1). Finally, we provide an example to illustrate our obtained results.

The rest of this paper is organized as follows. Some definitions from fractional calculus theory are recalled in Section 2. In Section 3, we will prove the existence, uniqueness and estimate of solutions for (1). In Section 4, we discuss the Ulam stability results. Finally, an example is given in Section 5 to illustrate the usefulness of our main results.

## 2. PRELIMINARIES

In this section we present some basic definitions, notations and results of fractional calculus which are used throughout this paper.

**Definition 1** ([30]). The fractional integral of order  $\alpha > 0$  of a function  $x : \mathbb{R}^+ \rightarrow \mathbb{R}$  is given by

$$I^\alpha x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) \, ds$$

provided the right side is pointwise defined on  $\mathbb{R}^+$ .

**Definition 2** ([30]). The Caputo fractional derivative of order  $\alpha > 0$  of a function  $x : \mathbb{R}^+ \rightarrow \mathbb{R}$  is given by

$${}^C D^\alpha x(t) = I^{n-\alpha} D^{(n)} x(t) = \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} x^{(n)}(s) \, ds,$$

where  $n = [\alpha] + 1$ , provided the right side is pointwise defined on  $\mathbb{R}^+$ .

**Lemma 1** ([30]). Let  $\alpha > 0$ . Suppose  $x \in C^{n-1}([0, +\infty), \mathbb{R})$  and  $x^{(n)}$  exists almost every-where on any bounded interval of  $\mathbb{R}^+$ . Then

$$(I^{\alpha C} D^\alpha x)(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} t^k.$$

In particular, when  $0 < \alpha < 1$ ,  $(I^{\alpha C} D^\alpha x)(t) = x(t) - x(0)$ .

**Lemma 2** ([30]). For all  $\alpha, \beta \in [0, \infty)$  and  $\mu > -1$ , then

$$\int_0^t (t-s)^{\beta-1} s^{\alpha-1} \, ds = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1},$$

and

$$(2) \quad I^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}.$$

The following generalization of Gronwall's lemma for singular kernels plays an important role in obtaining our main results.

**Lemma 3** ([29]). *Let  $x : [0, T] \rightarrow [0, \infty)$  be a real function and  $w$  is a nonnegative locally integrable function on  $[0, T]$ . Assume that there is a constant  $a > 0$  such that for  $0 < \alpha < 1$*

$$x(t) \leq w(t) + a \int_0^t (t-s)^{-\alpha} x(s) \, ds,$$

then, there exists a constant  $k = k(\alpha)$  such that

$$x(t) \leq w(t) + ka \int_0^t (t-s)^{-\alpha} w(s) \, ds.$$

To define Ulam's stability, we consider the following fractional differential equation

$$(3) \quad {}^C D^\alpha u(t) = f(t, u(t)), \quad 0 < \alpha \leq 1, \quad t \in [0, T].$$

**Definition 3** ([36]). The equation (3) is said to be Ulam-Hyers stable if there exists a real number  $k > 0$  such that for each  $\epsilon > 0$  and for each  $y \in C([0, T], \mathbb{R})$  solution of the inequality

$$(4) \quad |{}^C D^\alpha y(t) - f(t, y(t))| \leq \epsilon, \quad t \in [0, T],$$

there exists a solution  $u$  of the equation (3) with

$$|y(t) - u(t)| \leq k\epsilon, \quad t \in [0, T].$$

**Definition 4** ([36]). The equation (3) is said to be Ulam-Hyers-Rassias stable with respect to  $\phi \in C([0, T], \mathbb{R}^+)$  if there exists a real number  $k_{\phi, f} > 0$  such that for each  $\epsilon > 0$  and for each  $y \in C([0, T], \mathbb{R})$  solution of the inequality

$$(5) \quad |{}^C D^\alpha y(t) - f(t, y(t))| \leq \epsilon\phi(t), \quad t \in [0, T],$$

there exists a solution  $u \in C([0, T], \mathbb{R})$  of the equation (3) with

$$|y(t) - u(t)| \leq k_{\phi, f}\phi(t)\epsilon, \quad t \in [0, T].$$

**Remark 1** ([36]). If there is a function  $\Psi \in C([0, T], \mathbb{R})$  (dependent on  $y$ ), such that

- 1)  $|\Psi(t)| \leq \epsilon$ , for all  $t \in [0, T]$ ,
- 2)  ${}^C D^\alpha y(t) = f(t, y(t)) + \Psi(t)$ ,  $t \in [0, T]$ .

Then a function  $y \in C([0, T], \mathbb{R})$  is a solution of the inequality (4).

**Theorem 1** (Banach's fixed point theorem [37]). *Let  $\Omega$  be a non-empty closed convex subset of a Banach space  $(S, \|\cdot\|)$ , then any contraction mapping  $P$  of  $\Omega$  into itself has a unique fixed point.*

3. EXISTENCE AND UNIQUENESS

In this section, we give the equivalence of the initial value problem (1) and prove the existence, uniqueness and estimate of solution of (1).

The proof of the following lemma is close to the proof of Lemma 6.2 given in [19].

**Lemma 4.** *If the functions  $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$  and  $h : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous, then the initial value problem (1) is equivalent to the nonlinear fractional Volterra integro-differential equation*

$$x(t) = f(t, x(t)) + \theta g(t, x(t)) + \frac{g(t, x(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h\left(s, x(s), {}^C D^\alpha \left(\frac{x(s) - f(s, x(s))}{g(s, x(s))}\right)\right) ds,$$

for  $t \in [0, T]$ .

**Theorem 2.** *Let  $T > 0$ . Assume that the continuous functions  $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$  and  $h : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  satisfy the following conditions:*

(H1) *There exists  $M_g \in \mathbb{R}^+$  such that*

$$|g(t, u)| \leq M_g,$$

*for all  $u \in \mathbb{R}$  and  $t \in [0, T]$ .*

(H2) *There exists  $M_h \in \mathbb{R}^+$  such that*

$$|h(t, u, v)| \leq M_h,$$

*for all  $u, v \in \mathbb{R}$  and  $t \in [0, T]$ .*

(H3) *There exist  $K_1, K_2, K_3 \in \mathbb{R}^+$ ,  $K_4 \in (0, 1)$  with  $K_1 + K_2|\theta| \in (0, 1)$  such that*

$$\begin{aligned} |f(t, u) - f(t, u^*)| &\leq K_1 |u - v|, \\ |g(t, u) - g(t, u^*)| &\leq K_2 |u - v|, \end{aligned}$$

and

$$|h(t, u, v) - h(t, u^*, v^*)| \leq K_3 |u - u^*| + K_4 |v - v^*|,$$

*for all  $u, v, u^*, v^* \in \mathbb{R}$  and  $t \in [0, T]$ .*

If

$$(6) \quad \beta = K_1 + K_2|\theta| + M_h K_2 + \frac{M_g K_3}{(1 - K_4)} \frac{T^\alpha}{\Gamma(\alpha + 1)} < 1.$$

Then, the problem (1) has a unique solution  $x \in C([0, T], \mathbb{R})$ .

*Proof.* Let

$${}^C D^\alpha \left(\frac{x(t) - f(t, x(t))}{g(t, x(t))}\right) = z_x(t), \quad x(0) = \theta g(0, x(0)) + f(0, x(0)),$$

then by Lemma 4, we have

$$x(t) = f(t, x(t)) + \theta g(t, x(t)) + \frac{g(t, x(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z_x(s) \, ds,$$

where

$$z_x(t) = h(t, f(t, x(t)) + \theta g(t, x(t)) + g(t, x(t)) I^\alpha z_x(t), z_x(t)).$$

That is  $x(t) = f(t, x(t)) + \theta g(t, x(t)) + g(t, x(t)) I^\alpha z_x(t)$ . Define the mapping  $P : C([0, T], \mathbb{R}) \rightarrow C([0, T], \mathbb{R})$  as follows

$$(Px)(t) = f(t, x(t)) + \theta g(t, x(t)) + \frac{g(t, x(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z_x(s) \, ds.$$

It is clear that the fixed points of  $P$  are solutions of (1). Let  $x, y \in C([0, T], \mathbb{R})$ , then we have

$$\begin{aligned} & |(Px)(t) - (Py)(t)| \\ &= \left| f(t, x(t)) + \theta g(t, x(t)) + \frac{g(t, x(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z_x(s) \, ds \right. \\ &\quad \left. - f(t, y(t)) - \theta g(t, y(t)) - \frac{g(t, y(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z_y(s) \, ds \right| \\ &\leq |f(t, x(t)) - f(t, y(t))| + |\theta| |g(t, x(t)) - g(t, y(t))| \\ &\quad + |g(t, x(t)) - g(t, y(t))| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z_x(s)| \, ds \\ &\quad + |g(t, y(t))| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z_x(s) - z_y(s)| \, ds \\ &\leq K_1 |x(t) - y(t)| + K_2 |\theta| |x(t) - y(t)| \\ &\quad + K_2 |x(t) - y(t)| \frac{M_h}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, ds \\ (7) \quad &+ \frac{M_g}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z_x(s) - z_y(s)| \, ds, \end{aligned}$$

and

$$\begin{aligned} |z_x(t) - z_y(t)| &\leq |h(t, x(t), z_x(t)) - h(t, y(t), z_y(t))| \\ &\leq K_3 |x(t) - y(t)| + K_4 |z_x(t) - z_y(t)| \\ (8) \quad &\leq \frac{K_3}{1 - K_4} |x(t) - y(t)|. \end{aligned}$$

By replacing (8) in the inequality (7), we get

$$\begin{aligned} & |(Px)(t) - (Py)(t)| \leq \\ &\leq K_1 |x(t) - y(t)| + K_2 |\theta| |x(t) - y(t)| \\ &\quad + K_2 |x(t) - y(t)| \frac{M_h}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, ds \end{aligned}$$

$$\begin{aligned}
 & + \frac{M_g}{\Gamma(\alpha)} \frac{K_3}{1 - K_4} \int_0^t (t - s)^{\alpha-1} |x(s) - y(s)| \, ds \\
 & \leq K_1 \|x - y\| + K_2 \left( |\theta| + \frac{M_h t^\alpha}{\Gamma(\alpha + 1)} \right) \|x - y\| \\
 & \quad + \frac{M_g}{\Gamma(\alpha)} \frac{K_3}{1 - K_4} \left( \int_0^t (t - s)^{\alpha-1} \, ds \right) \|x - y\| \\
 & \leq \left( K_1 + K_2 |\theta| + \left( M_h K_2 + \frac{M_g K_3}{1 - K_4} \right) \frac{t^\alpha}{\Gamma(\alpha + 1)} \right) \|x - y\| \\
 & \leq \left( K_1 + K_2 |\theta| + \left( M_h K_2 + \frac{M_g K_3}{1 - K_4} \right) \frac{T^\alpha}{\Gamma(\alpha + 1)} \right) \|x - y\|.
 \end{aligned}$$

Then

$$\|Px - Py\| \leq \beta \|x - y\|.$$

By (6), the mapping  $P$  is a contraction in  $C([0, T], \mathbb{R})$ . Hence  $P$  has a unique fixed point  $x \in C([0, T], \mathbb{R})$ . Therefore (1) has a unique solution.  $\square$

**Theorem 3.** *Assume that  $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$  and  $h : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  satisfy (H1)-(H3) and (6) holds. Then (1) has a unique solution  $x$  and*

$$\begin{aligned}
 |x(t)| \leq & \left( \frac{(1 - K_4)(1 - (K_1 + K_2 |\theta|)) \Gamma(\alpha + 1) + M_g K_3 K T^\alpha}{(1 - K_4)(1 - (K_1 + K_2 |\theta|))^2 \Gamma(\alpha + 1)} \right) \\
 & \times \left( Q_1 + |\theta| Q_2 + \frac{M_g Q_3 T^\alpha}{(1 - K_4) \Gamma(\alpha + 1)} \right),
 \end{aligned}$$

where  $Q_1 = \sup_{t \in [0, T]} |f(t, 0)|$ ,  $Q_2 = \sup_{t \in [0, T]} |g(t, 0)|$ ,  $Q_3 = \sup_{t \in [0, T]} |h(t, 0, 0)|$  and  $K \in \mathbb{R}^+$  is a constant.

*Proof.* Theorem 2 shows that the problem (1) has a unique solution. Let

$${}^C D^\alpha \left( \frac{x(t) - f(t, x(t))}{g(t, x(t))} \right) = z_x(t), \quad x(0) = \theta g(0, x(0)) + f(0, x(0)),$$

then by Lemma 4,  $x(t) = f(t, x(t)) + \theta g(t, x(t)) + g(t, x(t)) I^\alpha z_x(t)$ . Then by (H1), (H2) and (H3), for any  $t \in [0, T]$  we have

$$\begin{aligned}
 |x(t)| & \leq |f(t, x(t))| + |\theta| |g(t, x(t))| + |g(t, x(t))| |I^\alpha z_x(t)| \\
 & \leq |f(t, x(t)) - f(t, 0)| + |f(t, 0)| \\
 & \quad + |\theta| (|g(t, x(t)) - g(t, 0)| + |g(t, 0)|) + M_g |I^\alpha z_x(t)| \\
 & \leq K_1 |x(t)| + Q_1 + |\theta| (K_2 |x(t)| + Q_2) + M_g |I^\alpha z_x(t)|.
 \end{aligned}$$

On the other hand, for any  $t \in [0, T]$  we get

$$\begin{aligned}
 |z_x(t)| & = |h(t, x(t), z_x(t))| \\
 & \leq |h(t, x(t), z_x(t)) - h(t, 0, 0)| + |h(t, 0, 0)|
 \end{aligned}$$

$$\begin{aligned} &\leq K_3 |x(t)| + K_4 |z_x(t)| + |h(t, 0, 0)| \\ &\leq \frac{K_3}{1 - K_4} |x(t)| + \frac{Q_3}{1 - K_4}. \end{aligned}$$

Therefore

$$\begin{aligned} |x(t)| &\leq K_1 |x(t)| + Q_1 + |\theta| (K_2 |x(t)| + Q_2) \\ &\quad + M_g I^\alpha \left( \frac{K_3}{1 - K_4} |x(t)| + \frac{Q_3}{1 - K_4} \right). \end{aligned}$$

Thus

$$\begin{aligned} &(1 - (K_1 + K_2 |\theta|)) |x(t)| \\ &\leq Q_1 + |\theta| Q_2 + \frac{M_g Q_3 T^\alpha}{(1 - K_4) \Gamma(\alpha + 1)} \\ &\quad + \left( \frac{M_g K_3}{(1 - K_4) (1 - (K_1 + K_2 |\theta|))} \right) (I^\alpha \{(1 - (K_1 + K_2 |\theta|)) |x(t)|\}). \end{aligned}$$

By Lemma 3, there is a constant  $K = K(\alpha)$  such that

$$\begin{aligned} &(1 - (K_1 + K_2 |\theta|)) |x(t)| \\ &\leq Q_1 + |\theta| Q_2 + \frac{M_g Q_3 T^\alpha}{(1 - K_4) \Gamma(\alpha + 1)} \\ &\quad + \left( \frac{M_g K_3 K T^\alpha}{(1 - K_4) (1 - (K_1 + K_2 |\theta|)) \Gamma(\alpha + 1)} \right) \\ &\quad \times \left( Q_1 + |\theta| Q_2 + \frac{M_g Q_3 T^\alpha}{(1 - K_4) \Gamma(\alpha + 1)} \right) \\ &\leq \left( \frac{(1 - K_4) (1 - (K_1 + K_2 |\theta|)) \Gamma(\alpha + 1) + M_g K_3 K T^\alpha}{(1 - K_4) (1 - (K_1 + K_2 |\theta|)) \Gamma(\alpha + 1)} \right) \\ &\quad \times \left( Q_1 + |\theta| Q_2 + \frac{M_g Q_3 T^\alpha}{(1 - K_4) \Gamma(\alpha + 1)} \right). \end{aligned}$$

Hence

$$\begin{aligned} |x(t)| &\leq \left( \frac{(1 - K_4) (1 - (K_1 + K_2 |\theta|)) \Gamma(\alpha + 1) + M_g K_3 K T^\alpha}{(1 - K_4) (1 - (K_1 + K_2 |\theta|))^2 \Gamma(\alpha + 1)} \right) \\ &\quad \times \left( Q_1 + |\theta| Q_2 + \frac{M_g Q_3 T^\alpha}{(1 - K_4) \Gamma(\alpha + 1)} \right). \end{aligned}$$

This completes the proof.  $\square$

#### 4. ULAM STABILITY

In this section, we study two types of Ulam stability of the hybrid implicit Caputo fractional differential equation (1) which are Ulam-Hyers and Ulam-Hyers-Rassias stabilities.

**Lemma 5.** Assume that  $g$  satisfies (H1). If  $y \in C([0, T], \mathbb{R})$  is a solution of the fractional differential inequality for each  $\epsilon > 0$

$$(9) \quad \left| {}^C D^\alpha \left( \frac{y(t) - f(t, y(t))}{g(t, y(t))} \right) - h \left( t, y(t), {}^C D^\alpha \left( \frac{y(t) - f(t, y(t))}{g(t, y(t))} \right) \right) \right| \leq \epsilon,$$

then,  $y$  is a solution of the following inequality

$$(10) \quad |y(t) - (Py)(t)| \leq \frac{\epsilon M_g T^\alpha}{\Gamma(\alpha + 1)}.$$

*Proof.* Let  $y \in C([0, T], \mathbb{R})$  be a solution of the inequality (9) for each  $\epsilon > 0$ . Then, from Remark 1 and Lemma 4 for some continuous function  $\Psi(t)$  such that  $|\Psi(t)| < \epsilon, t \in [0, T]$ , we have

$$\begin{aligned} y(t) &= f(t, y(t)) + \theta g(t, y(t)) + \frac{g(t, y(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \times \left( h \left( s, y(s), {}^C D^\alpha \left( \frac{y(s) - f(s, y(s))}{g(s, y(s))} \right) \right) + \Psi(s) \right) ds. \end{aligned}$$

Then, by Remark 1 and (H1), we obtain

$$\begin{aligned} |y(t) - (Py)(t)| &= \left| \frac{g(t, y(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Psi(s) ds \right| \\ &\leq \frac{|g(t, y(t))|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\Psi(s)| ds \\ &\leq \frac{\epsilon M_g T^\alpha}{\Gamma(\alpha + 1)}. \end{aligned}$$

which is satisfied inequality (10). This completes the proof. □

**Theorem 4.** Assume that the assumptions (H1)–(H3) are fulfilled and (6) holds. Then the problem (1) is Ulam-Hyers stable.

*Proof.* Under (H1)–(H3) and (6), the problem (1) has a unique solution in  $C([0, T], \mathbb{R})$ . Let  $y \in C([0, T], \mathbb{R})$  be a solution of the inequality (9), then for each  $t \in [0, T]$ , we have

$$\begin{aligned} &|y(t) - x(t)| \\ &= |y(t) - f(t, x(t)) + \theta g(t, x(t)) \\ &\quad + \frac{g(t, x(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h \left( s, x(s), {}^C D^\alpha \left( \frac{x(s) - f(s, x(s))}{g(s, x(s))} \right) \right) ds| \\ &= |y(t) - Py(t) + Py(t) - Px(t)| \\ &\leq |y(t) - Py(t)| + |Py(t) - Px(t)| \\ &\leq \frac{\epsilon M_g T^\alpha}{\Gamma(\alpha + 1)} + \beta \|y - x\|. \end{aligned}$$

Then

$$\|y - x\| \leq \frac{\epsilon M_g T^\alpha}{(1 - \beta) \Gamma(\alpha + 1)}.$$

By setting

$$k = \frac{M_g T^\alpha}{(1 - \beta) \Gamma(\alpha + 1)},$$

we obtain

$$|y(t) - x(t)| \leq k\epsilon.$$

Therefore, the problem (1) is Ulam-Hyers stable.

This completes the proof.  $\square$

In the next, we introduce the following function.

(H4) The function  $\phi \in C([0, T], \mathbb{R}^+)$  is increasing and there exists a constant  $\lambda_\phi > 0$  such that, for each  $t \in [0, T]$ , we have

$$I^\alpha \phi(t) \leq \lambda_\phi \phi(t).$$

**Lemma 6.** Assume that (H1) and (H4) are satisfied. If  $y \in C([0, T], \mathbb{R})$  is a solution of the fractional differential inequality for each  $\epsilon > 0$

$$(11) \quad \left| {}^C D^\alpha \left( \frac{y(t) - f(t, y(t))}{g(t, y(t))} \right) - h \left( t, y(t), {}^C D^\alpha \left( \frac{y(t) - f(t, y(t))}{g(t, y(t))} \right) \right) \right| \leq \epsilon \phi(t),$$

then,  $y$  is a solution of the following inequality

$$(12) \quad |y(t) - (Py)(t)| \leq \epsilon M_g \lambda_\phi \phi(t).$$

*Proof.* Let  $y \in C([0, T], \mathbb{R})$  be a solution of the inequality (11) for each  $\epsilon > 0$ . Then, from Remark 1 and Lemma 4 for some continuous function  $\Psi(t)$  such that  $|\Psi(t)| < \epsilon \phi(t)$ ,  $t \in [0, T]$ , we have

$$\begin{aligned} |y(t) - (Py)(t)| &= \left| \frac{g(t, y(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Psi(s) \, ds \right| \\ &\leq \frac{|g(t, y(t))|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\Psi(s)| \, ds \\ &\leq \frac{\epsilon M_g}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi(s) \, ds \\ &\leq \epsilon M_g \lambda_\phi \phi(t). \end{aligned}$$

which is satisfied inequality (12). This completes the proof.  $\square$

**Theorem 5.** Assume that the assumptions (H1)–(H4) are fulfilled and (6) holds. Then the problem (1) is Ulam-Hyers-Rassias stable.

*Proof.* Under (H1)–(H4) and (6), the problem has a unique solution in  $C([0, T], \mathbb{R})$ . Let  $y \in C([0, T], \mathbb{R})$  be a solution of the inequality (11), then for each  $t \in [0, T]$ , we have

$$|y(t) - x(t)|$$

$$\begin{aligned}
 &= \left| y(t) - f(t, x(t)) + \theta g(t, x(t)) + \frac{g(t, x(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \right. \\
 &\quad \left. \times h\left(s, x(s), {}^C D^\alpha \left( \frac{x(s) - f(s, x(s))}{g(s, x(s))} \right)\right) ds \right| \\
 &= |y(t) - (Py)(t) + (Py)(t) - (Px)(t)| \\
 &\leq |y(t) - (Py)(t)| + |(Py)(t) - (Px)(t)| \\
 &\leq \epsilon M_g \lambda_\phi \phi(t) + \beta \|y - x\|.
 \end{aligned}$$

Then

$$|y(t) - x(t)| \leq \frac{\epsilon M_g \lambda_\phi \phi(t)}{1 - \beta}.$$

By taking a constant

$$k_{\phi, f} = \frac{M_g \lambda_\phi}{1 - \beta},$$

we obtain

$$|y(t) - x(t)| \leq k_{\phi, f} \epsilon \phi(t).$$

Therefore, the problem (1) is Ulam-Hyers-Rassias stable. This completes the proof.  $\square$

### 5. EXAMPLE

Consider the following nonlinear problem

$$(13) \quad \begin{cases} {}^C D^{\frac{3}{4}} \left( \frac{x(t) - \frac{1}{4}t \sin(x(t))}{1 + \frac{1}{10} \exp(-t^2) \cos(x(t))} \right) \\ = \frac{1}{5}t \cos(x(t)) + \frac{1}{5 + \left| {}^C D^{\frac{3}{4}} \left( \frac{x(t) - \frac{1}{4}t \sin(x(t))}{1 + \frac{1}{10} \exp(-t^2) \cos(x(t))} \right) \right|}, \quad t \in [0, 1], \\ x(0) = g(0, x(0)) + f(0, x(0)). \end{cases}$$

where  $\theta = 1$ ,  $f(t, x(t)) = \frac{1}{4}t \sin(x(t))$ ,  $g(t, x(t)) = 1 + \frac{1}{10} \exp(-t^2) \cos(x(t))$  and

$$\begin{aligned}
 &h\left(t, x(t), {}^C D^\alpha \left( \frac{x(t) - f(t, x(t))}{g(t, x(t))} \right)\right) \\
 &= \frac{1}{5}t \cos(x(t)) + \frac{1}{5 + \left| {}^C D^{\frac{3}{4}} \left( \frac{x(t) - \frac{1}{4}t \sin(x(t))}{1 + \frac{1}{10} \exp(-t^2) \cos(x(t))} \right) \right|}.
 \end{aligned}$$

Set  $h(t, u, v) = \frac{1}{5}t \cos(u) + \frac{1}{5 + |v|}$ . For any  $u, v, u^*, v^* \in \mathbb{R}$  and  $t \in [0, 1]$ , we have

$$\begin{aligned}
 |f(t, u) - f(t, u^*)| &\leq \frac{1}{4} |u - u^*|, \\
 |g(t, u) - g(t, u^*)| &\leq \frac{1}{10} |u - u^*|, \\
 |h(t, u, v) - h(t, u^*, v^*)| &\leq \frac{1}{5} |u - u^*| + \frac{1}{25} |v - v^*|,
 \end{aligned}$$

and

$$|g(t, u)| \leq \frac{11}{10}, \quad |h(t, u, v)| \leq \frac{2}{5}.$$

Hence, all conditions of Theorem 2 are fulfilled and  $\beta = 0.6393 < 1$ , with

$$M_g = \frac{11}{10}, \quad M_h = \frac{2}{5}, \quad K_1 = \frac{1}{4}, \quad K_2 = \frac{1}{10}, \quad K_3 = \frac{1}{5}, \quad K_4 = \frac{1}{25}.$$

Then the problem (13) has a unique solution  $x \in C([0, T], \mathbb{R})$ . And from Theorem 4 we deduce that (13) is Ulam-Hyers stable.

Now, we choose  $\phi(t) = t^2$  and in view of (2), we have

$$I^\alpha \phi(t) = \frac{\Gamma(3)}{\Gamma(3+\alpha)} t^{2+\alpha} \leq \frac{2}{\Gamma(\frac{7}{2})} t^2 = \lambda_\phi \phi(t).$$

Thus condition (H4) is satisfied with  $\phi(t) = t^2$  and  $\lambda_\phi = \frac{2}{\Gamma(\frac{7}{2})} = \frac{16}{15\sqrt{\pi}}$ , it follows from Theorem 5 that the problem (13) is Ulam-Hyers-Rassias stable.

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