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On exponentially (h_1, h_2)-convex functions and fractional integral inequalities related

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Abstract. In this work the concept of exponentially (h_1, h_2)-convex function is introduced and using it, the Hermite-Hadamard inequality and some bounds for the right side of this inequality, via Raina's fractional integral operator and generalized convex functions, are established.

1. Introduction

In many practical investigations it is necessary to bound one quantity by another. The classical inequalities are very useful for this purpose. An enormous amount of efforts has been devoted to the extension of the classical inequalities and to the applications of the same in diverse areas of science: estimation of integrals, special functions of mathematical physics, electrostatic field and capacitance, signal analysis, dynamical system stability and control and others.

One of the most discussed inequalities in recent work is the classic Hermite-Hadamard inequality. In [7], J. Hadamard stated his famous inequality in this way.

Theorem 1. *Let f be a convex function over $[a,b]$, $a < b$. If f is integrable over $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

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The subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during almost the past five decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. It does indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables.

The inequalities involving more general fractional integral operators have also been considered in [2, 12, 16, 19]. Since work in this direction has received a lot of attention, as evidenced in the work of S. Turhan et. al. [13, 20] and J. E. Hernández Hernández and M. J. Vivas-Cortez [8, 9, 10, 21], in this work we establish a general expression of some Hermite-Hadamard type inequa-lities by the introduction of the concept of exponentially $(h_1, h_2)-$convex function and using the Raina's fractional integral operator.

2. Preliminaries

2.1. **About Fractional Integral Operator.** In [16], R. K. Raina introduced a class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) = \mathcal{F}^{\sigma(0),\sigma(1),..}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \tag{2}$$

where $\rho, \lambda > 0, |x| < R$, (R is the set of real numbers), $\sigma = (\sigma(1), .., \sigma(k), ..)$ is a bounded sequence of positive real numbers. Note that if we take in (2) $\rho = 1, \lambda = 0$ and $\sigma(k) = ((\alpha)_k(\beta)_k)/(\gamma)_k)$ for $k = 0, 1, 2, ...$, where α, β and γ are parameters which can take arbitrary real or complex values (provided that $\gamma \neq 0, -1, -2, ...$), and the symbol $(a)_k$ denote the quantity

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)...(a+k-1), \quad k = 0, 1, ...,$$

and restrict its domain to $|x| \leq 1$ (with $x \in \mathbb{C}$), then we have the classical Hypergeometric Function, that is

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) = F(\alpha, \beta; \gamma; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k)}{(\gamma)_k k!} x^k,$$

also, if $\sigma(k) = (1, 1, 1, ...)$ with $\rho = \alpha, (Re(\alpha) > 0)$, $\lambda = 1$ and restricting its domain to $z \in \mathbb{C}$ in (2) then we have the classical Mitag-Leffler function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} z^k.$$

When it is provided that the series converges uniformly then we can differentiate term wise, also integrate, to obtain

$$\left(\frac{d}{dx}\right)^n x^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}(wx^{\rho}) = x^{\lambda-n-1}\mathcal{F}^{\sigma}_{\rho,\lambda-n}(wx^{\rho})$$

and

$$\int_0^x ... \int_0^x t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}(wt^{\rho})(dt)^n = x^{\lambda+n-1}\mathcal{F}^{\sigma}_{\rho,\lambda+n}(wx^{\rho}).$$

Using (2), in [2], R. P. Agarwal et. al., defined the following left-sided and right-sided fractional integral operators respectively, as follows

$$(3)\qquad \left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi\right)(x) = \int_a^x (x-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x-t)^{\rho}\right]\varphi(t)dt, \quad (x>a)$$

and

$$(4)\qquad \left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi\right)(x) = \int_x^b (t-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(t-x)^{\rho}\right]\varphi(t)dt, \quad (x<b),$$

where $\lambda,\rho>0$, $w\in R$ and φ is such that the integral on the right side exits.

It is easy to verify that $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi$ and $\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi$ are bounded integral operators on $L_p(a,b)$,$(1\le p\le\infty)$, if

$$\mathfrak{M} := \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right] < \infty.$$

Indeed, for $\varphi\in L_p\left((a,b)\right)$ we have

$$\left\|\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi\right\|_p \le \mathfrak{M}\left\|\varphi\right\|_p$$

and

$$\left\|\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi\right\|_p \le \mathfrak{M}\left\|\varphi\right\|_p$$

where

$$\left\|\varphi\right\|_p = \left(\int_a^b \left|\varphi(x)\right|^p dx\right)^{1/p}.$$

Many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. By example, the classical Riemann-Liouville fractional integrals I^{α}_{a+} and I^{α}_{b-} of order α

$$\left(I^{\alpha}_{a+}\varphi\right)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\varphi(t)dt, \qquad (x>a,\alpha>0)$$

and

$$\left(I^{\alpha}_{b-}\varphi\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\varphi(t)dt, \qquad (x<b,\alpha>0)$$

follow from (3) and (4) setting $\lambda=\alpha,\sigma(0)=1$ and $w=0$.

The Hermite-Hadamard integral inequality for the Raina's fractional integral operator is established in [22] as follows.

Theorem 2. *Let $\lambda \in \mathbb{R}^+, a, b \in \mathbb{R}, a < b$ and $\phi : [a, b] \to \mathbb{R}$ be a convex function. Then*

$$\phi\left(\frac{a+b}{2}\right) \leq \frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+,w}\phi\right)(b) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-,w}\phi\right)(a)}{(b-a)^{\lambda}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\,(b-a)^{\rho}\right]} \leq \frac{\phi(a)+\phi(b)}{2}.$$

2.2. **About Generalized convexity.** The well known concept of convex function is due to W. Jensen and it is established as follow.

Definition 1. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called convex on the interval I, if the following inequality holds

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$$

for all $a, b \in I$ and $t \in [0, 1]$.

From the work of S. S. Dragomir et. al. [5], we extract the following definition.

Definition 2. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a non-negative function where I is an interval. It is said that f belongs to the class $P(I)$ or f is a $P-$convex if for all $a, b \in I$ and $t \in [0, 1]$ the following inequality holds

$$f(ta + (1-t)b) \leq f(a) + f(b).$$

Also, H. Hudzik and L. Maligranda, in [11], disused about some properties of the following generalized concept of convexity.

Definition 3. Let $0 < s \leq 1$. A function $f : \mathbb{R}_+ \to \mathbb{R}_+$, where $\mathbb{R}_+ = [0, \infty)$, is said to be s-convex in the first sense if

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t^s)f(b)$$

for all $a, b \in I$ and $t \in [0, 1]$. This is denoted by $f \in K^1_s$. A function $f : \mathbb{R}_+ \to \mathbb{R}$, is said to be s-convex in the second sense if

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b)$$

for all $a, b \in I$ and $t \in [0, 1]$. This is denoted by $f \in K^2_s$.

The first class of functions in Definition 3 were introduced by Orlicz W. in [15], and the second class by W. W. Breckner in [3].

G. Cristescu et. al., in [4], in order to study bounds of the second degree cumulative frontier gaps of functions with generalized convexity functions, introduced the so-called $(h_1, h_2)-$convex functions.

Definition 4. Let $h_1, h_2 : [0, 1] \to \mathbb{R}$ be two non-negative functions. A function $f : I \to \mathbb{R}_+$ is called an $(h_1, h_2)-$ convex function if the inequality

$$f(ta + (1-t)b) \leq h_1(t)f(a) + h_2(t)f(b)$$

holds for all $a, b \in I$ and $t \in [0,1]$. The functions that transform the inequality in an equality is called $(h_1, h_2)-$affine function.

Remark 1. If $h_1(t) = t$ and $h_2(t) = 1-t$ for all $t \in [0,1]$, then the $(h_1, h_2)-$convexity coincides with the classical convexity. If $h_1(t) = h_2(t) = 1$ for all $t \in [0,1]$ the it is obtained the $P-$convexity. If $h_1(t) = t^s$ and $h_2(t) = 1 - t^s$ for all $t \in [0,1]$, then we have the $s-$convexity in the first sense, and If $h_1(t) = t^s$ and $h_2(t) = (1-t)^s$ for all $t \in [0,1]$, we get the $s-$convexity in the second sense.

The exponentially convex functions are of interest for the development of this work. In the works of T. Antczac [1] and S. S. Dragomir [6] introduce this concept and find some results related to the Hermite-Hadamard inequality.

Definition 5. A positive function $f : I \to \mathbb{R}$ is said to be an exponentially convex function if the inequality

$$e^{f(ta+(1-t)b)} \leq te^{f(a)} + (1-t)e^{f(b)}$$

holds for all $a, b \in I$ and $t \in [0,1]$.

3. Main Results

Definition 6. Let $h_1, h_2 : [0,1] \to R$ be a two non negative functions. A positive function $f : I \to R$, where I is an interval include in $\mathbb{R}$, is called exponentially $(h_1, h_2)-$convex if the following inequality holds for all $x, y \in I$ and $t \in [0,1]$

$$e^{f(tx+(1-t)y)} \leq h_1(t)e^{f(x)} + h_2(t)e^{f(y)}.$$

Remark 2. Note that:

(1) If $h_1 = h_2 \equiv 1$ then we have an exponentially $P-$convex function.
(2) If $h_1(t) = t$ and $h_2(t) = 1-t$ for all $t \in [0,1]$ we obtain an exponentially convex function.
(3) If $h_1(t) = t^s$ and $h_2(t) = 1 - t^s$ for all $t \in [0,1]$ and some $0 < s \leq 1$, then we have the exponentially $s-$convexity in the first sense.
(4) If $h_1(t) = t^s$ and $h_2(t) = (1-t)^s$ for all $t \in [0,1]$ and some $0 < s \leq 1$, we get the exponentially $s-$convexity in the second sense.

First, we establish the Hermite-Hadamard inequality for exponentially convex function using Raina's fractional integral operator.

Theorem 3. *Let $\lambda, \rho > 0, w \in R$, and $\sigma = \{\sigma(k)\}_{k=0}^{\infty}$ a sequence of non-negatives real numbers. Let $f : [a,b] \to R$ be an exponentially $(h_1, h_2)-$convex function, then the following inequalities holds*

$$e^{f\left(\frac{a+b}{2}\right)} \leq \frac{\left(h_1(1/2)\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+,w}e^f\right)(b) + h_2(1/2)\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-,w}e^f\right)(a)\right)}{(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\,(b-a)^{\rho}\right]}$$

and

$$\frac{1}{(b-a)^{\lambda}}\left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+,w}e^{f}\right)(b)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-,w}e^{f}\right)(a)\right)$$
$$\leq\left(e^{f(a)}+e^{f(b)}\right)\left(I\left(h_1\right)+I\left(h_2\right)\right),$$

where

$$I\left(h_1\right)=\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(b-a\right)^{\rho}t^{\rho}\right]h_1(t)dt$$

and

$$I\left(h_2\right)=\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(b-a\right)^{\rho}t^{\rho}\right]h_2(t)dt.$$

Proof. Note that

$$\frac{a+b}{2}=\frac{ta+(1-t)b+tb+(1-t)a}{2}$$

for any $t\in[0,1]$. Consequently, using the exponentially $(h_1,h_2)-$convexity of f we have

$$e^{f\left(\frac{a+b}{2}\right)}=e^{f\left(\frac{ta+(1-t)b+tb+(1-t)a}{2}\right)}$$
$$\leq h_1(1/2)e^{f(ta+(1-t)b)}+h_2(1/2)e^{f(tb+(1-t)a)}.$$

Multiplying by $t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(b-a\right)^{\rho}t^{\rho}\right]$ in both sides of the above inequality

$$e^{f\left(\frac{a+b}{2}\right)}t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(b-a\right)^{\rho}t^{\rho}\right]$$
$$\leq t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(b-a\right)^{\rho}t^{\rho}\right]\left(h_1(1/2)e^{f(ta+(1-t)b)}+h_2(1/2)e^{f(tb+(1-t)a)}\right).$$

Integrating over $t\in[0,1]$ we have

$$e^{f\left(\frac{a+b}{2}\right)}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(b-a\right)^{\rho}\right]\tag{5}$$
$$\leq h_1(1/2)\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(b-a\right)^{\rho}t^{\rho}\right]e^{f(ta+(1-t)b)}dt$$
$$+h_2(1/2)\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(b-a\right)^{\rho}t^{\rho}\right]e^{f(tb+(1-t)a)}dt.$$

With a convenient change of variable we have

$$\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(b-a\right)^{\rho}t^{\rho}\right]e^{f(ta+(1-t)b)}dt\tag{6}$$
$$\leq\frac{1}{b-a}\int_a^b\left(\frac{b-x}{b-a}\right)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(b-a\right)^{\rho}\left(\frac{b-x}{b-a}\right)^{\rho}\right]e^{f(x)}dx$$
$$=\frac{1}{(b-a)^{\lambda}}\int_a^b\left(b-x\right)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(b-x\right)^{\rho}\right]e^{f(x)}dx$$

$$= \frac{1}{(b-a)^{\lambda}} \left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+,w} e^{f} \right)(b)$$

and

$$\text{(7)} \quad \int_0^1 t^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda} \left[w\,(b-a)^{\rho}\, t^{\rho} \right] e^{f(tb+(1-t)a)} dt$$
$$\leq \frac{1}{b-a} \int_a^b \left(\frac{x-a}{b-a} \right)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda} \left[w\,(b-a)^{\rho} \left(\frac{x-a}{b-a} \right)^{\rho} \right] e^{f(x)} dx$$
$$= \frac{1}{(b-a)^{\lambda}} \int_a^b (x-a)^{\lambda-1}\, \mathcal{F}^{\sigma}_{\rho,\lambda} \left[w\,(b-x)^{\rho} \right] e^{f(x)} dx$$
$$= \frac{1}{(b-a)^{\lambda}} \left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-,w} e^{f} \right)(a).$$

By replacement of (6) and (7) in (5) we have

$$e^{f\left(\frac{a+b}{2}\right)} \mathcal{F}^{\sigma}_{\rho,\lambda+1} \left[w\,(b-a)^{\rho} \right]$$
$$\leq \frac{1}{(b-a)^{\lambda}} \left(h_1(1/2) \left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+,w} e^{f} \right)(b) + h_2(1/2) \left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-,w} e^{f} \right)(a) \right).$$

For the right side of the proposed inequality we have

$$e^{f(ta+(1-t)b)} \leq h_1(t) e^{f(a)} + h_2(t) e^{f(b)}$$
$$e^{f(tb+(1-t)a)} \leq h_1(t) e^{f(b)} + h_2(t) e^{f(a)}.$$

Multiplying by $t^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda} \left[w\,(b-a)^{\rho}\, t^{\rho} \right]$ both inequalities

$$t^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda} \left[w\,(b-a)^{\rho}\, t^{\rho} \right] e^{f(ta+(1-t)b)}$$
$$\leq t^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda} \left[w\,(b-a)^{\rho}\, t^{\rho} \right] \left(h_1(t) e^{f(a)} + h_2(t) e^{f(b)} \right),$$

$$t^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda} \left[w\,(b-a)^{\rho}\, t^{\rho} \right] e^{f(tb+(1-t)a)}$$
$$\leq t^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda} \left[w\,(b-a)^{\rho}\, t^{\rho} \right] \left(h_1(t) e^{f(b)} + h_2(t) e^{f(a)} \right).$$

Adding these inequalities and integrating over $t \in [0,1]$ we obtain

$$\int_0^1 t^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda} \left[w\,(b-a)^{\rho}\, t^{\rho} \right] \left(e^{f(ta+(1-t)b)} + e^{f(tb+(1-t)a)} \right) dt$$
$$\leq \left(e^{f(a)} + e^{f(b)} \right) \left(\int_0^1 t^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda} \left[w\,(b-a)^{\rho}\, t^{\rho} \right] h_1(t) dt \right.$$
$$\left. + \int_0^1 t^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda} \left[w\,(b-a)^{\rho}\, t^{\rho} \right] h_2(t) dt \right).$$

With the change of variable $u = ta + (1-t)b$ and $v = tb + (1-t)a$ in the first integral of the above inequality it is obtained

$$\frac{1}{(b-a)^{\lambda}}\left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+,w}e^{f}\right)(b)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-,w}e^{f}\right)(a)\right)$$
$$\leq\left(e^{f(a)}+e^{f(b)}\right)\left(\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(b-a\right)^{\rho}t^{\rho}\right]h_1(t)dt\right.$$
$$\left.+\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(b-a\right)^{\rho}t^{\rho}\right]h_2(t)dt\right),$$

and letting

$$I\left(h_1\right)=\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(b-a\right)^{\rho}t^{\rho}\right]h_1(t)dt$$

and

$$I\left(h_2\right)=\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(b-a\right)^{\rho}t^{\rho}\right]h_2(t)dt$$

it is attained the desired result. □

Remark 3. Doing $\lambda=\alpha$, $w=1$ and $\sigma=(1,0,\dots)$, then from Theorem 3, we obtain the Riemann-Liouville fractional integral version:

$$e^{f\left(\frac{a+b}{2}\right)}\leq\frac{\left(h_1(1/2)\left(\mathcal{I}^{\alpha}_{a+}e^{f}\right)(b)+h_2(1/2)\left(\mathcal{I}^{\alpha}_{b-}e^{f}\right)(a)\right)}{(b-a)^{\alpha}\,\Gamma(\alpha)^{-1}}$$

and

$$\frac{1}{(b-a)^{\alpha}}\left(\left(\mathcal{I}^{\alpha}_{a+}e^{f}\right)(b)+\left(\mathcal{I}^{\alpha}_{b-}e^{f}\right)(a)\right)\leq\left(e^{f(a)}+e^{f(b)}\right)\left(I\left(h_1\right)+I\left(h_2\right)\right),$$

where $I\left(h_1\right)$ and $I\left(h_1\right)$ take the form

$$I\left(h_1\right)=\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}h_1(t)dt \text{ and } I\left(h_2\right)=\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}h_2(t)dt$$

additionally if $h_1(t)=t$ and $h(t)=1-t$ then

$$e^{f\left(\frac{a+b}{2}\right)}\leq\frac{\Gamma(\alpha+1)\left(\left(\mathcal{I}^{\alpha}_{a+}e^{f}\right)(b)+\left(\mathcal{I}^{\alpha}_{b-}e^{f}\right)(a)\right)}{2\left(b-a\right)^{\alpha}}\leq\frac{\left(e^{f(a)}+e^{f(b)}\right)}{\Gamma(\alpha+1)},$$

and if we choose $\alpha=1$ then it is obtained

$$2e^{f\left(\frac{a+b}{2}\right)}\leq\frac{1}{b-a}\int_a^b e^{f(t)}dt\leq 2\left(e^{f(a)}+e^{f(b)}\right)$$

making coincidence with Corollary 3.2 in [17]. If $h_1(t)=t^s$ and $h_2=(1-t)^s$ for $t\in[0,1]$ and some $s\in(0,1]$ with $\lambda=\alpha=1$, $w=1$ and $\sigma=(1,0,\dots)$ we find coincidence with Corollary 1 obtained by S. Rashid et.al. in [18].

The following Lemma will be useful to establish some others inequalities related with the right side of the Hermite-Hadamard inequality for exponentially convex functions using the Raina's fractional integral operator.

Lemma 3.1. *Let $\lambda, \rho > 0, w \in R$, and $\sigma = \{\sigma(k)\}_{k=0}^{\infty}$ a sequence of non-negatives real numbers. Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) with $a < b$ and $\lambda > 0$. If $e^f \in L_1([a,b])$ then the following equality for the Raina's fractional integral operator holds*

$$\begin{aligned}
&\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w} e^f\right)(b) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w} e^f\right)(a) \\
&\quad - \left(\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho})}{(b-a)^{1-\lambda}}\right)\left(e^{f(a)} + e^{f(b)}\right) \\
&= \int_0^1 t^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho} t^{\rho}) e^{f(ta+(1-t)b)} f'(ta+(1-t)b)dt \\
&\quad - \int t^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho} t^{\rho}) e^{f((1-t)a+tb)} f'((1-t)a+tb)dt.
\end{aligned}$$

Proof. Using integration by parts it follows that

$$\begin{aligned}
I_1 &= \int_0^1 t^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho} t^{\rho}) e^{f(ta+(1-t)b)} f'(ta+(1-t)b)dt \\
&= \left.\frac{t^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho} t^{\rho}) e^{f(ta+(1-t)b)}}{a-b}\right|_0^1 \\
&\quad - \frac{1}{a-b} \int_0^1 t^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}(w(b-a)^{\rho} t^{\rho}) e^{f(ta+(1-t)b)} dt \\
&= -\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}) e^{f(a)}}{b-a} + \frac{1}{(b-a)^{\lambda}} \left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w} e^f\right)(b)
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \int t^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho} t^{\rho}) e^{f((1-t)a+tb)} f'((1-t)a+tb)dt \\
&= \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}) e^{f(b)}}{b-a} \\
&\quad - \frac{1}{b-a} \int_0^1 t^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}(w(b-a)^{\rho} t^{\rho}) e^{f((1-t)a+tb)} dt \\
&= \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}) e^{f(b)}}{b-a} - \frac{1}{(b-a)^{\lambda}} \left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w} e^f\right)(a).
\end{aligned}$$

Subtracting I_2 from I_1 it is attained the desired result. □

Theorem 4. *Let $\lambda, \rho > 0, w \in R$, and σ a sequence of non-negatives real numbers. Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) with $a < b$, and exponentially $(h_1, h_2)-$convex. If $e^f \in L_1([a,b])$ and $|f'|$ is*

(g_1,g_2) *−convex then the following inequality for the Raina's fractional integral operator holds*

$$\begin{aligned}&\Big|\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}e^{f}\right)(b)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}e^{f}\right)(a)\\&\quad-\left(\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho})}{(b-a)^{1-\lambda}}\right)\left(e^{f(a)}+e^{f(b)}\right)\Big|\\&\le (e^{f(a)}|f'(a)|+e^{f(b)}|f'(b)|)(I(h_1,g_1)+I(h_2,g_2))\\&\quad+(e^{f(a)}|f'(b)|+e^{f(b)}|f'(a)|)(|(I(h_1,g_2)+I(h_2,g_1)),\end{aligned}$$

where

$$\begin{aligned}I(h_1,g_1)&=\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}t^{\rho})h_1(t)g_1(t)dt,\\I(h_2,g_2)&=\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}t^{\rho})h_2(t)g_2(t)dt,\\I(h_1,g_2)&=\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}t^{\rho})h_1(t)g_2(t)dt,\\I(h_2,g_1)&=\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}t^{\rho})h_2(t)g_1(t)dt.\end{aligned}$$

Proof. Using the Lemma 3.1 and the triangular inequality we have

$$\begin{aligned}&\Big|\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}e^{f}\right)(b)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}e^{f}\right)(a)\\&\quad-\left(\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho})}{(b-a)^{1-\lambda}}\right)\left(e^{f(a)}+e^{f(b)}\right)\Big|\\&\le\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}t^{\rho})|e^{f(ta+(1-t)b)}f'(ta+(1-t)b)|dt\\&\quad+\int t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}t^{\rho})|e^{f((1-t)a+tb)}f'((1-t)a+tb)|dt.\end{aligned}\tag{8}$$

Now, we discuse the integrals involve in (8) using the exponentially $(h_1,h_2)-$ convexity of f and the $(g_1,g_2)-$convexity of $|f'|$. First,

$$\begin{aligned}&\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}t^{\rho})|e^{f(ta+(1-t)b)}f'(ta+(1-t)b)|dt\\&\le\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\lambda+1}(w(b-a)^{\rho}t^{\rho})\times\\&\qquad\left(h_1(t)e^{f(a)}+h_2(t)e^{f(b)}\right)\left(g_1(t)|f'(a)|+g_2(t)|f'(b)|\right)dt\\&\le\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}t^{\rho})\times\end{aligned}$$

$$\Big(h_1(t)e^{f(a)}g_1(t)|f'(a)| + h_2(t)e^{f(b)}g_2(t)|f'(b)|$$
$$+h_1(t)e^{f(a)}g_2(t)|f'(b)| + h_2(t)e^{f(b)}g_1(t)|f'(a)|\Big)\,dt$$

$$= e^{f(a)}|f'(a)|\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}t^{\rho})h_1(t)g_1(t)dt$$
$$+ e^{f(b)}|f'(b)|\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}t^{\rho})h_2(t)g_2(t)dt$$
$$+ e^{f(a)}|f'(b)|\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}t^{\rho})h_1(t)g_2(t)dt$$
$$+ e^{f(b)}|f'(a)|\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}t^{\rho})h_2(t)g_1(t)dt$$
$$= e^{f(a)}|f'(a)|I(h_1,g_1) + e^{f(b)}|f'(b)|I(h_2,g_2) \tag{9}$$
$$+ e^{f(a)}|f'(b)|I(h_1,g_2) + e^{f(b)}|f'(a)|I(h_2,g_1),$$

where

$$I(h_1,g_1) = \int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}t^{\rho})h_1(t)g_1(t)dt,$$
$$I(h_2,g_2) = \int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}t^{\rho})h_2(t)g_2(t)dt,$$
$$I(h_1,g_2) = \int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}t^{\rho})h_1(t)g_2(t)dt,$$
$$I(h_2,g_1) = \int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}t^{\rho})h_2(t)g_1(t)dt.$$

Similarly, for the second integral we have

$$\int t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}t^{\rho})|e^{f((1-t)a+tb)}f'((1-t)a+tb)|dt$$
$$\leq e^{f(a)}|f'(a)|I(h_2,g_2) + e^{f(b)}|f'(b)|I(h_1,g_1) \tag{10}$$
$$+ e^{f(a)}|f'(b)|I(h_2,g_1) + e^{f(b)}|f'(a)|I(h_1,g_2).$$

By replacement of (9) and (10) in (8) then it follows the result. □

Using the previous Theorem some Corollary is deduced.

Corollary 1. *Let $\lambda,\rho>0, w\in R$, and $\sigma=\{\sigma(k)\}_{k=0}^{\infty}$ a sequence of non-negatives real numbers. Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) with $a<b$, and exponentially convex. If $e^f\in L_1([a,b])$ and $|f'|$ is a convex function then the following inequality for the Raina's fractional integral operator holds*

$$\left|\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}e^f\right)(b) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}e^f\right)(a)\right.$$

$$
\begin{aligned}
& - \left(\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho})}{(b-a)^{1-\lambda}} \right) \left(e^{f(a)} + e^{f(b)} \right) \Bigg| \\
& \leq (e^{f(a)}|f'(a)| + e^{f(b)}|f'(b)|)\mathcal{F}^{2\sigma_1}_{\rho,\lambda+1}(w(b-a)^{\rho}) \\
& \quad + (e^{f(a)}|f'(b)| + e^{f(b)}|f'(a)|)\mathcal{F}^{2\sigma_2}_{\rho,\lambda+1}(w(b-a)^{\rho}),
\end{aligned}
$$

where

$$
\left.\begin{aligned}
\sigma_1(k) &= \frac{\sigma(k)}{k\rho+\lambda+3}, \\
\sigma_2(k) &= \frac{\sigma(k)}{(k\rho+\lambda+3)(k\rho+\lambda+2)},
\end{aligned}\right. \qquad k = 0,1,2,\ldots
$$

Proof. Letting $h_1(t) = g_1(t) = t$ and $h_2(t) = g_2(t) = 1-t$ for all $t \in [0,1]$ then

$$
\begin{aligned}
I(h_1,g_1) = I(h_2,g_2) &= \int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}t^{\rho})t^2dt \\
&= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)} w^k(b-a)^{k\rho} \int_0^1 t^{k\rho+\lambda+2}dt \\
&= \mathcal{F}^{\sigma_1}_{\rho,\lambda+1}(w(b-a)^{\rho}),
\end{aligned}
$$

where

$$
\sigma_1(k) = \frac{\sigma(k)}{k\rho+\lambda+3} \text{ for } k = 0,1,2,\ldots
$$

Similarly

$$
\begin{aligned}
I(h_1,g_2) = I(h_2,g_1) &= \int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}t^{\rho})t(1-t)dt \\
&= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)} w^k(b-a)^{k\rho} \int_0^1 t^{k\rho+\lambda+1}(1-t)dt \\
&= \mathcal{F}^{\sigma_2}_{\rho,\lambda+1}(w(b-a)^{\rho})
\end{aligned}
$$

where

$$
\sigma_2(k) = \frac{\sigma(k)}{(k\rho+\lambda+3)(k\rho+\lambda+2)}, \quad k = 0,1,2,\ldots
$$

Making the corresponding substitutions in Theorem 4 it follows the desired result. □

Corollary 2. *Let $\lambda, \rho > 0, w \in R$, and $\sigma = \{\sigma(k)\}_{k=0}^{\infty}$ a sequence of non-negatives real numbers. Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) with $a < b$, and exponentially convex. If $e^f \in L_1([a,b])$ and $|f'|$ is convex and bounded by some $M > 0$, then the following inequality for the Raina's fractional integral operator holds*

$$
\left| \left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w} e^f \right)(b) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w} e^f \right)(a) \right.
$$

$$
-\left(\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho})}{(b-a)^{1-\lambda}}\right)\left(e^{f(a)}+e^{f(b)}\right)\Bigg|
\leq M(e^{f(a)}+e^{f(b)})\mathcal{F}^{2\sigma'}_{\rho,\lambda+1}(w(b-a)^{\rho}),
$$

where

$$
\sigma'(k)=\frac{\sigma(k)}{k\rho+\lambda+2},\quad k=0,1,2,\ldots
$$

Proof. Noting that

$$
\begin{aligned}
\sigma'(k) &= \sigma_1(k)+\sigma_2(k)\\
&= \frac{\sigma(k)}{k\rho+\lambda+2},\quad k=0,1,2,\ldots
\end{aligned}
$$

then, using Corollary 1 easily it finds the result. □

Corollary 3. *Let $\lambda,\rho>0, w\in R$, and $\sigma=\{\sigma(k)\}_{k=0}^{\infty}$ a sequence of non-negatives real numbers. Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) with $a<b$, and exponentially convex. If $e^{f}\in L_1([a,b])$ and $|f'|$ is a $P-$convex function then the following inequality for the Raina's fractional integral operator holds*

$$
\begin{aligned}
&\left|\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}e^{f}\right)(b)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}e^{f}\right)(a)\right.\\
&\quad\left.-\left(\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho})}{(b-a)^{1-\lambda}}\right)\left(e^{f(a)}+e^{f(b)}\right)\right|\\
&\leq \mathcal{F}^{\sigma'}_{\rho,\lambda+1}(w(b-a)^{\rho})\left[(e^{f(a)}+e^{f(b)})(|f'(a)|+|f'(b)|)\right],
\end{aligned}
$$

where

$$
\sigma'(k)=\frac{\sigma(k)}{k\rho+\lambda+1},\quad k=0,1,2,\ldots
$$

Proof. Following the same demonstration scheme used in the Corollary 1 with Letting $h_1(t)=t$, $h_2(t)=1-t$, $g_1(t)=g_2(t)=1$, for all $t\in[0,1]$ then it is obtained that

$$
\begin{aligned}
I(h_1,g_1) &= I(h_1,g_2)\\
&= \int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}t^{\rho})tdt\\
&= \mathcal{F}^{\sigma_1}_{\rho,\lambda+1}(w(b-a)^{\rho}),
\end{aligned}
$$

where

$$
\sigma_1(k)=\frac{\sigma(k)}{k\rho+\lambda+2},\quad k=0,1,2,\ldots
$$

and

$$
I(h_2,g_1)=I(h_2,g_2)=\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^{\rho}t^{\rho})(1-t)dt=\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}(w(b-a)^{\rho}),
$$

where

$$\sigma_2(k) = \frac{\sigma(k)}{(k\rho+\lambda+2)(k\rho+\lambda+1)}, \quad k = 0, 1, 2, \ldots$$

Note that

$$\begin{aligned} I(h_1, g_1) + I(h_2, g_2) &= \mathcal{F}^{\sigma_1}_{\rho,\lambda+1}(w(b-a)^\rho) + \mathcal{F}^{\sigma_2}_{\rho,\lambda+1}(w(b-a)^\rho) \\ &= \mathcal{F}^{\sigma'}_{\rho,\lambda+1}(w(b-a)^\rho), \end{aligned}$$

where

$$\sigma'(k) = \sigma_1(k) + \sigma_2(k) = \frac{\sigma(k)}{k\rho+\lambda+1}, \quad k = 0, 1, 2, \ldots$$

The proof is complete. □

Corollary 4. *Let $\lambda, \rho > 0, w \in R$, and $\sigma = \{\sigma(k)\}_{k=0}^{\infty}$ a sequence of non-negatives real numbers. Let $f : [a, b] \to R$ be a differentiable mapping on (a, b) with $a < b$, and exponentially convex. If $e^f \in L_1([a, b])$ and $|f'|$ is a $s-$convex function in the second sense then the following inequality for the Raina's fractional integral operator holds*

$$\begin{aligned} &\Bigg| \left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w} e^f\right)(b) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w} e^f\right)(a) \\ &\quad - \left(\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^\rho)}{(b-a)^{1-\lambda}}\right)\left(e^{f(a)} + e^{f(b)}\right)\Bigg| \\ &\le (e^{f(a)}|f'(a)| + e^{f(b)}|f'(b)|)\times \\ &\qquad (\mathcal{F}^{\sigma_{s;1,1}}_{\rho,\lambda+1}(w(b-a)^\rho) + \Gamma(s+2)\mathcal{F}^{\sigma_1}_{\rho,\lambda+s+4}(w(b-a)^\rho)) \\ &\quad + (e^{f(a)}|f'(b)| + e^{f(b)}|f'(a)|)\times \\ &\qquad (\Gamma(s+1)\mathcal{F}^{\sigma_2}_{\rho,\lambda+s+4}(w(b-a)^\rho) + \mathcal{F}^{\sigma_{s;2,1}}_{\rho,\lambda+1}(w(b-a)^\rho)), \end{aligned}$$

where

$$\begin{aligned} \sigma_{s;1,1}(k) &= \frac{\sigma(k)}{k\rho+\lambda+s+2}, \\ \sigma_1(k) &= \sigma(k)(k\rho+\lambda+1), \\ \sigma_2(k) &= \sigma(k)(k\rho+\lambda+2)(k\rho+\lambda+1), \\ \sigma_{s;2,1}(k) &= \frac{\sigma(k)}{(k\rho+\lambda+s+3)(k\rho+\lambda+s+2)}, \end{aligned}$$

for $k = 0, 1, 2, \ldots$

Proof. Letting $h_1(t) = t$, $h_2(t) = (1-t)$, $g_1(t) = t^s$ and $g_2(t) = (1-t)^s$ for all $t \in [0, 1]$ and some $s \in (0, 1]$ then

$$\begin{aligned} I(h_1, g_1) &= \int_0^1 t^\lambda \mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^\rho t^\rho) t^{s+1} dt \\ &= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)} w^k (b-a)^{k\rho} \int_0^1 t^{k\rho+\lambda+s+1} dt \end{aligned}$$

$$= \mathcal{F}^{\sigma_{s;1,1}}_{\rho,\lambda+1}(w(b-a)^\rho),$$

where

$$\sigma_{s;1,1}(k) = \frac{\sigma(k)}{k\rho+\lambda+s+2}, \quad k = 0,1,2,\dots,$$

$$\begin{aligned}
I(h_2,g_2) &= \int_0^1 t^\lambda \mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^\rho t^\rho)(1-t)^{s+1}dt \\
&= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)} w^k (b-a)^{k\rho} \int_0^1 t^{k\rho+\lambda+1}(1-t)^{s+1}dt \\
&= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)} w^k (b-a)^{k\rho} \frac{\Gamma(k\rho+\lambda+2)\Gamma(s+2)}{\Gamma(k\rho+\lambda+s+4)} \\
&= \Gamma(s+2)\mathcal{F}^{\sigma_1}_{\rho,\lambda+s+4}(w(b-a)^\rho)
\end{aligned}$$

where

$$\sigma_1(k) = \sigma(k)(k\rho+\lambda+1), \quad k = 0,1,2,\dots,$$

$$\begin{aligned}
I(h_1,g_2) &= \int_0^1 t^\lambda \mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^\rho t^\rho)t(1-t)^{s}dt \\
&= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)} w^k (b-a)^{k\rho} \int_0^1 t^{k\rho+\lambda+2}(1-t)^{s}dt \\
&= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)} w^k (b-a)^{k\rho} \frac{\Gamma(k\rho+\lambda+3)\Gamma(s+1)}{\Gamma(k\rho+\lambda+s+4)} \\
&= \Gamma(s+1)\mathcal{F}^{\sigma_2}_{\rho,\lambda+s+4}(w(b-a)^\rho)
\end{aligned}$$

where

$$\sigma_2(k) = \sigma(k)(k\rho+\lambda+2)(k\rho+\lambda+1), \quad k = 0,1,2,\dots,$$

and

$$\begin{aligned}
I(h_2,g_1) &= \int_0^1 t^\lambda \mathcal{F}^{\sigma}_{\rho,\lambda+1}(w(b-a)^\rho t^\rho)(1-t)t^{s}dt \\
&= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)} w^k (b-a)^{k\rho} \int_0^1 t^{k\rho+\lambda+s+1}(1-t)dt \\
&= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)} w^k (b-a)^{k\rho} \frac{\Gamma(k\rho+\lambda+s+2)\Gamma(2)}{\Gamma(k\rho+\lambda+s+4)} \\
&= \mathcal{F}^{\sigma_{s;2,1}}_{\rho,\lambda+1}(w(b-a)^\rho)
\end{aligned}$$

where

$$\sigma_{s;2,1}(k) = \frac{\sigma(k)}{(k\rho+\lambda+s+3)(k\rho+\lambda+s+2)}, \quad k = 0,1,2,\dots$$

By replacement of these values in Theorem 4 it is attained the result. □

Remark 4. Since in the preliminary section is mentioned the fact that from the Raina's fractional integral the fractional integrals of Riemann-Liouville and the classic integral of Riemman can be deduced then the results found in Theorem 4 and Corollaries 1,2, 3 and 4 are useful to express them in terms of these integrals.

4. Conclusion

In the present work we established the Hermite-Hadamard inequality for exponentially convex functions using the Raina's fractional integral and from this result we deduced some results found in [17, 18]. Also from Lemma 3.1 it was established a general theorem from which some fractional integral inequalities for exponentially convex functions, exponentially $P-$convex functions and exponentially $s-$convex functions in the second sense were found.

The usefulness of the theorems presented and the proposed technique can be applied to other types of generalized convex functions, for example, $MT-$convex functions [14].

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