Mathematica Moravica
Vol. **23**, No. **2** (2019), 87–96
doi: 10.5937/MatMor1902087B

Refinements of Hermite-Hadamard inequality for trigonometrically ρ-convex functions

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Abstract. In this study, we obtain some refinements of Hermite-Hadamard type inequalities for trigonometrically ρ-convex mappings.

1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g., [5], [15], [17, p. 137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality.

Over the last twenty years, numerous studies have focused on to establish generalization of the inequality (1) and to obtain new bounds for left hand side and right hand side of the inequality (1).

The following Lemma will be very useful when we prove the main theorems.

Lemma 1.1 ([20, 21]). *Let $f : [a,b] \to \mathbb{R}$ be a convex function and h be defined by*

$$h(t) = \frac{1}{2}\left[f\left(\frac{a+b}{2} - \frac{t}{2}\right) + f\left(\frac{a+b}{2} + \frac{t}{2}\right)\right].$$

Then h is convex, increasing on $[0, b-a]$ and for all $t \in [0, b-a]$,

$$f\left(\frac{a+b}{2}\right) \leq h(t) \leq \frac{f(a)+f(b)}{2}.$$

2010 *Mathematics Subject Classification.* Primary: 26D07; Secondary: 26D10, 26D15, 26A33.

Key words and phrases. Convex function, trigonometrically ρ-convex functions.

Full paper. Received 30 September 2019, accepted 5 December 2019, available online 19 December 2019.

In [6], Dragomir obtained following important inequalities which refines the first inequality of (1).

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a convex on $[a,b]$ and $f \in L_1[a,b]$. Then H is convex, increasing on $[0,1]$ and for all $t \in [0,1]$, we have*

$$(2) \qquad f\left(\frac{a+b}{2}\right) = H(0) \le H(t) \le H(1) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

where

$$H(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx.$$

Moreover, Yang and Hong [22] prove the following result which refines the second inequality of (1).

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$ be a convex on $[a,b]$ and $f \in L_1[a,b]$. Then P is convex, increasing on $[0,1]$ and for all $t \in [0,1]$, we have*

$$(3) \qquad \frac{1}{b-a}\int_a^b f(x)\,dx = P(0) \le P(t) \le P(1) = \frac{f(a)+f(b)}{2},$$

where

$$P(t) = \frac{1}{2(b-a)} \times \int_a^b \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right)\right]dx.$$

For the some refinements of the inequalities (1), please refer to [12], [13], [20], [21].

The definition of trigonometrically ρ-convex functions is given as follows:

Definition 1.1 ([1]). A function $f : I \to \mathbb{R}$ is said to be trigonometrically ρ-convex, if for any arbitrary closed subinterval $[a,b]$ of I such that $0 < \rho(b-a) < \pi$, we have

$$(4) \qquad f(x) \le \frac{\sin[\rho(b-x)]}{\sin[\rho(b-a)]}f(a) + \frac{\sin[\rho(x-a)]}{\sin[\rho(b-a)]}f(b)$$

for all $x \in [a,b]$. For the $x = (1-t)a + tb$, $t \in [0,1]$, then the condition (4) becomes

$$(5) \qquad f((1-t)a + tb) \le \frac{\sin[\rho(1-t)(b-a)]}{\sin[\rho(b-a)]}f(a) + \frac{\sin[\rho t(b-a)]}{\sin[\rho(b-a)]}f(b).$$

If the inequality (4) holds with "$\geq$", then the function will be called trigonometrically ρ-concave on I.

For some properties and results concerning the class of trigonometrically ρ-convex functions, see ([1], [2]-[4], [8]-[11], [14], [16], [18], [19]).

The following Hermite-Hadamard inequality for trigonometrically ρ-convex function is proved by S.S. Dragomir in [7].

Theorem 1.3. *Suppose that $f : I \to \mathbb{R}$ is trigonometrically ρ-convex on I. Then for any $a, b \in I$ with $0 < b - a < \frac{\pi}{\rho}$, we have*

(6)
$$\frac{2}{\rho} f\left(\frac{a+b}{2}\right) \sin\left[\frac{\rho(b-a)}{2}\right] \leq \int\limits_a^b f(x)dx \leq \frac{f(a)+f(b)}{\rho} \tan\left[\frac{\rho(b-a)}{2}\right].$$

Theorem 1.4. *Suppose that $f : I \to \mathbb{R}$ is trigonometrically ρ-convex on I. Then for any $a, b \in I$ with $0 < b - a < \frac{\pi}{\rho}$, we have*

(7)
$$f\left(\frac{a+b}{2}\right) \leq \int\limits_a^b \sin\left[\rho\left(x - \frac{a+b}{2}\right)\right] f(x)dx \leq \frac{f(a)+f(b)}{2} \sec\left[\frac{\rho(b-a)}{2}\right].$$

2. MAIN RESULTS

The following theorem refines the first inequality in (6).

Theorem 2.1. *Suppose that $f : [a,b] \to \mathbb{R}$ is a positive function with $0 < b - a < \frac{\pi}{\rho}$, then Λ_1 is monotonically increasing on $[0,1]$ and we have the following refinement inequality*

$$\frac{2}{\rho} f\left(\frac{a+b}{2}\right) \sin\left[\frac{\rho(b-a)}{2}\right] = \Lambda_1(0) \leq \Lambda_1(t) \leq \Lambda_1(1) = \int\limits_a^b f(x)dx,$$

where

$$\begin{aligned} \Lambda_1(t) &= \frac{1}{2}\int\limits_a^b \left[\cos\left(\frac{\rho(1-t)(b-x)}{2}\right) + \cos\left(\frac{\rho(1-t)(x-a)}{2}\right)\right] \\ &\quad \times f\left(tx + (1-t)\frac{a+b}{2}\right) dx. \end{aligned}$$

Proof. By using the change of variable, we obtain

$$\Lambda_1(t) = \frac{1}{2}\int_a^{\frac{a+b}{2}} \left[\cos\left(\frac{\rho(1-t)(b-x)}{2}\right) + \cos\left(\frac{\rho(1-t)(x-a)}{2}\right)\right]$$
$$\times f\left(tx + (1-t)\frac{a+b}{2}\right) dx$$
$$+ \frac{1}{2}\int_{\frac{a+b}{2}}^{b} \left[\cos\left(\frac{\rho(1-t)(b-x)}{2}\right) + \cos\left(\frac{\rho(1-t)(x-a)}{2}\right)\right]$$
$$\times f\left(tx + (1-t)\frac{a+b}{2}\right) dx$$
$$= \frac{1}{4}\int_0^{b-a} \left[\cos\left(\rho(1-t)\left(\frac{b-a}{4} + \frac{u}{4}\right)\right) + \cos\left(\rho(1-t)\left(\frac{b-a}{4} - \frac{u}{4}\right)\right)\right]$$
$$\times f\left(\frac{a+b}{2} - \frac{ut}{2}\right) du$$
$$+ \frac{1}{4}\int_0^{b-a} \left[\cos\left(\rho(1-t)\left(\frac{b-a}{4} - \frac{u}{4}\right)\right) + \cos\left(\rho(1-t)\left(\frac{b-a}{4} + \frac{u}{4}\right)\right)\right]$$
$$\times f\left(\frac{a+b}{2} + \frac{ut}{2}\right) du$$
$$= \frac{1}{4}\int_0^{b-a} \left[\cos\left(\rho(1-t)\left(\frac{b-a}{4} + \frac{u}{4}\right)\right) + \cos\left(\rho(1-t)\left(\frac{b-a}{4} - \frac{u}{4}\right)\right)\right]$$
$$\times \left[f\left(\frac{a+b}{2} - \frac{ut}{2}\right) + f\left(\frac{a+b}{2} + \frac{ut}{2}\right)\right] du.$$

From Lemma 1.1, we have $F(t) = \frac{1}{2}\left[f\left(\frac{a+b}{2} - \frac{t}{2}\right) + f\left(\frac{a+b}{2} + \frac{t}{2}\right)\right]$ is increasing on $[0, b-a]$. Since

$$\cos\left(\rho(1-t)\left(\frac{b-a}{4} + \frac{u}{4}\right)\right) + \cos\left(\rho(1-t)\left(\frac{b-a}{4} - \frac{u}{4}\right)\right)$$

is nonnegative for $u \in [0, b-a]$ with $0 < b-a < \frac{\pi}{\rho}$, thus $\Lambda_1(t)$ is increasing on $[0,1]$. As a result, using the facts that

$$\Lambda_1(0) = f\left(\frac{a+b}{2}\right)\frac{1}{2}\int_a^b \left[\cos\left(\frac{\rho(b-x)}{2}\right) + \cos\left(\frac{\rho(x-a)}{2}\right)\right] dx$$
$$= f\left(\frac{a+b}{2}\right)\frac{1}{2}\left[-\frac{2}{\rho}\sin\left(\frac{\rho(b-x)}{2}\right)\Bigg|_a^b + \frac{2}{\rho}\sin\left(\frac{\rho(x-a)}{2}\right)\Bigg|_a^b\right]$$

$$= \frac{2}{\rho} f\left(\frac{a+b}{2}\right) \sin\left[\frac{\rho(b-a)}{2}\right]$$

and

$$\Lambda_1(1) = \int_a^b f(x)dx,$$

we obtain the desired result. □

Remark 2.1. For $\rho \to 0$ we observe that

$$\lim_{\rho\to 0} \frac{2}{\rho} \sin\left[\frac{\rho(b-a)}{2}\right] = b-a$$

and

$$\lim_{\rho\to 0} \Lambda_1(t) = \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx.$$

Thus, refinement of Hermite-Hadamard inequality (2) follows from Theorem 2.1 in the limit $\rho \to 0$.

The following theorem refines the second inequality in (6).

Theorem 2.2. *Suppose that* $f : [a,b] \to \mathbb{R}$ *is a positive function with* $0 < b-a < \frac{\pi}{\rho}$, *then* Λ_2 *is monotonically increasing on* $[0,1]$ *and we have the following refinement inequality*

$$\int_a^b f(x)dx = \Lambda_2(0) \leq \Lambda_2(t) \leq \Lambda_2(1) = \frac{f(a)+f(b)}{\rho} \tan\left[\frac{\rho(b-a)}{2}\right],$$

where

$$\Lambda_2(t) = \frac{1}{4}\int_a^b \left[2 + \tan^2\left(\frac{\rho t(b-x)}{2}\right) + \tan^2\left(\frac{\rho t(x-a)}{2}\right)\right]$$
$$\times \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right)\right] dx.$$

Proof. By chance of variable, we have

$$\Lambda_2(t) = \frac{1}{4}\int_a^b \left[2 + \tan^2\left(\frac{\rho t(b-x)}{2}\right) + \tan^2\left(\frac{\rho t(x-a)}{2}\right)\right]$$
$$\times f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) dx$$

$$+\frac{1}{4}\int_a^b \left[2 + \tan^2\left(\frac{\rho t(b-x)}{2}\right) + \tan^2\left(\frac{\rho t(x-a)}{2}\right)\right]$$

$$\times f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)dx$$

$$= \frac{1}{4}\int_{0}^{b-a}\left[2+\tan^2\left(\frac{\rho t\,(b-a-u)}{2}\right)+\tan^2\left(\frac{\rho t u}{2}\right)\right]$$

$$\times f\left(a+\left(\frac{1-t}{2}\right)u\right)du$$

$$+\frac{1}{4}\int_{0}^{b-a}\left[2+\tan^2\left(\frac{\rho t u}{2}\right)+\tan^2\left(\frac{\rho t\,(b-a-u)}{2}\right)\right]$$

$$\times f\left(b+\left(\frac{1-t}{2}\right)u\right)du$$

$$= \frac{1}{4}\int_{0}^{b-a}\left[2+\tan^2\left(\frac{\rho t\,(b-a-u)}{2}\right)+\tan^2\left(\frac{\rho t u}{2}\right)\right]$$

$$\times\left[f\left(a+\left(\frac{1-t}{2}\right)u\right)+f\left(b+\left(\frac{1-t}{2}\right)u\right)\right]du$$

It follows that from Lemma 1.1 that $h(t)=\frac{1}{2}\left[f\left(\frac{a+b}{2}-\frac{t}{2}\right)+f\left(\frac{a+b}{2}+\frac{t}{2}\right)\right]$ and $k(t)=b-a-(1-t)u$ are increasing on $[0,b]$ and $[0,1]$, respectively. Thus, $h(k(t))=f\left(a+\left(\frac{1-t}{2}\right)u\right)+f\left(b-\left(\frac{1-t}{2}\right)u\right)$ is increasing on $[0,1]$. Since

$$2+\tan^2\left(\frac{\rho t\,(b-a-u)}{2}\right)+\tan^2\left(\frac{\rho t u}{2}\right)$$

is non negative for $u\in[0,b]$ with $0<b-a<\frac{\pi}{\rho}$, then we deduce that Λ_2 is monotonically increasing on $[0,1]$. Using the facts that

$$\Lambda_2(0)=\frac{1}{2}\left[\int_a^b f\left(\frac{a+x}{2}\right)dx+\int_a^b f\left(\frac{x+b}{2}\right)dx\right]=\int_a^b f(x)dx$$

and

$$\Lambda_2(1)=\frac{f(a)+f(b)}{4}\int_a^b\left[1+\tan^2\left(\frac{\rho\,(b-x)}{2}\right)+1+\tan^2\left(\frac{\rho\,(x-a)}{2}\right)\right]dx$$

$$=\frac{f(a)+f(b)}{4}$$

$$\times\left[-\frac{2}{\rho}\left[1+\tan\left(\frac{\rho\,(b-x)}{2}\right)\right]\Bigg|_a^b+\frac{2}{\rho}\left[1+\tan\left(\frac{\rho\,(x-a)}{2}\right)\right]\Bigg|_a^b\right]$$

$$= \frac{f(a)+f(b)}{\rho} \tan\left[\frac{\rho(b-a)}{2}\right],$$

then one can obtain the required result. $\square$

Remark 2.2. For $\rho \to 0$ we observe that

$$\lim_{\rho \to 0} \frac{1}{\rho} \tan\left[\frac{\rho(b-a)}{2}\right] = \frac{b-a}{2}$$

and

$$\lim_{\rho \to 0} \Lambda_2(t) =$$

$$= \frac{1}{2} \int_a^b \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right)\right] dx.$$

Thus, refinement of Hermite-Hadamard inequality (3) follows from Theorem 2.2 in the limit $\rho \to 0$.

The following theorem refines the first inequality in (7).

Theorem 2.3. *Suppose that $f : [a,b] \to \mathbb{R}$ is a positive function with $0 < b - a < \frac{\pi}{\rho}$, then Λ_3 is monotonically increasing on $[0,1]$ and we have the following refinement inequality*
(8)

$$f\left(\frac{a+b}{2}\right) = \Lambda_3(0) \le \Lambda_3(t) \le \Lambda_3(1) = \int_a^b f(x) \sin\left[\rho\left(x - \frac{a+b}{2}\right)\right] dx,$$

where

$$\Lambda_3(t) = \frac{1}{b-a} \int_a^b \sec\left[\rho t\left(x - \frac{a+b}{2}\right)\right] f\left(tx + (1-t)\frac{a+b}{2}\right) dx.$$

Proof. By using the change of variable and by using the fact that $\sec x$ is is an even function we obtain

$$\begin{aligned}
\Lambda_3(t) &= \frac{1}{b-a} \int_a^{\frac{a+b}{2}} \sec\left[\rho t\left(x - \frac{a+b}{2}\right)\right] f\left(tx + (1-t)\frac{a+b}{2}\right) dx \\
&\quad + \frac{1}{b-a} \int_{\frac{a+b}{2}}^b \sec\left[\rho t\left(x - \frac{a+b}{2}\right)\right] f\left(tx + (1-t)\frac{a+b}{2}\right) dx \\
&= \frac{1}{2(b-a)} \int_0^{b-a} \sec\left[-\frac{\rho t u}{2}\right] f\left(\frac{a+b}{2} - \frac{ut}{2}\right) du
\end{aligned}$$

$$+\frac{1}{2(b-a)}\int_0^{b-a}\sec\left[\frac{\rho t u}{2}\right] f\left(\frac{a+b}{2}+\frac{ut}{2}\right)du$$

$$=\frac{1}{2(b-a)}\int_0^{b-a}\sec\left[\frac{\rho t u}{2}\right]\left[f\left(\frac{a+b}{2}-\frac{ut}{2}\right)+f\left(\frac{a+b}{2}+\frac{ut}{2}\right)\right]du.$$

From Lemma 1.1, we have $h(t)=\frac{1}{2}\left[f\left(\frac{a+b}{2}-\frac{t}{2}\right)+f\left(\frac{a+b}{2}+\frac{t}{2}\right)\right]$ is increasing on $[0,b-a]$. Since $\sec\left[\frac{\rho t u}{2}\right]$ is nonnegative for $u\in[0,b-a]$ with $0<b-a<\frac{\pi}{\rho}$, thus $\Lambda_3(t)$ is increasing on $[0,1]$. This completes the proof. □

Remark 2.3. If we choose $\rho=1$ in Theorem 2.3, then the inequality (8) reduces to the inequality (2).

The following theorem refines the second inequality in (7).

Theorem 2.4. *Suppose that $f:[a,b]\to\mathbb{R}$ is a positive function with $0<b-a<\frac{\pi}{\rho}$, then Λ_4 is monotonically increasing on $[0,1]$ and we have the following refinement inequality*

$$\int_a^b \sec\left[\rho\left(x-\frac{a+b}{2}\right)\right]f(x)dx \tag{9}$$
$$=\Lambda_4(0)\le\Lambda_4(t)\le\Lambda_4(1)$$
$$=\frac{f(a)+f(b)}{2}\sec\left[\frac{\rho(b-a)}{2}\right],$$

where

$$\Lambda_4(t)=\frac{1}{2(b-a)}\int_a^b\sec\left[\rho\left(\frac{t(x-a)+(b-x)}{2}\right)\right]$$
$$\times f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right)$$
$$+\frac{1}{2(b-a)}\int_a^b\sec\left[\rho\left(\frac{t(b-x)+(x-a)}{2}\right)\right]$$
$$\times f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)dx.$$

Proof. Theorem 2.4 can be proven similar to Theorem 2.2. The detail is omitted. □

Remark 2.4. If we choose $\rho = 1$ in Theorem 2.4, then the inequality (9) reduces to the inequality (3).

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