

## Derivation $d_{a,\beta}$ of ordered $\Gamma$ -semirings

MARAPUREDDY MURALI KRISHNA RAO, KONA RAJENDR KUMAR,  
BOLINENI VENKATESWARLU, BANDARU RAVI KUMAR

ABSTRACT. In this paper, we introduce the concept of derivation  $d_{a,\beta}$  of ordered  $\Gamma$ -semiring. We study some of the properties of derivation  $d_{a,\beta}$  of ordered  $\Gamma$ -semirings. We prove that if a derivation  $d_{a,\beta}$  is nonzero on an integral  $\Gamma$ -semiring  $M$  then it is non-zero on any non-zero ideal of  $M$  and we characterize  $k$ -ideal and  $m - k$  ideal using derivation  $d_{a,\beta}$  of ordered  $\Gamma$ -semiring.

### 1. INTRODUCTION

In 1995, Murali Krishna Rao [12, 13, 16] introduced the notion of  $\Gamma$ -semiring as a generalization of  $\Gamma$ -ring, ternary semiring and semiring. Semiring is an algebraic structure as, a common generalization of ring and distributive lattice. Semiring was first introduced by American mathematician Vandiver [19] in 1934 but non trivial examples of semirings had appeared in the earlier studies on the theory of commutative ideals of rings by German mathematician Richard Dedekind in 19th century. Semiring is a universal algebra with two associative binary operations called addition and multiplication where one of them is distributive over the other. Bounded distributive lattices are commutative semirings which are both additively idempotent and multiplicatively idempotent. A natural example of semiring is the set of all natural numbers under usual addition and multiplication of numbers. In particular, if  $I$  is the unit interval on the real line then  $(I, \max, \min)$ , in which 0 is the additive identity and 1 is the multiplicative identity, is a semiring. The theory of rings and the theory of semigroups have considerable impact on the development of the theory of semirings. Additive and multiplicative structures of a semiring play an important role in determining the structure of a semiring. Semirings are used in the areas of theoretical computer science as well as in the solutions of graph theory and optimization theory and in particular for studying automata, coding theory and formal

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languages as the basic algebraic structure [6, 7]. Semiring theory has many applications in other branches [3, 4].

The notion of  $\Gamma$ -ring was introduced by Nobusawa [10] as a generalization of ring in 1964. Sen [18] introduced the notion of  $\Gamma$ -semigroup in 1981. The notion of ternary algebraic system was introduced by Lehmer [8] in 1932. Lister [9] introduced the notion of ternary ring. Dutta & Kar introduced the notion of ternary semiring which is a generalization of ternary ring and semiring. After the paper [12] was published, many mathematicians obtained interesting results on  $\Gamma$ -semirings.

Over the last few decades several authors have investigated the relationship between the commutativity of ring  $R$  and the existence of certain specified derivations of  $R$ . The first result in this direction is due to Posner [11] in 1957. In the year 1990, Bresar and Vukman [2] established that a prime ring must be a commutative if it admits a non zero left derivation. The notion of derivation of ring is useful for characterization of rings [1, 17, 5]. Murali Krishna Rao and Venkateswarlu [14, 15] introduced the concept of generalized right derivation of  $\Gamma$ -incline and the concept of right derivation of ordered  $\Gamma$ -semirings. In this paper, we introduce the concept of derivation  $d_{\alpha,\beta}$  of ordered  $\Gamma$ -semirings. We study some of the properties of derivation  $d_{\alpha,\beta}$  of ordered  $\Gamma$ -semirings and we prove that if a derivation  $d_{\alpha,\beta}$  is non-zero on an integral ordered  $\Gamma$ -semiring  $M$ , then it is non-zero on any non-zero ideal of  $M$ .

## 2. PRELIMINARIES

In this section we will recall some of the fundamental concepts and definitions necessary for this paper.

**Definition 2.1.** Let  $(M, +)$  and  $(\Gamma, +)$  be commutative semigroups. Then we call  $M$  as a  $\Gamma$ -semiring, if there exists a mapping  $M \times \Gamma \times M \rightarrow M$  is written  $(x, \alpha, y)$  as  $x\alpha y$  such that it satisfies the following axioms

- (i)  $x\alpha(y + z) = x\alpha y + x\alpha z$ ,
- (ii)  $(x + y)\alpha z = x\alpha z + y\alpha z$ ,
- (iii)  $x(\alpha + \beta)y = x\alpha y + x\beta y$ ,
- (iv)  $x\alpha(y\beta z) = (x\alpha y)\beta z$ , for all  $x, y, z \in M$  and  $\alpha, \beta \in \Gamma$ .

Every semiring  $R$  is a  $\Gamma$ -semiring with  $\Gamma = R$  and ternary operation  $x\gamma y$  as the usual semiring multiplication.

We illustrate the definition of  $\Gamma$ -semiring by the following example

**Example 2.1.** Let  $S$  be a semiring and  $M_{p,q}(S)$  denote the additive abelian semigroup of all  $p \times q$  matrices with identity element whose entries are from  $S$ . Then  $M_{p,q}(S)$  is a  $\Gamma$ -semiring with  $\Gamma = M_{p,q}(S)$  ternary operation is defined by  $x\alpha z = x(\alpha^t)z$  as the usual matrix multiplication, where  $\alpha^t$  denote the transpose of the matrix  $\alpha$ ; for all  $x, y$  and  $\alpha \in M_{p,q}(S)$ .

**Definition 2.2.** A  $\Gamma$ -semiring  $M$  is called an ordered  $\Gamma$ -semiring if it admits a compatible relation  $\leq$ . i.e.  $\leq$  is a partial ordering on  $M$  satisfies the following conditions. If  $a \leq b$  and  $c \leq d$  then

- (i)  $a + c \leq b + d$ ,
- (ii)  $a\alpha c \leq b\alpha d$ ,
- (iii)  $c\alpha a \leq d\alpha b$ , for all  $a, b, c, d \in M, \alpha \in \Gamma$ .

**Definition 2.3.** A non-empty subset  $A$  of ordered  $\Gamma$ -semiring  $M$  is called a  $\Gamma$ -subsemiring  $M$  if  $(A, +)$  is a subsemigroup of  $(M, +)$  and  $a\alpha b \in A$  for all  $a, b \in A$  and  $\alpha \in \Gamma$ .

**Definition 2.4.** A non-empty subset  $A$  of an ordered  $\Gamma$ -semiring.  $M$  is called a left (right) ideal of ordered  $\Gamma$ -semiring  $M$  if  $A$  satisfies the following conditions.

- (i)  $A$  is closed under addition,
- (ii)  $M\Gamma A \subseteq A$  ( $A\Gamma M \subseteq A$ ),
- (iii) if for any  $a \in M, b \in A, a \leq b$  implies  $a \in A$ .

$A$  is called an ideal of  $M$  if it is both a left ideal and a right ideal of  $M$ .

**Definition 2.5.** A non-empty subset  $A$  of an ordered  $\Gamma$ -semiring  $M$  is called a  $k$ -ideal if  $A$  is an ideal and  $x \in M, x + y \in A, y \in A$  then  $x \in A$ .

**Definition 2.6.** An ordered  $\Gamma$ -semiring  $M$  is said to have zero element if there exists an element  $0 \in M$  such that  $0 + x = x = x + 0$  and  $0\alpha x = x\alpha 0 = 0$ , for all  $x \in M, \alpha \in \Gamma$ .

**Definition 2.7.** An element  $1 \in M$  is said to be unity if for each  $x \in M$  there exists  $\alpha \in \Gamma$  such that  $x\alpha 1 = 1\alpha x = x$ .

**Definition 2.8.** An element  $a \in M$  is said to be idempotent of  $M$  if there exists  $\alpha \in \Gamma$  such that  $a = a\alpha a$  and  $a$  is also said to be  $\alpha$  idempotent.

**Definition 2.9.** A semigroup  $(M, +)$  is said to be band if  $a+a = a$ , for all  $a \in M$ .

**Definition 2.10.** Every element of an ordered  $\Gamma$ -semiring  $M$  is an idempotent of  $M$  then  $M$  is said to be idempotent ordered  $\Gamma$ -semiring  $M$ .

**Definition 2.11.** A non zero element  $a$  in an ordered  $\Gamma$ -semiring  $M$  is said to be zero divisor if there exists non zero element  $b \in M, \alpha \in \Gamma$  such that  $a\alpha b = b\alpha a = 0$ .

**Definition 2.12.** An ordered  $\Gamma$ -semiring  $M$  with unity  $1$  and zero element  $0$  is called an integral ordered  $\Gamma$ -semiring if it has no zero divisors.

**Definition 2.13.** An ordered  $\Gamma$ -semiring  $M$  is said to be totally ordered  $\Gamma$ -semiring  $M$  if any two elements of  $M$  are comparable.

**Definition 2.14.** In an ordered  $\Gamma$ -semiring  $M$

- (i) the semigroup  $(M, +)$  is said to be positively ordered , if  $a \leq a + b$  and  $b \leq a + b$ , for all  $a, b \in M$ .
- (ii) the semigroup  $(M, +)$  is said to be negatively ordered, if  $a + b \leq a$  and  $a + b \leq b$ , for all  $a, b \in M$ .
- (iii) the  $\Gamma$ -semigroup  $M$  is said to be positively ordered, if  $a \leq a\alpha b$  and  $b \leq a\alpha b$ , for all  $\alpha \in \Gamma, a, b \in M$ .
- (iv)  $\Gamma$ -semigroup  $M$  is said to be negatively ordered if  $a\alpha b \leq a$  and  $a\alpha b \leq b$  for all  $\alpha \in \Gamma, a, b \in M$ .

**Definition 2.15.** Let  $M$  and  $N$  be ordered  $\Gamma$ - semirings. A mapping  $f : M \rightarrow N$  is called a homomorphism if

- (i)  $f(a + b) = f(a) + f(b)$ ,
- (ii)  $f(a\alpha b) = f(a)\alpha f(b)$ , for all  $a, b \in M, \alpha \in \Gamma$ .

**Definition 2.16.** Let  $M$  be an ordered  $\Gamma$ -semiring. A mapping  $d : M \rightarrow M$  is called a derivation if it satisfies the following conditions.

- (i)  $d(x + y) = d(x) + d(y)$ ,
- (ii)  $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$  for all  $x, y \in M$  and  $\alpha \in \Gamma$ .

### 3. DERIVATION $d_{a,\beta}$ OF ORDERED $\Gamma$ -SEMIRINGS

In this section, we introduce the notion of derivation of the form  $d_{a,\beta}$  of ordered  $\Gamma$ -semirings. We study some of the properties of derivation  $d_{a,\beta}$  of ordered  $\Gamma$ -semirings.

**Definition 3.1.** Let  $M$  be an ordered  $\Gamma$ -semiring. Then for any  $a \in M$  and  $\beta \in \Gamma$ , we define a mapping  $d : M \rightarrow M$  by  $d(x) = x\beta a$ , for all  $x \in M$ . This function  $d$  is denoted by  $d_{a,\beta}$ .

**Definition 3.2.** Let  $M$  be an ordered  $\Gamma$ -semiring and  $d_{a,\beta}$  be a function. Then  $d_{a,\beta}$  is said to be derivation of  $M$  if

- (i)  $d_{a,\beta}(x + y) = d_{a,\beta}(x) + d_{a,\beta}(y)$ ,
- (ii)  $d_{a,\beta}(x\alpha y) = d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y)$ , for all  $x, y \in M, \alpha \in \Gamma$ .

**Example 3.1.** Let  $M = \{0, a, b, 1\}$  and  $\Gamma = \{\alpha, \beta\}$ . If we define the the additive operations on  $M$  and  $\Gamma$ , by

+	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

+	$\alpha$	$\beta$
$\alpha$	$\alpha$	$\beta$
$\beta$	$\beta$	$\alpha$

ternary operation, by

$\alpha$	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

$\beta$	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

and  $x \leq y$  if and only if  $x + y = y$ , for all  $x, y \in M$  then  $M$  is an ordered  $\Gamma$ -semiring .

Let  $a \in M$  and  $\beta \in \Gamma$ . Define  $d_{a,\beta} = x\beta a$ , for all  $x \in M$ . Obviously  $d_{a,\beta}$  is a derivation of  $M$

**Example 3.2.** Let  $M = [0, 1]$  and  $\Gamma =$  set of all natural numbers. Define the binary operations  $+$  on  $M$  and  $\Gamma$ , by  $a + b = \max\{a, b\}$ ,  $\alpha + \beta = \max\{\alpha, \beta\}$ , and ternary operation by  $a\alpha b = \min\{a, \alpha, b\}$ , for all  $a, b \in M, \alpha \in \Gamma$  and  $a \leq b$  if and only if  $a + b = b$ , for all  $a, b \in M, \alpha, \beta \in \Gamma$  Then  $M$  is an ordered  $\Gamma$ -semiring .

Let  $a \in M$  and  $\beta \in \Gamma$ . Define  $d_{a,\beta}(x) = x\beta a$ , for all  $x \in M$ .

Obviously  $d_{a,\beta}$  is a derivation of  $M$ .

**Theorem 3.1.** Let  $M$  be an ordered commutative  $\Gamma$ -semiring in which semi-group  $(M, +)$  is a band .Then  $d_{a,\beta}$  is a derivation of  $M$ .

*Proof.* Let  $M$  be an ordered commutative  $\Gamma$ -semiring in which semigroup  $(M, +)$  is a band ,  $x, y \in M$  and  $\alpha, \beta \in \Gamma$ . Then

$$\begin{aligned}
 d_{a,\beta}(x\alpha y) &= (x\alpha y)\beta a \\
 d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y) &= (x\beta a)\alpha y + x\alpha y\beta a \\
 &= y\alpha(x\beta a) + x\alpha y\beta a \\
 &= (y\alpha x)\beta a + x\alpha y\beta a \\
 &= x\alpha y\beta a + x\alpha y\beta a \\
 &= x\alpha y\beta a.
 \end{aligned}$$

Hence  $d_{a,\beta}$  is a derivation of  $M$ . □

**Theorem 3.2.** Let  $M$  be an ordered commutative  $\Gamma$ -semiring in which semi-group  $(M, +)$  is a band with unity element 1. Then for each  $x \in M$ , there exists a derivation  $d_{1,\beta}$  ,  $\beta \in \Gamma$  such that  $d_{1,\beta}(x) = x$ .

*Proof.* Let  $x \in M$ . Then there exists  $\beta \in \Gamma$  such that  $x\beta 1 = x$ . By Theorem 3.1  $d_{1,\beta}$  is a derivation and  $d_{1,\beta}(x) = x\beta 1 = x$  □

**Theorem 3.3.** Let  $M$  be an ordered  $\Gamma$ -semiring in which semigroup  $(M, +)$  is is a band and positively ordered,  $\Gamma$ -semigroup  $M$  is negatively ordered and  $d_{a,\beta}$  be a derivation. Then

- (i)  $d_{a,\beta}(x\alpha y) \leq d_{a,\beta}(x) + d_{a,\beta}(y)$ ,
- (ii)  $d_{a,\beta}(x) \leq x$ ,

(iii) if  $x \leq y$  then  $d_{a,\beta}(x\alpha y) \leq y$ .

*Proof.* (i) Let  $x, y \in M, \alpha \in \Gamma$  then

$$\begin{aligned} d_{a,\beta}(x\alpha y) &= (x\alpha y)\beta a \\ &= (x + x)\alpha y\beta a \\ &= x\alpha y\beta a + x\alpha y\beta a \\ &\leq x\beta a + y\beta a \\ &= d_{a,\beta}(x) + d_{a,\beta}(y). \end{aligned}$$

(ii)  $d_{a,\beta}(x) = x\beta a \leq x$ .

(iii) Suppose  $x \leq y$ . Then

$$\begin{aligned} x + y &\leq y + y, \\ x + y &\leq y \leq x + y, \\ x + y &= y. \\ d_{a,\beta}(x\alpha y) &\leq d_{a,\beta}(x) + d_{a,\beta}(y) \\ &\leq x + y = y. \end{aligned}$$

This completes the proof.  $\square$

**Theorem 3.4.** Let  $d_{a,\beta}$  be a derivation of an ordered  $\Gamma$ -semiring  $M$ .

Then  $d_{a,\beta}(0) = 0$ .

*Proof.* By Definition 3.1,  $d_{a,\beta}(x) = x\beta a$ , for all  $x \in M$ . Then  $d_{a,\beta}(0) = 0\beta a = 0$ .

Therefore  $d_{a,\beta}(0) = 0$ .  $\square$

**Theorem 3.5.** Let  $d_{a,\beta}$  be a derivation of an idempotent ordered  $\Gamma$ -semiring  $M$  in which  $\Gamma$ -semigroup  $M$  is negatively ordered, semigroup  $(M, +)$  is a band. Then  $d_{a,\beta}(x) \leq x$ , for all  $x \in M$ .

*Proof.* Let  $d_{a,\beta}$  be a derivation of an idempotent ordered  $\Gamma$ -semiring  $M$  in which  $\Gamma$ -semigroup  $M$  is negatively ordered. Suppose  $x \in M$ . Then there exists  $\alpha \in \Gamma$  such that  $x\alpha x = x$ .

$$\begin{aligned} d_{a,\beta}(x) &= d_{a,\beta}(x\alpha x) \\ &= d_{a,\beta}(x)\alpha x + x\alpha d_{a,\beta}(x) \\ &\leq x + x. \end{aligned}$$

Therefore  $d_{a,\beta}(x) \leq x$ .

This completes the proof.  $\square$

**Theorem 3.6.** Let  $M$  be an ordered  $\Gamma$ -semiring in which  $\Gamma$ -semigroup  $M$  is negatively ordered. Then  $d_{a,\beta}(x\alpha y) \leq d_{a,\beta}(x + y)$ , for all  $x, y \in M, \alpha \in \Gamma$ .

*Proof.* Let  $M$  be an ordered  $\Gamma$ -semiring in which  $\Gamma$ -semigroup  $M$  is negatively ordered. Suppose  $x, y \in M, \alpha \in \Gamma$ . Then

$$d_{a,\beta}(x)\alpha y \leq d_{a,\beta}(x), \quad x\alpha d_{a,\beta}(y) \leq d_{a,\beta}(y).$$

Therefore

$$\begin{aligned} d_{a,\beta}(x\alpha y) &= d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y) \\ &\leq d_{a,\beta}(x) + d_{a,\beta}(y) \\ &= d_{a,\beta}(x + y). \end{aligned}$$

This completes the proof. □

**Theorem 3.7.** *Let  $M$  be an idempotent ordered  $\Gamma$ -semiring in which  $\Gamma$ -semigroup  $M$  is negatively ordered and semigroup  $(M, +)$  is a band. Then the following hold for all  $x, y \in M, \alpha \in \Gamma$ .*

- (i)  $d_{a,\beta}(x\alpha y) \leq d_{a,\beta}(x) + d_{a,\beta}(y)$ ,
- (ii) If  $x \leq y$  then  $d_{a,\beta}(x\alpha y) \leq d_{a,\beta}(y)$ ,
- (iii)  $d_{a,\beta}(x) \leq x$ .

*Proof.* (i) The following holds

$$\begin{aligned} d_{a,\beta}(x\alpha y) &= d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y) \\ &\leq d_{a,\beta}(x) + d_{a,\beta}(y). \end{aligned}$$

(ii) Suppose  $x \leq y$ , then

$$\begin{aligned} x\alpha d_{a,\beta}(y) &\leq y\alpha d_{a,\beta}(y) \leq y, \\ d_{a,\beta}(x)\alpha y &\leq y, \\ d_{a,\beta}(x\alpha y) &= d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y) \\ &\leq y + y = y. \end{aligned}$$

(iii) Let  $x \in M$ . Then there exists  $\alpha \in \Gamma$  such that  $x\alpha x = x$ .

$$\begin{aligned} d_{a,\beta}(x) &= d_{a,\beta}(x\alpha x) \\ &= d_{a,\beta}(x)\alpha x + x\alpha d_{a,\beta}(x) \\ &\leq x + x = x. \end{aligned}$$

This completes the proof. □

**Theorem 3.8.** *Let  $M$  be an idempotent ordered  $\Gamma$ -semiring with unity 1 in which semigroup  $(M, +)$  is a band and positively ordered,  $\Gamma$ -semigroup  $M$  is negatively ordered and  $d_{a,\beta}$  be a derivation of  $M$ . Then the following hold for all  $x \in M$ ,*

- (i)  $x\alpha d_{a,\beta}(1) \leq d_{a,\beta}(x), \alpha \in \Gamma$ ,
- (ii) If  $d_{a,\beta}(1) = 1$  then  $d_{a,\beta}(x) = x$ , for all  $x \in M$ .

*Proof.* (i) Let  $x \in M$ . Then there exists  $\alpha \in \Gamma$  such that  $x\alpha 1 = x$ .

Therefore

$$\begin{aligned} d_{a,\beta}(x\alpha 1) &= d_{a,\beta}(x), \\ d_{a,\beta}(x)\alpha 1 + x\alpha d_{a,\beta}(1) &= d_{a,\beta}(x), \\ x\alpha d_{a,\beta}(1) &\leq d_{a,\beta}(x). \end{aligned}$$

(ii) Suppose  $d_{a,\beta}(1) = 1$ . We have

$$\begin{aligned}x\alpha d_{a,\beta}(1) &\leq d_{a,\beta}(x), \\x\alpha 1 &\leq d_{a,\beta}(x).\end{aligned}$$

Therefore  $x \leq d_{a,\beta}(x)$ .

By Theorem 3.7,  $d_{a,\beta}(x) \leq x$ . Hence  $d_{a,\beta}(x) = x$ , for all  $x \in M$ .  $\square$

**Theorem 3.9.** *Let  $M$  be an ordered  $\Gamma$ -semiring with unity 1 in which semigroup  $(M, +)$  is a band and positively ordered,  $\Gamma$ -semigroup  $M$  is negatively ordered and  $d_{a,\beta}$  be a derivation of  $M$ . If  $x \in M$  then there exists  $\alpha \in \Gamma$  such that*

- (i)  $x\alpha d_{a,\beta}(1) \leq d_{a,\beta}(x)$ ,
- (ii) If  $d_{a,\beta}(1) = 1$  then  $x \leq d_{a,\beta}(x)$ .

*Proof.* (i) Let  $M$  be an ordered  $\Gamma$ -semiring with unity 1,  $d_{a,\beta}$  be a derivation of  $M$  and  $x \in M$ . Then there exists  $\alpha \in \Gamma$  such that  $x\alpha 1 = x$ .

$$\begin{aligned}d_{a,\beta}(x) &= d_{a,\beta}(x\alpha 1) \\&= d_{a,\beta}(x)\alpha 1 + x\alpha d_{a,\beta}(1), \\x\alpha d_{a,\beta}(1) &\leq d_{a,\beta}(x)\alpha 1 + x\alpha d_{a,\beta}(1) = d_{a,\beta}(x).\end{aligned}$$

(ii) Suppose  $d_{a,\beta}(1) = 1$  and  $x\alpha d_{a,\beta}(1) \leq d_{a,\beta}(x)$ . Then  $\Rightarrow x\alpha 1 \leq d_{a,\beta}(x)$  and  $x \leq d_{a,\beta}(x)$ .

This completes the proof.  $\square$

**Theorem 3.10.** *Let  $M$  be an idempotent ordered  $\Gamma$ -semiring in which  $\Gamma$ -semigroup  $M$  is negatively ordered and semigroup  $(M, +)$  is a band. If  $d_{a,\beta}^2(x) = d_{a,\beta}(d_{a,\beta}(x)) = d_{a,\beta}(x)$  then  $d_{a,\beta}(x\alpha d_{a,\beta}(x)) \leq d_{a,\beta}(x)$ , for all  $x \in M$ .*

*Proof.* Let  $M$  be an idempotent ordered  $\Gamma$ -semiring and

$$d_{a,\beta}^2(x) = d_{a,\beta}(d_{a,\beta}(x)) = d_{a,\beta}(x),$$

for all  $x \in M$ . Then

$$\begin{aligned}d_{a,\beta}(x\alpha d_{a,\beta}(x)) &= d_{a,\beta}(x)\alpha d_{a,\beta}(x) + x\alpha d_{a,\beta}(d_{a,\beta}(x)) \\&= d_{a,\beta}(x) + x\alpha d_{a,\beta}(x) \\&\leq d_{a,\beta}(x) + d_{a,\beta}(x) \\&= d_{a,\beta}(x).\end{aligned}$$

Therefore  $d_{a,\beta}(x\alpha d_{a,\beta}(x)) \leq d_{a,\beta}(x)$ .

This completes the proof.  $\square$

**Theorem 3.11.** *Let  $d_{a,\beta}$  be a derivation of an ordered integral- $\Gamma$ -semiring  $M$  with unity and  $a \in M$ . If  $a\alpha d_{a,\beta}(x) = 0$  for all  $x \in M$ ,  $\alpha \in \Gamma$  then either  $a = 0$  or  $d_{a,\beta} = 0$ .*



*Proof.* Suppose  $a\alpha d_{a,\beta}(x) = 0$ , for all  $x \in M$ ,  $\alpha \in \Gamma$ .

Let  $y \in M$  and  $\gamma \in \Gamma$ . If we replace  $x$  by  $x\gamma y$ , then we obtain  $a\alpha d_{a,\beta}(x\gamma y) = 0$  and

$$\begin{aligned} a\alpha[d_{a,\beta}(x)\gamma y + x\beta d_{a,\gamma}(y)] &= 0, \\ a\alpha x\gamma d_{a,\beta}(y) &= 0, \quad \gamma \in \Gamma, \\ a\alpha 1\gamma d_{a,\beta}(y) &= 0, \\ a\alpha d_{a,\beta}(y) &= 0. \end{aligned}$$

Therefore  $a = 0$  or  $d_{a,\beta}(y) = 0$  since  $M$  has no zero divisors.

This completes the proof. □

**Definition 3.3.** An ideal  $I$  of an ordered  $\Gamma$ -semiring  $M$  is said to be  $m - k$  ideal if  $x\alpha y \in I$ ,  $x \in I$ ,  $1 \neq y \in M$  and  $\alpha \in \Gamma$  then  $y \in I$ .

**Definition 3.4.** Let  $d_{a,\beta}$  be a derivation of an ordered  $\Gamma$ -semiring  $M$ . Derivation  $d_{a,\beta}$  is called an isotone derivation if  $x \leq y$  then  $d_{a,\beta}(x) \leq d_{a,\beta}(y)$  for all  $x, y \in M$ .

**Theorem 3.12.** Let  $d_{a,\beta}$  be an isotone derivation of an ordered  $\Gamma$ -semiring  $M$ . Define  $\ker d_{a,\beta} = \{x \in M / d_{a,\beta}(x) = 0\}$ . Then  $\ker d_{a,\beta}$  is a  $k$ -ideal of an ordered  $\Gamma$ -semiring  $M$ .

*Proof.* Let  $x, y \in \ker d_{a,\beta}$  and  $\alpha \in \Gamma$ . Then

$$\begin{aligned} x\beta a &= y\beta a = 0, \\ d_{a,\beta}(x + y) &= (x + y)\beta a = 0. \end{aligned}$$

Therefore  $x + y \in \ker d_{a,\beta}$ .

$$\begin{aligned} d_{a,\beta}(x\alpha y) &= d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y) \\ &= (x\beta a)\alpha y + x\alpha(y\beta a) \\ &= 0\alpha y + x\alpha 0 = 0. \end{aligned}$$

Therefore  $x\alpha y \in \ker d_{a,\beta}$ .

Suppose  $y \in \ker d_{a,\beta}$ ,  $x \in M$  and  $x \leq y$ . Then

$$\begin{aligned} d_{a,\beta}(x) &\leq d_{a,\beta}(y), \\ x\beta a &\leq y\beta a = 0, \\ x\beta a &= 0, \\ x &\in \ker d_{a,\beta}. \end{aligned}$$

Hence  $\ker d_{a,\beta}$  is an ideal.

Suppose  $x + y \in \ker d_{a,\beta}$  and  $y \in \ker d_{a,\beta}$ . Then

$$\begin{aligned} d_{a,\beta}(x + y) &= 0, \\ d_{a,\beta}(x) + d_{a,\beta}(y) &= 0, \\ d_{a,\beta}(x) &= 0, \end{aligned}$$

$$x \in \ker d_{a,\beta}.$$

This completes the proof.  $\square$

**Theorem 3.13.** *Let  $d_{a,\beta}$  be an isotone derivation of an integral ordered  $\Gamma$ -semiring  $M$ . Then  $\ker d_{a,\beta}$  is a  $m - k$  ideal of  $M$ .*

*Proof.* By Theorem 3.12,  $\ker d_{a,\beta}$  is an ideal of an ordered  $\Gamma$ -semiring  $M$ .

Let  $0 \neq y \in \ker d_{a,\beta}$ ,  $x \in M$ ,  $\alpha \in \Gamma$  and  $x\alpha y \in \ker d_{a,\beta}$ . Then

$$\begin{aligned} d_{a,\beta}(x\alpha y) &= 0, \\ d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y) &= 0, \\ d_{a,\beta}(x)\alpha y &= 0, \\ d_{a,\beta}(x) &= 0, \end{aligned}$$

since  $M$  is an integral ordered  $\Gamma$ -semiring.

Therefore  $\ker d_{a,\beta}$  is a  $m - k$  ideal of  $M$ .  $\square$

**Theorem 3.14.** *Let  $d_{a,\beta}$  be a derivation of multiplicatively cancellative commutative idempotent ordered  $\Gamma$ -semiring  $M$  where  $(M, +)$  is positively ordered and band,  $\Gamma$ -semigroup  $M$  is negatively ordered and  $d_{a,\beta}(1) = 1$ . Define a set  $Fixd_{a,\beta}(M) = \{x \in M \mid d_{a,\beta}(x) = x\}$ . Then  $Fixd_{a,\beta}(M)$  is a  $m - k$  ideal.*

*Proof.* Obviously  $Fixd_{a,\beta}(M) = \{x \in M \mid d_{a,\beta}(x) = x\}$  is an ideal of  $M$ . Suppose  $x\alpha y \in Fixd_{a,\beta}(M)$ ,  $x \in Fixd_{a,\beta}(M)$  and  $\alpha \in \Gamma$ . Then

$$\begin{aligned} d_{a,\beta}(x\alpha y) &= x\alpha y, \\ d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y) &= x\alpha y, \\ x\alpha y + x\alpha d_{a,\beta}(y) &= x\alpha y, \\ x\alpha[y + d_{a,\beta}(y)] &= x\alpha y, \\ y + d_{a,\beta}(y) &= y, \\ d_{a,\beta}(y) &\leq y + d_{a,\beta}(y) = y. \end{aligned}$$

By Theorem 3.9, we have  $y \leq d_{a,\beta}(y)$ . Hence  $d_{a,\beta}(y) = y$ ,  $y \in Fixd_{a,\beta}(M)$ .

Therefore,  $Fixd_{a,\beta}(M)$  is a  $m - k$ -ideal of  $M$ .  $\square$

**Theorem 3.15.** *Let  $M$  be an ordered commutative  $\Gamma$ -semiring and  $d_{a,\beta}$ ,  $d_{b,\gamma}$  be derivations of  $M$ . If  $d_{a,\beta} d_{b,\gamma} = 0$  then  $d_{b,\gamma} d_{a,\beta}$  is a derivation of  $M$ .*

*Proof.* Let  $x, y \in M$  and  $\alpha \in \Gamma$ . Then

$$\begin{aligned} 0 &= d_{a,\beta} d_{b,\gamma} (x\alpha y) \\ &= d_{a,\beta} [d_{b,\gamma} (x)\alpha y + x\alpha d_{b,\gamma} (y)] \\ &= d_{a,\beta} (d_{b,\gamma} (x))\alpha y + d_{b,\gamma} (x)\alpha d_{a,\beta}(y) + d_{a,\beta}(x)\alpha d_{b,\gamma} (y) + x\alpha d_{a,\beta} (d_{b,\gamma} (y)), \\ 0 &= d_{b,\gamma} (x)\alpha d_{a,\beta}(y) + d_{a,\beta}(x)\alpha d_{b,\gamma} (y). \end{aligned}$$

$$\begin{aligned} d_{b,\gamma} d_{a,\beta}(x + y) &= d_{b,\gamma} [d_{a,\beta}(x) + d_{a,\beta}(y)] \\ &= d_{b,\gamma} (d_{a,\beta}(x)) + d_{b,\gamma} (d_{a,\beta}(y)) \end{aligned}$$

and

$$\begin{aligned} d_{b,\gamma} d_{a,\beta}(x\alpha y) &= d_{b,\gamma} [d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y)] \\ &= d_{b,\gamma} d_{a,\beta}(x)\alpha y + d_{a,\beta}(x)\alpha d_{b,\gamma} (y) + d_{b,\gamma} (x)\alpha d_{a,\beta}(y) \\ &\quad + x\alpha d_{b,\gamma} (d_{a,\beta}(y)) \\ &= d_{b,\gamma} (d_{a,\beta}(x))\alpha y + x\alpha d_{b,\gamma} (d_{a,\beta}(y)). \end{aligned}$$

Hence  $d_{b,\gamma} d_{a,\beta}$  is a derivation of  $M$ . □

**Theorem 3.16.** *Let  $d_{a,\beta}$  be a derivation of an ordered integral  $\Gamma$ -semiring  $M$  with unity and  $b \in M, \alpha \in \Gamma$ . Then if  $b\alpha d_{a,\beta}(x) = 0$ , for all  $x \in M, \alpha \in \Gamma$ . Then either  $b = 0$  or  $d_{a,\beta}$  is zero.*

*Proof.* Suppose  $b\alpha d_{a,\beta}(x) = 0$ , for all  $x \in M$ . Let  $y \in M$ . Replace  $x$  by  $x\alpha y$ , then

$$\begin{aligned} b\alpha d_{a,\beta}(x\alpha y) &= 0, \\ b\alpha [d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y)] &= 0, \\ b\alpha x\alpha d_{a,\beta}(y) &= 0, \quad \alpha \in \Gamma. \end{aligned}$$

Since  $b \in M$ , there exists  $\gamma \in \Gamma$  such that  $b\gamma 1 = b$ , then

$$\begin{aligned} b\gamma d_{a,\beta}(y) &= 0, \\ b = 0 \text{ or } d_{a,\beta}(y) &= 0, \\ b = 0 \text{ or } d_{a,\beta} &= 0. \end{aligned}$$

This completes the proof. □

**Theorem 3.17.** *Let  $M$  be an ordered  $\Gamma$ -semiring in which  $(M, +)$  is positively ordered and  $\text{band}, \Gamma$ -semigroup  $M$  is negatively ordered and  $d_{a,\beta}$  be a derivation of  $M$ . Then the following hold,*

- (i)  $d_{a,\beta}(x\alpha y) \leq d_{a,\beta}(x)$ ,
- (ii)  $d_{a,\beta}(x\alpha y) \leq d_{a,\beta}(y)$ ,
- (iii)  $x \leq y$  then  $d_{a,\beta}(x) \leq d_{a,\beta}(y)$ , for all  $x, y \in M, \alpha \in \Gamma$ .

*Proof.* (i) Let  $x, y \in M, \alpha \in \Gamma$ . Then

$$d_{a,\beta}(x\alpha y) = (x\alpha y)\beta a \leq x\beta a = d_{a,\beta}(x).$$

(ii) Similarly we can prove  $d_{a,\beta}(x\alpha y) \leq d_{a,\beta}(y)$ .

(iii) Suppose  $x \leq y$ . Then

$$\begin{aligned} x + y &\leq y + y, \\ x + y &\leq y \leq x + y, \\ x + y &= y, \\ d_{a,\beta}(x + y) &= d_{a,\beta}(y), \end{aligned}$$

$$d_{\alpha,\beta}(x) + d_{\alpha,\beta}(y) = d_{\alpha,\beta}(y).$$

Therefore  $d_{\alpha,\beta}(x) \leq d_{\alpha,\beta}(y)$ .  $\square$

**Theorem 3.18.** *Let  $d_{\alpha,\beta}$  be a derivation of an integral ordered  $\Gamma$ -semiring  $M$  in which semigroup  $(M, +)$  is a band. Define  $d_{\alpha,\beta}^2(x) = d_{\alpha,\beta}(d_{\alpha,\beta}(x))$ , for all  $x \in M$ . If  $d_{\alpha,\beta}^2 = 0$  then  $d_{\alpha,\beta} = 0$ .*

*Proof.* Let  $x, y \in M$  and  $\alpha \in \Gamma$ . Then

$$\begin{aligned} d_{\alpha,\beta}^2(x\alpha y) &= 0, \\ d_{\alpha,\beta}[d_{\alpha,\beta}(x\alpha y)] &= 0, \\ d_{\alpha,\beta}[d_{\alpha,\beta}(x)\alpha y + x\alpha d_{\alpha,\beta}(y)] &= 0, \\ d_{\alpha,\beta}^2(x)\alpha y + d_{\alpha,\beta}(x)\alpha d_{\alpha,\beta}(y) + d_{\alpha,\beta}(x)\alpha d_{\alpha,\beta}(y) + x\alpha d_{\alpha,\beta}^2(y) &= 0, \\ d_{\alpha,\beta}(x)\alpha d_{\alpha,\beta}(y) + d_{\alpha,\beta}(x)\alpha d_{\alpha,\beta}(y) &= 0, \\ d_{\alpha,\beta}(x)\alpha d_{\alpha,\beta}(y) &= 0, \\ d_{\alpha,\beta}(x) = 0 \text{ or } d_{\alpha,\beta}(y) &= 0, \end{aligned}$$

for all  $x, y \in M$ . Therefore in both the cases we have  $d_{\alpha,\beta} = 0$ .

This completes the proof.  $\square$

**Theorem 3.19.** *Let  $I$  be a non-zero ideal of an integral ordered  $\Gamma$ -semiring  $M$  in which  $\Gamma$ -semigroup  $M$  is negatively ordered. If  $d_{\alpha,\beta}$  is a non-zero derivation of  $M$  then  $d_{\alpha,\beta}$  is non-zero on  $I$ .*

*Proof.* Suppose  $d_{\alpha,\beta}$  is a non-zero derivation of  $M$  and  $d_{\alpha,\beta}(x) = 0$  for all  $x \in I$ . Let  $y \in M, \alpha \in \Gamma$  and  $x \in I$ . Then  $x\alpha y \leq x$ . Therefore  $x\alpha y \in I$ , and

$$\begin{aligned} d_{\alpha,\beta}(x\alpha y) &= 0, \\ d_{\alpha,\beta}(x)\alpha y + x\alpha d_{\alpha,\beta}(y) &= 0, \\ x\alpha d_{\alpha,\beta}(y) &= 0. \end{aligned}$$

Since  $M$  has no zero divisors, we have  $x = 0$  or  $d_{\alpha,\beta}(y) = 0$ , for all  $y \in M$ .

Further, since  $I$  is a non-zero ideal, we get  $d_{\alpha,\beta}(y) = 0$ , for all  $y \in M$ , which is a contradiction to  $d_{\alpha,\beta} \neq 0$  on  $M$ . Hence  $d_{\alpha,\beta}$  is non-zero derivation on  $I$ .

This completes the proof.  $\square$

**Theorem 3.20.** *Let  $d_{\alpha,\beta}$  be a non-zero derivation of an integral ordered  $\Gamma$ -semiring  $M$ . If  $I$  is a non-zero ideal of  $M$  and  $t \in M$  such that  $t\alpha d_{\alpha,\beta}(I) = 0$  then  $t = 0$ .*

*Proof.* By Theorem 3.19, there exists  $x \in I$  such that  $d_{\alpha,\beta}(x) \neq 0$ .

Suppose  $t\alpha d_{\alpha,\beta}(I) = 0$ . Then

$$\begin{aligned} t\alpha d_{\alpha,\beta}(x\alpha x) &= 0, \\ t\alpha[d_{\alpha,\beta}(x)\alpha x + x\alpha d_{\alpha,\beta}(x)] &= 0, \end{aligned}$$

$$t\alpha d_{a,\beta}(x)\alpha x + t\alpha(x\alpha d_{a,\beta}(x)) = 0,$$

$$t\alpha(x\alpha d_{a,\beta}(x)) = 0.$$

Therefore  $t = 0$ .

This completes the proof.  $\square$

#### 4. CONCLUSION

In this paper, we introduced the concept of derivation  $d_{a,\beta}$  of ordered  $\Gamma$ -semiring. We studied some of the properties of derivation  $d_{a,\beta}$  of ordered  $\Gamma$ -semirings. We proved that if a derivation  $d_{a,\beta}$  is non-zero on an integral  $\Gamma$ -semiring  $M$ , then it is non-zero on any non-zero ideal of  $M$  and we characterized  $k$ -ideal and  $m - k$  ideal using derivation  $d_{a,\beta}$  of ordered  $\Gamma$ -semiring.

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**MARAPUREDDY MURALI KRISHNA RAO**

DEPARTMENT OF MATHEMATICS

GIT, GITAM UNIVERSITY

VISAKHAPATNAM – 530 045

INDIA

*E-mail address:* mmarapureddy@gmail.com

**KONA RAJENDR KUMAR**

DEPARTMENT OF MATHEMATICS

GIT, GITAM UNIVERSITY

VISAKHAPATNAM – 530 045

INDIA

*E-mail address:* rkkona72@rediffmail.com

**BOLINENI VENKATESWARLU**

DEPARTMENT OF MATHEMATICS

GST, GITAM UNIVERSITY

DODDABALLAPURA – 561 203

BENGALURU RUAL

INDIA

*E-mail address:* bvlmaths@gmail.com

**BANDARU RAVI KUMAR**

DEPARTMENT OF MATHEMATICS

GST, GITAM UNIVERSITY

MEDAK DISTRICT – 502329

HYDERBAD

INDIA

*E-mail address:* ravimaths83@gmail.com