

Some common fixed point theorems on partial metric spaces satisfying implicit relation

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ABSTRACT. The aim of this article is to prove some fixed point and common fixed point theorems on partial metric spaces satisfying implicit relation. Our results extend and generalize several results from the existing literature.

1. INTRODUCTION AND PRELIMINARIES

The well known Banach contraction principle is the most celebrated result in fixed point theory and its applications. Banach contraction principle guarantees that any contraction map on complete metric space has a unique fixed point. In 1969, Kannan [15] was the first, who gave a new contractive condition for which a map need not be continuous even if it has fixed point in complete metric space. In 1972, Chatterjæ [13] also generalized the Banach contraction condition and obtained a fixed point result. The well-known Banach contraction theorem [8] is stated as follows.

Theorem 1. *Let (X, d) be a complete metric space and let $T: X \rightarrow X$ be a mapping satisfying the contractive condition*

$$(1) \quad d(T(x), T(y)) \leq \alpha d(x, y)$$

for all $x, y \in X$, where $\alpha \in [0, 1)$ is a constant. Then T has a unique fixed point in X and that point can be obtained as a limit of repeated iteration of the mapping at any point of X .

A new space called a partial metric space (PMS) has been introduced by Matthews ([24, 25]). After introducing partial metric space, Matthews [25] also proved the partial metric version of the Banach fixed point theorem which is an analogous of Banach contraction principle, this makes the partial metric space relevant in fixed point theory. In 1999, Heckmann [14]

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established some results using a generalization of the partial metric function called a weak partial metric function. In 2004, Oltra and Valero [29] also generalized the Matthews's fixed point theorem in a complete partial metric space in the sense of O'Neill. Then many authors gave some generalizations of the results of Matthews and proved some fixed point theorems on this space (see, e.g., [1, 2, 3, 4, 5, 16, 17, 18, 20, 29, 39], see, also [11, 12, 28, 34, 35, 37]). Also, the concept of PMS provide to study denotational semantics of dataflow networks [24, 25, 38, 40].

In 2016, Kir and Kiziltunc introduced the concept of T_F contractive mappings in partial metric space and established a generalization of Banach's, Kannan's and Chatterjæ's fixed point theorems for the concept of partial metric space.

The study of common fixed point theorems using implicit relations was introduced by V. Popa [31] in 1999. Later on various authors used implicit relations and proved fixed point and common fixed point theorems in different spaces (see, e.g., [7, 9, 10, 23, 27, 30, 32, 33, 36] and many others).

The definition of a partial metric space is given by Matthews (see [24]) as follows.

Definition 1 ([24]). Let X be a nonempty set and let $p: X \times X \rightarrow \mathbb{R}^+$ be a function satisfy:

- (pm1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,
- (pm2) $p(x, x) \leq p(x, y)$,
- (pm3) $p(x, y) = p(y, x)$,
- (pm4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$,

for all $x, y, z \in X$. Then p is called partial metric on X and the pair (X, p) is called partial metric space.

It is clear that if $p(x, y) = 0$, then from (pm1) and (pm2) we obtain $x = y$. But if $x = y$, $p(x, y)$ may not be zero. Various applications of this space has been extensively investigated by many authors (see [22], [39] for details).

Example 1 ([6]). Let $X = \mathbb{R}^+$ and $p: X \times X \rightarrow \mathbb{R}^+$ given by $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. Then (\mathbb{R}^+, p) is a partial metric space.

Example 2 ([6]). Let $X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$. Then $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$ defines a partial metric p on X .

Remark 1 ([19]). Let (X, p) be a partial metric space.

- (1) The function $d_p: X \times X \rightarrow \mathbb{R}^+$ defined as $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ is a (usual) metric on X and (X, d_p) is a (usual) metric space.
- (2) The function $d_q: X \times X \rightarrow \mathbb{R}^+$ defined as $d_q(x, y) = \max\{p(x, y) - p(x, x), p(x, y) - p(y, y)\}$ is a (usual) metric on X and (X, d_q) is a (usual) metric space.

Note that each partial metric p on X generates a T_0 topology τ_p on X , whose base is a family of open p -balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x, \varepsilon) = \{y \in X : p(x, y) \leq p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

On a partial metric space the notions of convergence, the Cauchy sequence, completeness and continuity are defined as follows [24].

Definition 2 ([24]). Let (X, p) be a partial metric space. Then:

- (a₁) a sequence $\{x_n\}$ in (X, p) is said to be convergent to a point $x \in X$ if and only if $p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x)$,
- (a₂) a sequence $\{x_n\}$ is called a Cauchy sequence if $\lim_{m, n \rightarrow \infty} p(x_m, x_n)$ exists and finite,
- (a₃) (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ with respect to τ_p .

Furthermore,

$$\lim_{m, n \rightarrow \infty} p(x_m, x_n) = \lim_{n \rightarrow \infty} p(x_n, x) = p(x, x).$$

- (a₄) A mapping $F: X \rightarrow X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $F(B_p(x_0, \delta)) \subset B_p(F(x_0), \varepsilon)$.

Definition 3 ([26]). Let (X, p) be a partial metric space. Then:

- (b₁) a sequence $\{x_n\}$ in (X, p) is called 0-Cauchy if $\lim_{m, n \rightarrow \infty} p(x_m, x_n) = 0$,
- (b₂) (X, p) is said to be 0-complete if every 0-Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$, such that $p(x, x) = 0$.

Lemma 1 ([24, 25]). Let (X, p) be a partial metric space. Then:

- (c₁) a sequence $\{x_n\}$ in (X, p) is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X, d_p) ,
- (c₂) (X, p) is complete if and only if the metric space (X, d_p) is complete,
- (c₃) a subset E of a partial metric space (X, p) is closed if a sequence $\{x_n\}$ in E such that $\{x_n\}$ converges to some $x \in X$, then $x \in E$.

Lemma 2 ([2]). Assume that $x_n \rightarrow z$ as $n \rightarrow \infty$ in a partial metric space (X, p) such that $p(z, z) = 0$. Then $\lim_{n \rightarrow \infty} p(x_n, y) = p(z, y)$ for every $y \in X$.

Now, we introduce an implicit relation to investigate some fixed point and common fixed point theorems in partial metric spaces.

Definition 4 (Implicit Relation). Let Ψ be the family of all real valued continuous functions $\psi: \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ non-decreasing in the first argument for three variables. For some $\mu \in [0, 1)$, we consider the following conditions.

- (Ir1) For $x, y \in \mathbb{R}_+$, if $y \leq \psi(x, \frac{x+y}{2}, \frac{x+y}{2})$, then $y \leq \mu x$,
- (Ir2) For $x \in \mathbb{R}_+$, if $y \leq \psi(y, 0, y)$, then $y = 0$ since $\mu \in [0, 1)$,
- (Ir3) For $x \in \mathbb{R}_+$, if $y \leq \psi(0, \frac{y}{2}, \frac{y}{2})$, then $y = 0$.

The purpose of this paper is to establish some fixed point and common fixed point theorems in the setting of partial metric spaces satisfying implicit relation. Our results extend and generalize several results from the existing literature.

2. MAIN RESULTS

In this section we shall prove some fixed point and common fixed point theorems using implicit relation in the framework of partial metric spaces.

Theorem 2. *Let (X, p) be a complete partial metric space and let $T: X \rightarrow X$ be a mapping satisfying the inequality*

$$(2) \quad p(Tx, Ty) \leq \psi \left\{ p(x, y), \frac{1}{2}[p(x, Tx) + p(y, Ty)], \frac{1}{2}[p(x, Ty) + p(y, Tx)] \right\},$$

for all $x, y \in X$ and some $\psi \in \Psi$. If ψ satisfies the conditions (Ir1), (Ir2) and (Ir3), then T has a unique fixed point in X .

Proof. For each $x_0 \in X$ and $n \in \mathbb{N}$, put $x_{n+1} = Tx_n$. It follows from (2) and (pm4) that

$$(3) \quad \begin{aligned} p(x_n, x_{n+1}) &= p(Tx_{n-1}, Tx_n) \\ &\leq \psi \left\{ p(x_{n-1}, x_n), \frac{1}{2}[p(x_{n-1}, Tx_{n-1}) + p(x_n, Tx_n)], \right. \\ &\quad \left. \frac{1}{2}[p(x_{n-1}, Tx_n) + p(x_n, Tx_{n-1})] \right\} \\ &\leq \psi \left\{ p(x_{n-1}, x_n), \frac{1}{2}[p(x_{n-1}, x_n) + p(x_n, x_{n+1})], \right. \\ &\quad \left. \frac{1}{2}[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)] \right\} \\ &\leq \psi \left\{ p(x_{n-1}, x_n), \frac{1}{2}[p(x_{n-1}, x_n) + p(x_n, x_{n+1})], \right. \\ &\quad \left. \frac{1}{2}[p(x_{n-1}, x_n) + p(x_n, x_{n+1}) - p(x_n, x_n)] \right\} \\ &\leq \psi \left\{ p(x_{n-1}, x_n), \frac{1}{2}[p(x_{n-1}, x_n) + p(x_n, x_{n+1})], \right. \\ &\quad \left. \frac{1}{2}[p(x_{n-1}, x_n) + p(x_n, x_{n+1})] \right\} \end{aligned}$$

Since ψ satisfies the condition (Ir1), there exists $\mu \in [0, 1)$ such that

$$(4) \quad p(x_n, x_{n+1}) \leq \mu p(x_{n-1}, x_n) \leq \mu^n p(x_0, x_1).$$

Set $F_n = p(x_n, x_{n+1})$ and $F_{n-1} = p(x_{n-1}, x_n)$, then from (4), we obtain

$$F_n \leq \mu F_{n-1} \leq \mu^2 F_{n-2} \leq \cdots \leq \mu^n F_0.$$

Now we show that $\{x_n\}$ is a Cauchy sequence in X . Let $m, n > 0$ with $m > n$, then by using (pm4) and equation (4), we have

$$\begin{aligned}
 p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{n+m-1}, x_m) \\
 &\quad - p(x_{n+1}, x_{n+1}) - p(x_{n+2}, x_{n+2}) - \cdots - p(x_{n+m-1}, x_{n+m-1}) \\
 &\leq \mu^n p(x_0, x_1) + \mu^{n+1} p(x_0, x_1) + \cdots + \mu^{n+m-1} p(x_0, x_1) \\
 &= \mu^n [p(x_0, x_1) + \mu p(x_0, x_1) + \cdots + \mu^{m-1} p(x_0, x_1)] \\
 &= \mu^n [1 + \mu + \cdots + \mu^{m-1}] F_0 \\
 &\leq \mu^n \left(\frac{1 - \mu^{m-1}}{1 - \mu} \right) F_0.
 \end{aligned}$$

Taking $n, m \rightarrow \infty$ in the above inequality, we get $p(x_n, x_m) \rightarrow 0$ since $0 < \mu < 1$, hence $\{x_n\}$ is a Cauchy sequence in X . Thus by Lemma 1 this sequence will also be Cauchy in (X, d_p) . In addition, since (X, p) is complete, (X, d_p) is also complete. Thus there exists $u \in X$ such that $x_n \rightarrow u$ as $n \rightarrow \infty$. Moreover by Lemma 1,

$$(5) \quad p(u, u) = \lim_{n \rightarrow \infty} p(u, x_n) = \lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0,$$

implies

$$(6) \quad \lim_{n \rightarrow \infty} d_p(u, x_n) = 0.$$

Now, we show that u is a fixed point of T . Notice that due to (5), we have $p(u, u) = 0$. By using inequality (2), we get

$$\begin{aligned}
 p(x_{n+1}, Tu) &= p(Tx_n, Tu) \\
 &\leq \psi \left\{ p(x_n, u), \frac{1}{2} [p(x_n, Tx_n) + p(u, Tu)], \right. \\
 &\quad \left. \frac{1}{2} [p(x_n, Tu) + p(u, Tx_n)] \right\} \\
 &= \psi \left\{ p(x_n, z), \frac{1}{2} [p(x_n, x_{n+1}) + p(u, Tu)], \right. \\
 &\quad \left. \frac{1}{2} [p(x_n, Tu) + p(u, x_{n+1})] \right\}.
 \end{aligned}$$

Note that $\psi \in \Psi$, then taking the limit as $n \rightarrow \infty$ and using (5) and Lemma 2, we get

$$p(u, Tu) \leq \psi \left\{ 0, \frac{1}{2} p(u, Tu), \frac{1}{2} p(u, Tu) \right\}.$$

Since ψ satisfies the condition (Ir3), then $p(u, Tu) = 0$. This shows that $u = Tu$. Thus u is a fixed point of T .

Now to show that the fixed point of T is unique. Let u_1, u_2 be fixed points of T with $u_1 \neq u_2$. We shall prove that $u_1 = u_2$. It follows from equation (2) and (5) that

$$p(u_1, u_2) = p(Tu_1, Tu_2)$$

$$\begin{aligned}
&\leq \psi \left\{ p(u_1, u_2), \frac{1}{2} [p(u_1, Tu_1) + p(u_2, Tu_2)], \right. \\
&\quad \left. \frac{1}{2} [p(u_1, Tu_2) + p(u_2, Tu_1)] \right\} \\
&= \psi \left\{ p(u_1, u_2), \frac{1}{2} [p(u_1, u_1) + p(u_2, u_2)], \right. \\
&\quad \left. \frac{1}{2} [p(u_1, u_2) + p(u_2, u_1)] \right\} \\
&= \psi \left\{ p(u_1, u_2), 0, p(u_1, u_2) \right\}.
\end{aligned}$$

Since ψ satisfies the condition (Ir2), then we get

$$\begin{aligned}
p(u_1, u_2) &\leq \mu p(u_1, u_2) \\
&\Rightarrow p(u_1, u_2) = 0, \quad \text{since } 0 < \mu < 1.
\end{aligned}$$

This shows that $u_1 = u_2$. Thus the fixed point of T is unique. This completes the proof. \square

Theorem 3. *Let F and G be two mappings on a complete partial metric space (X, p) and*

$$\begin{aligned}
(7) \quad p(Fx, Gy) &\leq \psi \left\{ p(x, y), \frac{1}{2} [p(x, Fx) + p(y, Gy)], \right. \\
&\quad \left. \frac{1}{2} [p(x, Gy) + p(y, Fx)] \right\}
\end{aligned}$$

for all $x, y \in X$ and some $\psi \in \Psi$. Then F and G have a unique common fixed point in X .

Proof. For each $x_0 \in X$. Put $x_{2n+1} = Fx_{2n}$ and $x_{2n+2} = Gx_{2n+1}$ for $n = 0, 1, 2, \dots$. It follows from (7), (pm4) and Lemma 1 that

$$\begin{aligned}
p(x_{2n+1}, x_{2n}) &= p(Fx_{2n}, Gx_{2n-1}) \\
&\leq \psi \left\{ p(x_{2n}, x_{2n-1}), \frac{1}{2} [p(x_{2n}, Fx_{2n}) + p(x_{2n-1}, Gx_{2n-1})], \right. \\
&\quad \left. \frac{1}{2} [p(x_{2n}, Gx_{2n-1}) + p(x_{2n-1}, Fx_{2n})] \right\} \\
&= \psi \left\{ p(x_{2n}, x_{2n-1}), \frac{1}{2} [p(x_{2n}, x_{2n+1}) + p(x_{2n-1}, x_{2n})], \right. \\
&\quad \left. \frac{1}{2} [p(x_{2n}, x_{2n}) + p(x_{2n-1}, x_{2n+1})] \right\} \\
&\leq \psi \left\{ p(x_{2n}, x_{2n-1}), \frac{1}{2} [p(x_{2n}, x_{2n+1}) + p(x_{2n-1}, x_{2n})], \right. \\
&\quad \left. \frac{1}{2} [p(x_{2n-1}, x_{2n}) + p(x_{2n}, x_{2n+1}) - p(x_{2n}, x_{2n})] \right\} \\
&\leq \psi \left\{ p(x_{2n}, x_{2n-1}), \frac{1}{2} [p(x_{2n}, x_{2n+1}) + p(x_{2n-1}, x_{2n})], \right.
\end{aligned}$$

$$(8) \quad \left. \frac{1}{2}[p(x_{2n-1}, x_{2n}) + p(x_{2n}, x_{2n+1})] \right\}.$$

Since ψ satisfies the condition (Ir1), there exists $\mu \in [0, 1)$ such that

$$(9) \quad p(x_{2n+1}, x_{2n}) \leq \mu p(x_{2n}, x_{2n-1}) \leq \mu^{2n} p(x_1, x_0).$$

Now we show that $\{x_n\}$ is a Cauchy sequence in X . Let $m, n > 0$ with $m > n$, then by using (pm4) and equation (9), we have

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{n+m-1}, x_m) \\ &\quad - p(x_{n+1}, x_{n+1}) - p(x_{n+2}, x_{n+2}) - \cdots - p(x_{n+m-1}, x_{n+m-1}) \\ &\leq \mu^n p(x_0, x_1) + \mu^{n+1} p(x_0, x_1) + \cdots + \mu^{n+m-1} p(x_0, x_1) \\ &= \mu^n [p(x_0, x_1) + \mu(x_0, x_1) + \cdots + \mu^{m-1} p(x_0, x_1)] \\ &= \mu^n [1 + \mu + \cdots + \mu^{m-1}] p(x_0, x_1) \\ &\leq \mu^n \left(\frac{1 - \mu^{m-1}}{1 - \mu} \right) p(x_0, x_1). \end{aligned}$$

Taking $n, m \rightarrow \infty$ in the above inequality, we get $p(x_n, x_m) \rightarrow 0$ since $0 < \mu < 1$, hence $\{x_n\}$ is a Cauchy sequence in X . Thus by Lemma 1 this sequence will also be Cauchy in (X, d_p) . In addition, since (X, p) is complete, (X, d_p) is also complete. Thus there exists $v \in X$ such that $x_n \rightarrow v$ as $n \rightarrow \infty$. Moreover by Lemma 1,

$$(10) \quad p(v, v) = \lim_{n \rightarrow \infty} p(v, x_n) = \lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0,$$

implies

$$(11) \quad \lim_{n \rightarrow \infty} d_p(v, x_n) = 0.$$

Now we have to prove that v is a common fixed point of F and G . For this, consider

$$\begin{aligned} p(x_{2n+1}, Fv) &= p(Fx_{2n}, Fv) \\ &\leq \psi \left\{ p(x_{2n}, v), \frac{1}{2}[p(x_{2n}, Fx_{2n}) + p(v, Fv)], \right. \\ &\quad \left. \frac{1}{2}[p(x_{2n}, Fv) + p(v, Fx_{2n})] \right\} \\ &= \psi \left\{ p(x_{2n}, v), \frac{1}{2}[p(x_{2n}, x_{2n+1}) + p(v, Fv)], \right. \\ &\quad \left. \frac{1}{2}[p(x_{2n}, Fv) + p(v, x_{2n+1})] \right\}. \end{aligned}$$

Note that $\psi \in \Psi$, then using (10), Lemma 2 and taking the limit as $n \rightarrow \infty$, we get

$$p(v, Fv) \leq \psi \left(0, 0, \frac{p(v, Fv)}{2}, \frac{p(v, Fv)}{2} \right).$$

Since ψ satisfies the condition (Ir3), then $p(v, Fv) = 0$. This shows that $v = Fv$ for all $v \in X$. Similarly, we can show that $v = Gv$. Thus v is a common fixed point of F and G .

Now to show that the common fixed point of F and G is unique. For this, let v' be another common fixed point of F and G , that is, $Fv' = Gv' = v'$ with $v' \neq v$. Then we have to show that $v = v'$. It follows from equation (7) and (10) that

$$\begin{aligned} p(v, v') &= p(Fv, Gv') \\ &\leq \psi \left\{ p(v, v'), \frac{1}{2} [p(v, Fv) + p(v', Gv')], \right. \\ &\quad \left. \frac{1}{2} [p(v, Gv') + p(v', Fv)] \right\} \\ &= \psi \left\{ p(v, v'), \frac{1}{2} [p(v, v) + p(v', v')], \right. \\ &\quad \left. \frac{1}{2} [p(v, v') + p(v', v)] \right\} \\ &= \psi \left\{ p(v, v'), 0, p(v, v') \right\}. \end{aligned}$$

Since ψ satisfies the condition (Ir2), then we get

$$\begin{aligned} p(v, v') &\leq \mu p(v, v') \\ &\Rightarrow p(v, v') = 0, \text{ since } 0 < \mu < 1. \end{aligned}$$

Thus, we get $v = v'$. This shows that v is the unique common fixed point of F and G . This completes the proof. \square

Theorem 4. Let F_1 and F_2 be two continuous mappings on a complete partial metric space (X, p) and

$$(12) \quad \begin{aligned} p(F_1^m x, F_2^n y) &\leq \psi \left\{ p(x, y), \frac{1}{2} [p(x, F_1^m x) + p(y, F_2^n y)], \right. \\ &\quad \left. \frac{1}{2} [p(x, F_2^n y) + p(y, F_1^m x)] \right\} \end{aligned}$$

for all $x, y \in X$, where m and n are some integers and some $\psi \in \Psi$. Then F_1 and F_2 have a unique common fixed point in X .

Proof. Since F_1^m and F_2^n satisfy the conditions of Theorem 3. So F_1^m and F_2^n have a unique common fixed point. Let z be the common fixed point. Then, we have

$$\begin{aligned} F_1^m z = z &\Rightarrow F_1(F_1^m z) = F_1 z \\ &\Rightarrow F_1^m(F_1 z) = F_1 z. \end{aligned}$$

If $F_1 z = r_0$, then $F_1^m r_0 = r_0$. So, $F_1 z$ is a fixed point of F_1^m . Similarly, $F_2(F_2^n z) = F_2 z$. Now, using equation (12) and Lemma 1, we obtain

$$p(z, F_1 z) = p(F_1^m z, F_1^m(F_1 z))$$

$$\begin{aligned}
&\leq \psi \left\{ p(z, F_1 z), \frac{1}{2} [p(z, F_1^m z) + p(F_1 z, F_1^m(F_1 z))] \right\}, \\
&\quad \frac{1}{2} [p(z, F_1^m(F_1 z)) + p(F_1 z, F_1^m z)] \Big\} \\
&= \psi \left\{ p(z, F_1 z), \frac{1}{2} [p(z, z) + p(F_1 z, F_1 z)], \right. \\
&\quad \left. \frac{1}{2} [p(z, F_1 z) + p(F_1 z, z)] \right\} \\
&= \psi \left\{ p(z, F_1 z), 0, p(z, F_1 z) \right\}.
\end{aligned}$$

Since ψ satisfies the condition (Ir2), then we get

$$\begin{aligned}
p(z, F_1 z) &\leq \mu p(z, F_1 z) \\
&\Rightarrow p(z, F_1 z) = 0, \text{ since } 0 < \mu < 1.
\end{aligned}$$

Thus, we have $z = F_1 z$ for all $z \in X$. Similarly, we can show that $z = F_2 z$. This shows that z is a common fixed point of F_1 and F_2 . For uniqueness of z , let $z' \neq z$ be another common fixed point of F_1 and F_2 . Then clearly z' is also a common fixed point of F_1^m and F_2^n which implies $z' = z$. Hence F_1 and F_2 have a unique common fixed point. This completes the proof. \square

Theorem 5. Let $\{U_\alpha\}$ be a family of continuous mappings on a complete partial metric space (X, p) satisfying

$$(13) \quad p(U_\alpha x, U_\beta y) \leq \psi \left\{ p(x, y), \frac{1}{2} [p(x, U_\alpha x) + p(y, U_\beta y)], \right. \\
\left. \frac{1}{2} [p(x, U_\beta y) + p(y, U_\alpha x)] \right\}$$

for $\alpha, \beta \in \Psi$ with $\alpha \neq \beta$ and $x, y \in X$. Then there exists a unique $z \in X$ satisfying $U_\alpha z = z$ for all $\alpha \in \Psi$.

Proof. For $x_0 \in X$, we define a sequence as follows:

$$x_{2n+1} = U_\alpha x_{2n}, \quad x_{2n+2} = U_\beta x_{2n+1}, \quad n = 0, 1, 2, \dots$$

It follows from (13), (pm4) and Lemma 1 that

$$\begin{aligned}
p(x_{2n+1}, x_{2n}) &= p(U_\alpha x_{2n}, U_\beta x_{2n-1}) \\
&\leq \psi \left\{ p(x_{2n}, x_{2n-1}), \frac{1}{2} [p(x_{2n}, U_\alpha x_{2n}) + p(x_{2n-1}, U_\beta x_{2n-1})], \right. \\
&\quad \left. \frac{1}{2} [p(x_{2n}, U_\beta x_{2n-1}) + p(x_{2n-1}, U_\alpha x_{2n})] \right\} \\
&= \psi \left\{ p(x_{2n}, x_{2n-1}), \frac{1}{2} [p(x_{2n}, x_{2n+1}) + p(x_{2n-1}, x_{2n})], \right. \\
&\quad \left. \frac{1}{2} [p(x_{2n}, x_{2n}) + p(x_{2n-1}, x_{2n+1})] \right\} \\
&\leq \psi \left\{ p(x_{2n}, x_{2n-1}), \frac{1}{2} [p(x_{2n}, x_{2n+1}) + p(x_{2n-1}, x_{2n})], \right.
\end{aligned}$$

$$\begin{aligned}
& \left. \frac{1}{2}[p(x_{2n-1}, x_{2n}) + p(x_{2n}, x_{2n+1}) - p(x_{2n}, x_{2n})] \right\} \\
& \leq \psi \left\{ p(x_{2n}, x_{2n-1}), \frac{1}{2}[p(x_{2n}, x_{2n+1}) + p(x_{2n-1}, x_{2n})], \right. \\
(14) \quad & \left. \frac{1}{2}[p(x_{2n-1}, x_{2n}) + p(x_{2n}, x_{2n+1})] \right\}.
\end{aligned}$$

Since ψ satisfies the condition (Ir1), there exists $\mu \in (0, 1)$ such that

$$(15) \quad p(x_{2n+1}, x_{2n}) \leq \mu p(x_{2n}, x_{2n-1}) \leq \mu^{2n} p(x_1, x_0).$$

Now we show that $\{x_n\}$ is a Cauchy sequence in X . Let $m, n > 0$ with $m > n$, then by using (pm4) and equation (15), we have

$$\begin{aligned}
p(x_n, x_m) & \leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{n+m-1}, x_m) \\
& \quad - p(x_{n+1}, x_{n+1}) - p(x_{n+2}, x_{n+2}) - \cdots - p(x_{n+m-1}, x_{n+m-1}) \\
& \leq \mu^n p(x_0, x_1) + \mu^{n+1} p(x_0, x_1) + \cdots + \mu^{n+m-1} p(x_0, x_1) \\
& = \mu^n [p(x_0, x_1) + \mu p(x_0, x_1) + \cdots + \mu^{m-1} p(x_0, x_1)] \\
& = \mu^n [1 + \mu + \cdots + \mu^{m-1}] p(x_0, x_1) \\
& \leq \mu^n \left(\frac{1 - \mu^m}{1 - \mu} \right) p(x_0, x_1).
\end{aligned}$$

Taking $n, m \rightarrow \infty$ in the above inequality, we get $p(x_n, x_m) \rightarrow 0$ since $0 < \mu < 1$, hence $\{x_n\}$ is a Cauchy sequence in X . Thus by Lemma 1 this sequence will also be Cauchy in (X, d_p) . In addition, since (X, p) is complete, (X, d_p) is also complete. Thus there exists $r \in X$ such that $x_n \rightarrow r$ as $n \rightarrow \infty$. Moreover by Lemma 1,

$$(16) \quad p(r, r) = \lim_{n \rightarrow \infty} p(r, x_n) = \lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0,$$

implies

$$(17) \quad \lim_{n \rightarrow \infty} d_p(r, x_n) = 0.$$

By the continuity of U_α and U_β , it is clear that $U_\alpha r = U_\beta r = r$. Therefore r is a common fixed point of U_α for all $\alpha \in \Psi$.

In order to prove the uniqueness, let us take another common fixed point r' of U_α and U_β where $r \neq r'$. Then from equation (13) and (16), we obtain

$$\begin{aligned}
p(r, r') & = p(U_\alpha r, U_\beta r') \\
& \leq \psi \left\{ p(r, r'), \frac{1}{2}[p(r, F_\alpha r) + p(r', U_\beta r')], \right. \\
& \quad \left. \frac{1}{2}[p(r, U_\beta r') + p(r', U_\alpha r)] \right\} \\
& = \psi \left\{ p(r, r'), \frac{1}{2}[p(r, r) + p(r', r')], \right. \\
& \quad \left. \frac{1}{2}[p(r, r') + p(r', r)] \right\}
\end{aligned}$$

$$= \psi \left\{ p(r, r'), 0, p(r, r') \right\}.$$

Since ψ satisfies the condition (Ir2), then we get

$$\begin{aligned} p(r, r') &\leq \mu p(r, r') \\ \Rightarrow p(r, r') &= 0, \text{ since } 0 < \mu < 1. \end{aligned}$$

Thus, we get $r = r'$ for all $r \in X$. This shows that r is a unique common fixed point of U_α for all $\alpha \in \Psi$. This completes the proof. \square

Next, we give an analogues of fixed point theorems in metric spaces for partial metric spaces by combining Theorem 1 with $\psi \in \Psi$ and ψ satisfies conditions (Ir1), (Ir2) and (Ir3). The following corollary is an analogues of Banach's contraction principle.

Corollary 2.1. *Let (X, p) be a complete partial metric space. Suppose that the mapping $T: X \rightarrow X$ satisfies the following condition:*

$$p(Tx, Ty) \leq f p(x, y)$$

for all $x, y \in X$, where $f \in [0, 1)$ is a constant. Then T has a unique fixed point in X . Moreover, T is continuous at the fixed point.

Proof. The assertion follows using Theorem 2 with $\psi(v_1, v_2, v_3) = f v_1$ for some $f \in [0, 1)$ and all $v_1, v_2, v_3 \in \mathbb{R}_+$. This completes the proof. \square

The following corollary is an analogues of R. Kannan's result in [15].

Corollary 2.2. *Let (X, p) be a complete partial metric space. Suppose that the mapping $T: X \rightarrow X$ satisfies the following condition:*

$$p(Tx, Ty) \leq g [p(x, Tx) + p(y, Ty)]$$

for all $x, y \in X$, where $g \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X . Moreover, T is continuous at the fixed point.

Proof. The assertion follows using Theorem 2 with $\psi(v_1, v_2, v_3) = g v_2$ for some $g \in [0, 1)$ and all $v_1, v_2, v_3 \in \mathbb{R}_+$. Indeed, ψ is continuous. First, we have $\psi(x, \frac{x+y}{2}, \frac{x+y}{2}) = g \left(\frac{x+y}{2} \right)$. So, if $y \leq \psi(x, \frac{x+y}{2}, \frac{x+y}{2})$, then $y \leq \left(\frac{g}{2-g} \right) x$ with $\left(\frac{g}{2-g} \right) < 1$. Thus, T satisfies the condition (Ir1).

Next, if $y \leq \psi(y, 0, y) = g \cdot 0 = 0$, then $y = 0$. Thus, T satisfies the condition (Ir2).

Finally, if $y \leq \psi(0, \frac{y}{2}, \frac{y}{2}) = g \left(\frac{y}{2} \right) = 0$, then $y = 0$ since $g < \frac{1}{2} < 1$. Thus, T satisfies the condition (Ir3). \square

The following corollary is an analogues of S. K. Chatterjæ's result in [13].

Corollary 2.3. *Let (X, p) be a complete partial metric space. Suppose that the mapping $T: X \rightarrow X$ satisfies the following condition:*

$$p(Tx, Ty) \leq h [p(x, Ty) + p(y, Tx)]$$

for all $x, y \in X$, where $h \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X . Moreover, T is continuous at the fixed point.

Proof. The assertion follows using Theorem 2 with $\psi(v_1, v_2, v_3) = hv_3$ for some $h \in [0, 1)$ and all $v_1, v_2, v_3 \in \mathbb{R}_+$. Indeed, ψ is continuous. First, we have $\psi(x, \frac{x+y}{2}, \frac{x+y}{2}) = h\left(\frac{x+y}{2}\right)$. So, if $y \leq \psi(x, \frac{x+y}{2}, \frac{x+y}{2})$, then $y \leq \left(\frac{h}{2-h}\right)x$ with $\left(\frac{h}{2-h}\right) < 1$. Thus, T satisfies the condition (Ir1).

Next, if $y \leq \psi(y, 0, y) = h \cdot 0 = 0$, then $y = 0$. Thus, T satisfies the condition (Ir2).

Finally, if $y \leq \psi(0, \frac{y}{2}, \frac{y}{2}) = h\left(\frac{y}{2}\right) = 0$, then $y = 0$ since $h < \frac{1}{2} < 1$. Thus, T satisfies the condition (Ir3). \square

Example 3. Let $X = [0, 1]$. Define $p: X \times X \rightarrow \mathbb{R}^+$ as $p(x, y) = \max\{x, y\}$ with $T: X \rightarrow X$ by $T(x) = \frac{x}{4}$. Clearly (X, p) is a partial metric space. Now, let $x \leq y$. Then choose $x = \frac{1}{2}$ and $y = 1$, we have $p(Tx, Ty) = \frac{y}{4}$, $p(x, y) = y$, $p(x, Tx) = x$, $p(y, Ty) = y$, $p(x, Ty) = x$, $p(y, Tx) = y$.

(i) Now, we consider

$$p(Tx, Ty) = \frac{y}{4} \leq fy,$$

or $f \geq \frac{1}{4}$. If take $0 \leq f < 1$, then T satisfies all the conditions of Corollary 2.1. Hence by application of Corollary 2.1, T has a unique fixed point. Here it is seen that $0 \in X$ is the unique fixed point of T .

(ii) Now, we consider

$$p(Tx, Ty) = \frac{y}{4} \leq g(x + y),$$

putting $x = \frac{1}{2}$ and $y = 1$ in the above inequality, we get

$$\frac{1}{4} \leq \frac{3}{2}g,$$

or $g \geq \frac{1}{6}$. If take $0 \leq g < \frac{1}{2}$, then T satisfies all the conditions of Corollary 2.2. Hence by application of Corollary 2.2, T has a unique fixed point and the unique fixed point T is $0 \in X$.

(iii) Now, we consider

$$p(Tx, Ty) = \frac{y}{4} \leq h(x + y),$$

putting $x = \frac{1}{2}$ and $y = 1$ in the above inequality, we get

$$\frac{1}{4} \leq \frac{3}{2}h,$$

or $h \geq \frac{1}{6}$. If take $0 \leq h < \frac{1}{2}$, then T satisfies all the conditions of Corollary 2.3. Hence by application of Corollary 2.3, T has a unique fixed point and it is $0 \in X$.

3. CONCLUSION

In this article, we prove some fixed point and common fixed point theorems satisfying the implicit relation in the framework of complete partial metric spaces. Our results extend, unify and generalize several results from the existing literature in the context of partial metric spaces.

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