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Starlike functions of complex order with bounded radius rotation by using quantum calculus

Asena Çetİnkaya, Oya Mert

Abstract. In the present paper, we study on the subclass of starlike functions of complex order with bounded radius rotation using $q-$difference operator denoted by $\mathcal{R}_k(q,b)$ where $k \geq 2$, $q \in (0,1)$ and $b \in \mathbb{C}\backslash\{0\}$. We investigate coefficient inequality, distortion theorem and radius of starlikeness for the class $\mathcal{R}_k(q,b)$.

1. Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $\mathbb{D} := \{z : |z| < 1\}$ and satisfy the condition $f(0) = f'(0) - 1 = 0$ for every $z \in \mathbb{D}$. We say that f_1 is subordinate to f_2, written as $f_1 \prec f_2$, if there exists a Schwarz function ϕ which is analytic in $\mathbb{D}$ with $\phi(0) = 0$ and $|\phi(z)| < 1$ such that $f_1(z) = f_2(\phi(z))$. In particular, when f_2 is univalent, then the subordination is equivalent to $f_1(0) = f_2(0)$ and $f_1(\mathbb{D}) \subset f_2(\mathbb{D})$ (Subordination principle [4]).

In 1909 and 1910, Jackson [5, 6] initiated a study of $q-$ difference operator by

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad \text{for } z \in B\diagdown\{0\},$$

where B is a subset of complex plane $\mathbb{C}$, called $q-$ geometric set if $qz \in B$, whenever $z \in B$. Note that if a subset B of $\mathbb{C}$ is $q-$ geometric, then it contains all geometric sequences $\{zq^n\}_0^{\infty}$, $zq \in B$. Obviously, $D_q f(z) \to f'(z)$ as $q \to 1^-$. Note that such an operator plays an important role in the theory of hypergeometric series and quantum physics (see for instance [1, 3, 7]).

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For a function $f(z) = z^n$, we observe that

$$D_q z^n = \frac{1-q^n}{1-q} z^{n-1}.$$

Therefore we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} a_n \frac{1-q^n}{1-q} z^{n-1},$$

where $[n]_q = \frac{1-q^n}{1-q}$. Clearly, as $q \to 1^-$, $[n]_q \to n$.

Denote by $\mathcal{P}_q$ the family of functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots$, analytic in $\mathbb{D}$ and satisfying the condition

$$\left| p(z) - \frac{1}{1-q} \right| \le \frac{1}{1-q},$$

where $q \in (0,1)$ is a fixed real number.

Lemma 1.1 ([2]). *$p \in \mathcal{P}_q$ if and only if $p(z) \prec \frac{1+z}{1-qz}$. This result is sharp for the functions $p(z) = \dfrac{1+\phi(z)}{1-q\phi(z)}$, where ϕ is a Schwarz function.*

A function p analytic in $\mathbb{D}$ with $p(0) = 1$ is said to be in the class $\mathcal{P}_k(q)$, $k \ge 2$, $q \in (0,1)$ if and only if there exists $p_1^{(1)}, p_2^{(2)} \in \mathcal{P}_q$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1^{(1)}(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2^{(2)}(z).$$

For $q \to 1^-$, $\mathcal{P}_k(q) \equiv \mathcal{P}_k$, (see [10]); for $k = 2$, $q \to 1^-$, $\mathcal{P}_k(q) \equiv \mathcal{P}$ is the well known class of functions with positive real part. Also, for $k = 2$, $\mathcal{P}_k(q) \equiv \mathcal{P}_q$ consists of all functions subordinate to $\frac{1+z}{1-qz}$, $z \in \mathbb{D}$.

Definition 1.1. Let f of the form (1) be an element of $\mathcal{A}$. If f satisfies the condition

$$z \frac{D_q f(z)}{f(z)} = p(z), \quad p \in \mathcal{P}_k(q),$$

with $k \ge 2, q \in (0,1)$, then f is called $q-$ starlike function with bounded radius rotation denoted by $\mathcal{R}_k(q)$. This class was introduced and studied by Noor et al. [9].

Definition 1.2. Let f of the form (1) be an element of $\mathcal{A}$. If f satisfies the condition

$$1 + \frac{1}{b}\left(z \frac{D_q f(z)}{f(z)} - 1 \right) = p(z), \quad p \in \mathcal{P}_k(q),$$

with $k \ge 2, q \in (0,1)$, $b \in \mathbb{C} \backslash \{0\}$, then f is called $q-$ starlike function of complex order with bounded radius rotation denoted by $\mathcal{R}_k(q,b)$. When $q \to 1^-$, $b = 1$, the class $\mathcal{R}_k(q,b)$ reduces to $\mathcal{R}_k$ (see [10]). For $k = 2$, $q \to 1^-$, the class $\mathcal{R}_k(q,b)$ reduces to $\mathcal{S}^*(1-b)$ (see [8]). For $k = 2$, $q \to 1^-$, $b = 1$ the class $\mathcal{R}_k(q,b)$ reduces to traditional class of the starlike functions $\mathcal{S}^*$.

We investigate coefficient inequality, distortion theorem and radius of starlikeness for the class $\mathcal{R}_k(q,b)$.

2. Main Results

We first prove coefficient inequality for the class $\mathcal{R}_k(q,b)$. For our main theorem, we need the following lemma.

Lemma 2.1 ([11]). *Let $p(z)=1+p_1z+p_2z^2+\dots$ be an element of $\mathcal{P}_k(q)$, then*

$$|p_n|\leq\frac{k}{2}(1+q).$$

This result is sharp for the functions

$$p(z)=\left(\frac{k}{4}+\frac{1}{2}\right)p_1^{(1)}(z)-\left(\frac{k}{4}-\frac{1}{2}\right)p_2(z)^{(2)},$$

where $p_1^{(1)},p_2^{(2)}\in\mathcal{P}_q$.

Theorem 2.1. *If $f\in\mathcal{R}_k(q,b)$, then*

$$|a_n|\leq\frac{1}{([n]_q-1)!}\prod_{\nu=1}^{n-1}\left(([\nu]_q-1)+\frac{k}{2}|b|(1+q)\right). \tag{2}$$

This inequality is sharp for every $n\geq 2$.

Proof. In view of definition of the class $\mathcal{R}_k(q,b)$ and subordination principle, we can write

$$1+\frac{1}{b}\left(z\frac{D_qf(z)}{f(z)}-1\right)=p(z),$$

where $p\in\mathcal{P}_k(q)$ with $p(0)=1$. Since $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ and $p(z)=1+p_1z+p_2z^2+\dots$, then we have

$$\begin{aligned}&z+[2]_qa_2z^2+[3]_qa_3z^3+[4]_qa_4z^4+\dots\\&=z+(a_2+bp_1)z^2+(a_3+bp_1a_2+bp_2)z^3+\\&(a_4+bp_1a_3+bp_2a_2+bp_3)z^4+\dots.\end{aligned}$$

Comparing the coefficients of z^n on both sides, we obtain

$$[n]_qa_n=a_n+bp_1a_{n-1}+bp_2a_{n-2}+\dots+bp_{n-2}a_2+bp_{n-1}$$

for all integer $n\geq 2$. In view of Lemma 2.1, we get

$$([n]_q-1)|a_n|\leq\frac{k}{2}|b|(1+q)(|a_{n-1}|+\dots+|a_2|+1),$$

or equivalently

$$|a_n|\leq\frac{\frac{k}{2}|b|(1+q)}{[n]_q-1}\sum_{\nu=1}^{n-1}|a_\nu|,\quad |a_1|=1. \tag{3}$$

In order to prove (2), we will use process of iteration. Let $c = \frac{k}{2}|b|(1+q)$ and use our assumption $|a_1| = 1$ in (3), we obtain successively

$$\text{for } n=2, \quad |a_2| \leq \frac{1}{[2]_q - 1} c,$$
$$\text{for } n=3, \quad |a_3| \leq \frac{1}{([3]_q - 1)!} c(([2]_q - 1) + c),$$
$$\text{for } n=4, \quad |a_4| \leq \frac{1}{([4]_q - 1)!} c(([2]_q - 1) + c)(([3]_q - 1) + c).$$

Hence induction shows that for n, we obtain

$$|a_n| \leq \frac{1}{([n]_q - 1)!} c(([2]_q - 1) + c)(([3]_q - 1) + c) \cdots (([n-1]_q - 1) + c).$$

This proves (2).

This inequality is sharp, because extremal function is the solution of the $q-$ differential equation

$$1 + \frac{1}{b}\left(z\frac{D_q f(z)}{f(z)} - 1\right) = \left(\frac{k}{4} + \frac{1}{2}\right)\frac{1+z}{1-qz} - \left(\frac{k}{4} - \frac{1}{2}\right)\frac{1-z}{1+qz}. \qquad \square$$

Corollary 2.1. *Taking $q \to 1^-$ and choosing $k = 2$, $b = 1$ in (2), we get $|a_n| \leq n$ for every $n \geq 2$. This is well known coefficient inequality for starlike functions.*

We now introduce distortion theorem and radius of $q-$ starlikeness for the class $\mathcal{R}_k(q, b)$.

Lemma 2.2 ([9]). *Let $f \in \mathcal{R}_k(q)$. Then for $k \geq 2$ and $q \in (0, 1)$, we have*

$$\left| z\frac{D_q f(z)}{f(z)} - \frac{1 + qr^2}{1 - q^2r^2} \right| \leq \frac{\frac{k}{2}(1+q)r}{1 - q^2r^2}.$$

Theorem 2.2. *If f is an element of $\mathcal{R}_k(q, b)$, then*

$$(4) \quad \Big(rF(k, q, \operatorname{Re} b, |b|, -r)\Big)^{\frac{1-q}{\log q^{-1}}} \leq |f(z)| \leq \Big(rF(k, q, \operatorname{Re} b, |b|, r)\Big)^{\frac{1-q}{\log q^{-1}}},$$

where

$$F(k, q, \operatorname{Re} b, |b|, r) = \frac{\big(1 + qr\big)^{\frac{1+q}{q}(\frac{k}{4}|b| - \frac{1}{2}\operatorname{Re} b)}}{\big(1 - qr\big)^{\frac{1+q}{q}((\frac{k}{4}|b| + \frac{1}{2}\operatorname{Re} b)}}.$$

This bound is sharp.

Proof. In view of Lemma 2.2 and subordination principle, we write

$$\left| 1 + \frac{1}{b}\left(z\frac{D_q f(z)}{f(z)} - 1\right) - \frac{1 + qr^2}{1 - q^2r^2} \right| \leq \frac{\frac{k}{2}(1+q)r}{1 - q^2r^2}.$$

Therefore, after routine calculations, we get

$$\left|z\frac{D_qf(z)}{f(z)}-\frac{1+(b(q+q^2)-q^2)r^2}{1-q^2r^2}\right|\leq\frac{\frac{k}{2}|b|(1+q)r}{1-q^2r^2}. \tag{5}$$

After calculations in (5), we obtain

$$\begin{aligned}\operatorname{Re}\big(z\frac{D_qf(z)}{f(z)}\big)&\leq\frac{1+\frac{k}{2}|b|(1+q)r+((q^2+q)\operatorname{Re}b-q^2)r^2}{(1-qr)(1+qr)},\\ \operatorname{Re}\big(z\frac{D_qf(z)}{f(z)}\big)&\geq\frac{1-\frac{k}{2}|b|(1+q)r+((q^2+q)\operatorname{Re}b-q^2)r^2}{(1-qr)(1+qr)}.\end{aligned} \tag{6}$$

On the other hand, we have

$$\operatorname{Re}\left(z\frac{D_qf(z)}{f(z)}\right)=r\frac{\partial_q}{\partial r}\log|f(z)|. \tag{7}$$

Considering (6) and (7) together, respectively, we get

$$\begin{aligned}\frac{\partial_q}{\partial r}\log|f(z)|&\leq\frac{1}{r}+\frac{(1+q)(\frac{1}{2}\operatorname{Re}b+\frac{k}{4}|b|)}{(1-qr)}-\frac{(1+q)(\frac{1}{2}\operatorname{Re}b-\frac{k}{4}|b|)}{(1+qr)},\\ \frac{\partial_q}{\partial r}\log|f(z)|&\geq\frac{1}{r}+\frac{(1+q)(\frac{1}{2}\operatorname{Re}b-\frac{k}{4}|b|)}{(1-qr)}-\frac{(1+q)(\frac{1}{2}\operatorname{Re}b+\frac{k}{4}|b|)}{(1+qr)}.\end{aligned}$$

Taking $q-$ integral on both sides of the last inequalities, we get (4).

This bound is sharp, because extremal function is the solution of the $q-$ differential equation

$$1+\frac{1}{b}\left(z\frac{D_qf(z)}{f(z)}-1\right)=\left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+z}{1-qz}-\left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-z}{1+qz}. \qquad \square$$

Corollary 2.2. *Taking $q\to 1^-$ and $b=1$ in Theorem 2.2, we get the following well known result:*

$$\frac{z\big(1-z\big)^{(\frac{k}{2}-1)}}{\big(1+z\big)^{(\frac{k}{2}+1)}}\leq|f(z)|\leq\frac{z\big(1+z\big)^{(\frac{k}{2}-1)}}{\big(1-z\big)^{(\frac{k}{2}+1)}}.$$

Let $f\in\mathcal{A}$, then the real number

$$r^*(f)=\sup\left\{r>0,\operatorname{Re}\left(z\frac{D_qf(z)}{f(z)}\right)>0\quad\text{for all}\quad z\in\mathbb{D}\right\}$$

is called the starlikeness of the class $\mathcal{A}$. The second inequality of (6) gives the starlikeness of the class $\mathcal{R}_k(q,b)$ as below:

$$r^*(f)=\frac{k|b|(1+q)-\sqrt{k^2|b|^2(1+q)^2-16((q^2+q)\operatorname{Re}b-q^2)}}{4((q^2+q)\operatorname{Re}b-q^2)}.$$

If $q\to 1^-$, $b=1$, then this radius reduces to $r^*(f)=\frac{k-\sqrt{k^2-4}}{2}$. This is the radius of the class $\mathcal{R}_k$ which was obtained by Pinchuk (see [10]). For $q\to 1^-$, $k=2$, we get the starlikennes of starlike functions of complex order as $r_f^* = \frac{|b| - \sqrt{|b|^2 - 2\operatorname{Re} b + 1}}{2\operatorname{Re} b - 1}$. If $k = 2$, $b = 1$ then the radius of the starlikeness of the class $\mathcal{S}_q^*$ is $r^*(f) = \frac{1}{q}$.

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ASENA ÇETİNKAYA
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES
ISTANBUL KÜLTÜR UNIVERSITY
ISTANBUL
TURKEY
E-mail address: asnfigen@hotmail.com

OYA MERT
DEPARTMENT OF BASIC SCIENCES
ALTINBAŞ UNIVERSITY
ISTANBUL
TURKEY
E-mail address: oya.mert@altinbas.edu.tr