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Different common fixed point theorems of integral type for pairs of subcompatible mappings

Hakima Bouhadjera*

Abstract. In this paper, a general common fixed point theorem for two pairs of subcompatible mappings satisfying integral type implicit relations is obtained in a metric space. Our result improves several results especially the result of Pathak et al. [6]. Also, another common fixed point theorem of Greguš type for four mappings satisfying a contractive condition of integral type in a metric space using the concept of subcompatibility is established which generalizes the result of Djoudi and Aliouche [1] and others. Again a third common fixed point theorem for two pairs of near-contractive subcompatible mappings is given which enlarges the result of Mbarki [5] and references therein.

1. Introduction

Let $(\mathcal{X}, d)$ be a metric space and let f, g be two mappings from $\mathcal{X}$ into itself. f and g commute if $fgx = gfx$ for all $x \in \mathcal{X}$.

This commutativity was weakened in 1982 by Sessa [7] with the notion of weakly commuting mappings. f and g above are weakly commuting if $d(fgx, gfx) \leq d(gx, fx)$ for all x in $\mathcal{X}$.

Later on, Jungck [3] enlarged the class of commuting and weakly commuting mappings by compatible mappings which asserts that the above mappings f and g are compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$.

This concept was further improved by Jungck [4] with the notion of weakly compatible mappings. f and g are weakly compatible if $ft = gt$ for some $t \in \mathcal{X}$ implies that $fgt = gft$.

Recently in 2007, Pathak et al. [6] stated and proved a general common fixed point theorem of integral type for two pairs of weakly compatible mappings satisfying integral type implicit relations in a symmetric space.

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Our aim here is to improve and extend the result of [6] by using the new concept of mappings called subcompatibility which enlarges the concept of weakly compatible mappings.

We introduce the notion of subcompatible mappings as follows: Let f and g be two self-mappings of a metric space $(\mathcal{X}, d)$. f and g are subcompatible if and only if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$ and $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$.

It is clear to see that weakly compatible mappings are subcompatible, however the implication is not reversible.

Example 1.1. Let $\mathcal{X} = [0, \infty)$ with the usual metric d. Define $f,\, g : \mathcal{X} \to \mathcal{X}$ as follows

$$fx = x^2 \text{ and } gx = \begin{cases} x+12, & \text{if } x \in [0,16] \cup (25,\infty)\,, \\ x+240, & \text{if } x \in (16,25]\,. \end{cases}$$

Let $\{x_n\}$ be a sequence in $\mathcal{X}$ defined by $x_n = 4 + \frac{1}{n}$ for $n \in \mathbb{N}^* = \{1, 2, ...\}$. Then, we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_n^2 = 16 = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} (x_n + 12)$$

and

$$\begin{aligned} fgx_n &= f(x_n+12) = (x_n+12)^2 \to 256 \text{ as } n \to \infty, \\ gfx_n &= g(x_n^2) = x_n^2 + 240 \to 256 \text{ as } n \to \infty. \end{aligned}$$

Therefore, $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$. Hence, f and g are subcompatible mappings.

On the other hand, we have $fx = gx$ if and only if $x = 4$ but

$$fg(4) = f(16) = 256 \neq 28 = gf(4) = g(16).$$

Thus, f and g are not weakly compatible.

For our first main result we need the following implicit relations.

2. Implicit relations

Let $\mathbb{R}_+$ be the set of all nonnegative real numbers, Ψ be the family of all $\psi : \mathbb{R}_+ \to \mathbb{R}$ Lebesgue-integrable and summable mappings and Φ be the set of all real continuous functions $\varphi : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(φ_1) for all $u, v \geq 0$, if

(φ_a) $\displaystyle\int_0^{\varphi(u,v,v,u,0,u+v)} \psi(t)\,\mathrm{d}\,t \leq 0$ or

(φ_b) $\displaystyle\int_0^{\varphi(u,v,u,v,u+v,0)} \psi(t)\,\mathrm{d}\,t \leq 0$,

we have $u \leq v$,

(φ_2) $\displaystyle\int_0^{\varphi(u,u,0,0,u,u)} \psi(t)\,\mathrm{d}\,t > 0$, for $u > 0$.

Example 2.1. Let $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\right\}$, where $k \in (0, 1)$ and $\psi(t) = t$. Then φ is continuous and ψ is a Lebesgue-integrable mapping which is summable. We have

(φ_1) Let $u > 0$ and $v \geq 0$. If $u > v$ then

$$\begin{aligned}\varphi(u, v, v, u, 0, u + v) &= \varphi(u, v, u, v, u + v, 0)\\ &= u - k \max\left\{u, v, \frac{u + v}{2}\right\}\\ &= u(1 - k),\end{aligned}$$

then

$$\int_0^{u(1-k)} t\,\mathrm{d}\,t = \frac{1}{2}u^2(1 - k)^2 \leq 0$$

impossible, hence $u \leq v$. If $u = 0$, then $u \leq v$.

(φ_2) $\varphi(u, u, 0, 0, u, u) = u(1 - k)$, so

$$\int_0^{u(1-k)} t\,\mathrm{d}\,t = \frac{1}{2}u^2(1 - k)^2 > 0,$$

for $u > 0$.

Example 2.2. $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = (1 + \alpha t_2)t_1 - \alpha \max\{t_3t_4, t_5t_6\}$ $-\beta \max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\}$, where $\alpha \geq 0$ and $0 < \beta < 1$ and $\psi(t) = 1$.

(φ_1) Let $u > 0$ and $v \geq 0$. Suppose that $u > v$, then

$$\begin{aligned}&\varphi(u, v, v, u, 0, u + v) = \varphi(u, v, u, v, u + v, 0)\\ &= (1 + \alpha v)u - \alpha \max\{uv, 0\} - \beta \max\left\{v, u, \frac{u + v}{2}\right\}\\ &= u(1 - \beta),\end{aligned}$$

then

$$\int_0^{u(1-\beta)} \mathrm{d}\,t = u(1 - \beta) \leq 0,$$

which is impossible. Thus, $u \leq v$. If $u = 0$, then $u \leq v$.

(φ_2) $\varphi(u, u, 0, 0, u, u) = u(1 - \beta)$, then

$$\int_0^{u(1-\beta)} \mathrm{d}\,t = u(1 - \beta) > 0, \quad \text{for all } u > 0.$$

Now, we state and prove our main results. We begin by the first one.

3. Main results

Theorem 3.1. *Let f, g, h and k be four mappings of a metric space $(\mathcal{X}, d)$ into itself such that*

$$\int_0^{\varphi(d(fx,gy),d(hx,ky),d(fx,hx),d(gy,ky),d(ky,fx),d(hx,gy))} \psi(t)\,\mathrm{d}\,t \leq 0, \tag{1}$$

for all x, y in $\mathcal{X}$, where $\varphi \in \Phi$ and $\psi \in \Psi$. Suppose that (f,h) and (g,k) are subcompatible and h and k are continuous, then, f, g, h and k have a unique common fixed point.

Proof. Since the pairs (f,h) and (g,k) are subcompatible, then, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = t$ for some $t \in \mathcal{X}$ and $\lim_{n\to\infty} d(fhx_n, hfx_n) = 0$; $\lim_{n\to\infty} gy_n = \lim_{n\to\infty} ky_n = z$ for some $z \in \mathcal{X}$ and $\lim_{n\to\infty} d(gky_n, kgy_n) = 0$.

First we prove that $z = t$. Indeed, by inequality (1) we get

$$\int_0^{\varphi(d(fx_n,gy_n),d(hx_n,ky_n),d(fx_n,hx_n),d(gy_n,ky_n),d(ky_n,fx_n),d(hx_n,gy_n))} \psi(t)\,\mathrm{d}\,t \leq 0.$$

Since φ is continuous, we obtain at infinity

$$\int_0^{\varphi(d(t,z),d(t,z),0,0,d(z,t),d(t,z))} \psi(t)\,\mathrm{d}\,t \leq 0,$$

which contradicts (φ_2) if $d(t,z) > 0$. Then, $z = t$.

Since h is continuous, then $h^2x_n \to ht$, $hfx_n \to ht$. Also we have

$$d(fhx_n, ht) \leq d(fhx_n, hfx_n) + d(hfx_n, ht).$$

Since f and h are subcompatible, taking the limit as $n \to \infty$ in the above inequality we have $\lim_{n\to\infty} fhx_n = ht$. The use of condition (1) gives

$$\int_0^{\varphi(d(fhx_n,gy_n),d(h^2x_n,ky_n),d(fhx_n,h^2x_n),d(gy_n,ky_n),d(ky_n,fhx_n),d(h^2x_n,gy_n))} \psi(t)\,\mathrm{d}\,t \leq 0.$$

At infinity we obtain

$$\int_0^{\varphi(d(t,z),d(t,z),0,0,d(z,t),d(t,z))} \psi(t)\,\mathrm{d}\,t \leq 0,$$

which contradicts (φ_2). Hence $ht = t$.

Again using (1) we get

$$\int_0^{\varphi(d(ft,gy_n),d(ht,ky_n),d(ft,ht),d(gy_n,ky_n),d(ky_n,ft),d(ht,gy_n))} \psi(t)\,\mathrm{d}\,t \leq 0.$$

Taking the limit as $n \to \infty$, we get

$$\int_0^{\varphi(d(ft,t),0,d(ft,t),0,d(t,ft),0)} \psi(t)\,\mathrm{d}\,t \leq 0,$$

which implies $d(ft,t)=0$ by using condition (φ_b). Thus, $ft=t$.

Now, since k is continuous we have $\lim\limits_{n\to\infty} k^2y_n = \lim\limits_{n\to\infty} kgy_n = kt$. Also we have

$$d(gky_n, kt) \le d(gky_n, kgy_n) + d(kgy_n, kt).$$

Since the pair (g,k) is subcompatible we obtain at infinity $\lim\limits_{n\to\infty} gky_n = kt$. Using condition (1) we have

$$\int_0^{\varphi(d(ft,gky_n),d(ht,k^2y_n),d(ft,ht),d(gky_n,k^2y_n),d(k^2y_n,ft),d(ht,gky_n))} \psi(t)\,\mathrm{d}\,t \le 0.$$

When n tends to infinity, we get

$$\int_0^{\varphi(d(t,kt),d(t,kt),0,0,d(kt,t),d(t,kt))} \psi(t)\,\mathrm{d}\,t \le 0,$$

which contradicts (φ_2) when $d(t,kt)>0$. Hence, $kt=t$.

If $gt \neq t$, using inequality (1) we have

$$\int_0^{\varphi(d(ft,gt),d(ht,kt),d(ft,ht),d(gt,kt),d(kt,ft),d(ht,gt))} \psi(t)\,\mathrm{d}\,t \le 0,$$

i.e.,

$$\int_0^{\varphi(d(t,gt),0,0,d(gt,t),0,d(t,gt))} \psi(t)\,\mathrm{d}\,t \le 0,$$

which implies $d(t,gt)=0$ by using condition (φ_a). Thus, $gt=t$.

For the uniqueness of common fixed point t, let $z \neq t$ be another common fixed point of f, g, h and k. Then using (1) we obtain

$$\int_0^{\varphi(d(ft,gz),d(ht,kz),d(ft,ht),d(gz,kz),d(kz,ft),d(ht,gz))} \psi(t)\,\mathrm{d}\,t \le 0,$$

that is,

$$\int_0^{\varphi(d(t,z),d(t,z),0,0,d(z,t),d(t,z))} \psi(t)\,\mathrm{d}\,t \le 0,$$

which is a contradiction of (φ_2). Therefore $z=t$. □

Corollary 3.1. *Let $(\mathcal{X},d)$ be a metric space and let f and h be two mappings from $\mathcal{X}$ into itself satisfying the condition*

$$\int_0^{\varphi(d(fx,fy),d(hx,hy),d(fx,hx),d(fy,hy),d(hy,fx),d(hx,fy))} \psi(t)\,\mathrm{d}\,t \le 0,$$

for all x, y in $\mathcal{X}$, where $\varphi \in \Phi$ and $\psi \in \Psi$. If h is continuous and the pair (f,h) is subcompatible, then, f and h have a unique common fixed point.

Corollary 3.2. *Let $(\mathcal{X},d)$ be a metric space and let f, g and h be three self-mappings of $\mathcal{X}$ such that*

(i) *h is continuous,*

(ii) *the pairs (f,h) and (g,h) are subcompatible and*

(iii) *the inequality*

$$\int_0^{\varphi(d(fx,gy),d(hx,hy),d(fx,hx),d(gy,hy),d(hy,fx),d(hx,gy))} \psi(t)\,\mathrm{d}\,t \leq 0,$$

holds for all x, y in $\mathcal{X}$, where $\varphi \in \Phi$ and $\psi \in \Psi$, then, f, g and h have a unique common fixed point.

Now, we give a generalization of Theorem 3.1.

Theorem 3.2. *Let h, k and $\{f_n\}_{n\in\mathbb{N}^*}$ be mappings from a metric space $(\mathcal{X}, d)$ into itself such that*

(i) *the pairs (f_n, h) and (f_{n+1}, k) are subcompatible,*
(ii) *the inequality*

$$\int_0^{\varphi(d(f_nx,f_{n+1}y),d(hx,ky),d(f_nx,hx),d(f_{n+1}y,ky),d(ky,f_nx),d(hx,f_{n+1}y)))} \psi(t)\,\mathrm{d}\,t \leq 0$$

holds for all x, y in $\mathcal{X}$, each $n \in \mathbb{N}^*$, $\varphi \in \Phi$ and $\psi \in \Psi$. If h and k are continuous, then, h, k and $\{f_n\}_{n\in\mathbb{N}^*}$ have a unique common fixed point.

Now, let $\mathcal{F}$ be the family of mappings $F : \mathbb{R}_+ \to \mathbb{R}_+$ such that each F is upper semi-continuous and $F(t) < t$ for all $t > 0$ and let Ω be the family of $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that every ω is a Lebesgue-integrable mapping which is summable and $\int_0^\epsilon \omega(t)\,\mathrm{d}\,t > 0$ for each $\epsilon > 0$.

In their paper [1], Djoudi and Aliouche proved a common fixed point theorem of Greguš type for four mappings satisfying a contractive condition of integral type in a metric space using the concept of weak compatibility.

Our objective here is to improve, extend and generalize the result of [1] by using the notion of subcompatibility.

Theorem 3.3. *Let f, g, h and k be mappings from a metric space $(\mathcal{X}, d)$ into itself satisfying inequality*

$$\begin{aligned}(2)\qquad &\left(\int_0^{d(fx,gy)} \omega(t)\,\mathrm{d}\,t\right)^p \\ \leq\ & F\left[a\left(\int_0^{d(hx,ky)} \omega(t)\,\mathrm{d}\,t\right)^p + (1-a)\max\left\{\int_0^{d(fx,hx)} \omega(t)\,\mathrm{d}\,t,\right.\right. \\ & \int_0^{d(gy,ky)} \omega(t)dt, \left(\int_0^{d(fx,hx)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}} \left(\int_0^{d(fx,ky)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}}, \\ & \left.\left.\left(\int_0^{d(hx,gy)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}} \left(\int_0^{d(fx,ky)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}}\right\}^p\right],\end{aligned}$$

for all x, y in $\mathcal{X}$, where $0 < a < 1$, p is an integer such that $p \geq 1$, $F \in \mathcal{F}$ and $\omega \in \Omega$. If h and k are continuous and the pairs (f, h) and (g, k) are subcompatible, then, f, g, h and k have a unique common fixed point.

Proof. Since the pair (f, h) as well as (g, k) is subcompatible, then, there are two sequenses $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} hx_n = \lim_{n\to\infty} fx_n = t$ for some $t \in \mathcal{X}$ and $\lim_{n\to\infty} d(fhx_n, hfx_n) = 0$; $\lim_{n\to\infty} gy_n = \lim_{n\to\infty} ky_n = z$ for some $z \in \mathcal{X}$ and $\lim_{n\to\infty} d(gky_n, kgy_n) = 0$.

First, we prove that $z = t$. If $t \neq z$, using inequality (2) we get

$$\begin{aligned}
&\left(\int_0^{d(fx_n,gy_n)} \omega(t)\,\mathrm{d}\,t\right)^p \\
\leq\ & F\Bigg[a\left(\int_0^{d(hx_n,ky_n)} \omega(t)dt\right)^p + (1-a)\max\Bigg\{\int_0^{d(fx_n,hx_n)} \omega(t)\,\mathrm{d}\,t, \\
&\int_0^{d(gy_n,ky_n)} \omega(t)dt, \left(\int_0^{d(fx_n,hx_n)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}} \left(\int_0^{d(fx_n,ky_n)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}}, \\
&\left(\int_0^{d(hx_n,gy_n)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}} \left(\int_0^{d(fx_n,ky_n)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}}\Bigg\}^p\Bigg].
\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned}
&\left(\int_0^{d(t,z)} \omega(t)\,\mathrm{d}\,t\right)^p \\
\leq\ & F\left[a\left(\int_0^{d(t,z)} \omega(t)\,\mathrm{d}\,t\right)^p + (1-a)\left(\int_0^{d(t,z)} \omega(t)\,\mathrm{d}\,t\right)^p\right] \\
=\ & F\left[\left(\int_0^{d(t,z)} \omega(t)\,\mathrm{d}\,t\right)^p\right] < \left(\int_0^{d(t,z)} \omega(t)\,\mathrm{d}\,t\right)^p,
\end{aligned}$$

which is a contradiction, then $\int_0^{d(t,z)} \omega(t)\,\mathrm{d}\,t = 0$, hence $z = t$.

Since h is continuous, then we have $h^2x_n \to ht$, $hfx_n \to ht$. Also, we have

$$d(fhx_n, ht) \leq d(fhx_n, hfx_n) + d(hfx_n, ht).$$

As f and h are subcompatible, letting n tends to infinity in the above inequality, we obtain $\lim_{n\to\infty} fhx_n = ht$. The use of condition (2) gives

$$\begin{aligned}
&\left(\int_0^{d(fhx_n,gy_n)} \omega(t)\,\mathrm{d}\,t\right)^p \\
\leq\ & F\Bigg[a\left(\int_0^{d(h^2x_n,ky_n)} \omega(t)\,\mathrm{d}\,t\right)^p + (1-a)\max\Bigg\{\int_0^{d(fhx_n,h^2x_n)} \omega(t)\,\mathrm{d}\,t,
\end{aligned}$$

$$\int_0^{d(gy_n,ky_n)} \omega(t)\,\mathrm{d}\,t, \left(\int_0^{d(fhx_n,h^2x_n)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}} \left(\int_0^{d(fhx_n,ky_n)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}},$$
$$\left.\left.\left(\int_0^{d(h^2x_n,gy_n)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}} \left(\int_0^{d(fhx_n,ky_n)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}}\right\}^p\right].$$

We obtain at infinity

$$\begin{aligned}&\left(\int_0^{d(ht,t)} \omega(t)\,\mathrm{d}\,t\right)^p\\ \leq\ & F\left[a\left(\int_0^{d(ht,t)} \omega(t)\,\mathrm{d}\,t\right)^p + (1-a)\left(\int_0^{d(ht,t)} \omega(t)\,\mathrm{d}\,t\right)^p\right]\\ =\ & F\left[\left(\int_0^{d(ht,t)} \omega(t)dt\right)^p\right] < \left(\int_0^{d(ht,t)} \omega(t)\,\mathrm{d}\,t\right)^p,\end{aligned}$$

which is a contradiction, therefore $ht = t$.

Again by inequality (2) we have

$$\begin{aligned}&\left(\int_0^{d(ft,gy_n)} \omega(t)\,\mathrm{d}\,t\right)^p\\ \leq\ & F\left[a\left(\int_0^{d(ht,ky_n)} \omega(t)\,\mathrm{d}\,t\right)^p + (1-a)\max\left\{\int_0^{d(ft,ht)} \omega(t)\,\mathrm{d}\,t,\right.\right.\\ & \int_0^{d(gy_n,ky_n)} \omega(t)\,\mathrm{d}\,t, \left(\int_0^{d(ft,ht)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}} \left(\int_0^{d(ft,ky_n)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}},\\ & \left.\left.\left(\int_0^{d(ht,gy_n)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}} \left(\int_0^{d(ft,ky_n)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}}\right\}^p\right].\end{aligned}$$

At infinity we obtain

$$\begin{aligned}\left(\int_0^{d(ft,t)} \omega(t)\,\mathrm{d}\,t\right)^p &\leq F\left[(1-a)\left(\int_0^{d(ft,t)} \omega(t)\,\mathrm{d}\,t\right)^p\right]\\ &< (1-a)\left(\int_0^{d(ft,t)} \omega(t)\,\mathrm{d}\,t\right)^p\\ &< \left(\int_0^{d(ft,t)} \omega(t)\,\mathrm{d}\,t\right)^p,\end{aligned}$$

which is a contradiction. Hence $ft = t$.

Now, since k is continuous, then, we have $k^2y_n \to kt$ and $kgy_n \to kt$ and

$$d(gky_n, kt) \leq d(gky_n, kgy_n) + d(kgy_n, kt).$$

Since the pair (g, k) is subcompatible, we get at infinity $\lim_{n\to\infty} gky_n = kt$. Using (2) we have

$$\begin{aligned}
&\left(\int_0^{d(ft,gky_n)} \omega(t)\,\mathrm{d}\,t\right)^p \\
\leq\ & F\left[a\left(\int_0^{d(ht,k^2y_n)} \omega(t)\,\mathrm{d}\,t\right)^p + (1-a)\max\left\{\int_0^{d(ft,ht)} \omega(t)\,\mathrm{d}\,t,\right.\right. \\
&\int_0^{d(gky_n,k^2y_n)} \omega(t)\,\mathrm{d}\,t, \left(\int_0^{d(ft,ht)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}} \left(\int_0^{d(ft,k^2y_n)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}}, \\
&\left.\left.\left(\int_0^{d(ht,gky_n)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}} \left(\int_0^{d(ft,k^2y_n)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}}\right\}^p\right].
\end{aligned}$$

We get at infinity

$$\begin{aligned}
&\left(\int_0^{d(t,kt)} \omega(t)\,\mathrm{d}\,t\right)^p \\
\leq\ & F\left[a\left(\int_0^{d(t,kt)} \omega(t)\,\mathrm{d}\,t\right)^p + (1-a)\left(\int_0^{d(t,kt)} \omega(t)\,\mathrm{d}\,t\right)^p\right] \\
=\ & F\left[\left(\int_0^{d(t,kt)} \omega(t)\,\mathrm{d}\,t\right)^p\right] < \left(\int_0^{d(t,kt)} \omega(t)\,\mathrm{d}\,t\right)^p.
\end{aligned}$$

This contradiction implies that $kt = t$.

Suppose that $gt \neq t$, the use of inequality (2) gives

$$\begin{aligned}
&\left(\int_0^{d(ft,gt)} \omega(t)\,\mathrm{d}\,t\right)^p \\
\leq\ & F\left[a\left(\int_0^{d(ht,kt)} \omega(t)\,\mathrm{d}\,t\right)^p + (1-a)\max\left\{\int_0^{d(ft,ht)} \omega(t)\,\mathrm{d}\,t,\right.\right. \\
&\int_0^{d(gt,kt)} \omega(t)\,\mathrm{d}\,t, \left(\int_0^{d(ft,ht)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}} \left(\int_0^{d(ft,kt)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}}, \\
&\left.\left.\left(\int_0^{d(ht,gt)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}} \left(\int_0^{d(ft,kt)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}}\right\}^p\right],
\end{aligned}$$

i.e.,

$$\begin{aligned}\left(\int_0^{d(t,gt)}\omega(t)\,\mathrm{d}\,t\right)^p &\le F\left[(1-a)\left(\int_0^{d(t,gt)}\omega(t)\,\mathrm{d}\,t\right)^p\right]\\ &< (1-a)\left(\int_0^{d(t,gt)}\omega(t)\,\mathrm{d}\,t\right)^p\\ &< \left(\int_0^{d(t,gt)}\omega(t)\,\mathrm{d}\,t\right)^p,\end{aligned}$$

which is a contradiction. Hence $gt = t$. Therefore $t = z$ is a common fixed point of both f, g, h and k.

Suppose that f, g, h and k have another common fixed point $z \neq t$. Then, by inequality (2) we get

$$\begin{aligned}&\left(\int_0^{d(ft,gz)}\omega(t)\,\mathrm{d}\,t\right)^p\\ \le\ & F\left[a\left(\int_0^{d(ht,kz)}\omega(t)\,\mathrm{d}\,t\right)^p + (1-a)\max\left\{\int_0^{d(ft,ht)}\omega(t)\,\mathrm{d}\,t,\right.\right.\\ &\int_0^{d(gz,kz)}\omega(t)\,\mathrm{d}\,t, \left(\int_0^{d(ft,ht)}\omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}}\left(\int_0^{d(ft,kz)}\omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}},\\ &\left.\left.\left(\int_0^{d(ht,gz)}\omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}}\left(\int_0^{d(ft,kz)}\omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}}\right\}^p\right],\end{aligned}$$

that is

$$\begin{aligned}\left(\int_0^{d(t,z)}\omega(t)\,\mathrm{d}\,t\right)^p &\le F\left[\left(\int_0^{d(t,z)}\omega(t)\,\mathrm{d}\,t\right)^p\right]\\ &< \left(\int_0^{d(t,z)}\omega(t)\,\mathrm{d}\,t\right)^p.\end{aligned}$$

This contradiction implies that $z = t$. □

If $f = g$ and $h = k$ in Theorem 3.3, we get the next result:

Corollary 3.3. *Let f and h be two self-mappings of a metric space $(\mathcal{X}, d)$ such that*

$$\begin{aligned}&\left(\int_0^{d(fx,fy)}\omega(t)\,\mathrm{d}\,t\right)^p\\ \le\ & F\left[a\left(\int_0^{d(hx,hy)}\omega(t)\,\mathrm{d}\,t\right)^p + (1-a)\max\left\{\int_0^{d(fx,hx)}\omega(t)\,\mathrm{d}\,t,\right.\right.\end{aligned}$$

$$\int_0^{d(fy,hy)}\omega(t)\,\mathrm{d}\,t, \left(\int_0^{d(fx,hx)}\omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}}\left(\int_0^{d(fx,hy)}\omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}},$$
$$\left.\left.\left(\int_0^{d(hx,fy)}\omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}}\left(\int_0^{d(fx,hy)}\omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}}\right\}^p\right],$$

for all x, y in $\mathcal{X}$, where $0 < a < 1$, p is an integer such that $p \geq 1$, $F \in \mathcal{F}$ and $\omega \in \Omega$. If h is continuous and the pair (f, h) is subcompatible, then, f and h have a unique common fixed point.

If we let in Theorem 3.3 $h = k$, then we get the following corollary:

Corollary 3.4. *Let f, g and h be three self-mappings of a metric space $(\mathcal{X}, d)$ such that*

$$\left(\int_0^{d(fx,gy)}\omega(t)\,\mathrm{d}\,t\right)^p$$
$$\leq F\left[a\left(\int_0^{d(hx,hy)}\omega(t)\,\mathrm{d}\,t\right)^p + (1-a)\max\left\{\int_0^{d(fx,hx)}\omega(t)\,\mathrm{d}\,t,\right.\right.$$
$$\int_0^{d(gy,hy)}\omega(t)\,\mathrm{d}\,t, \left(\int_0^{d(fx,hx)}\omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}}\left(\int_0^{d(fx,hy)}\omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}},$$
$$\left.\left.\left(\int_0^{d(hx,gy)}\omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}}\left(\int_0^{d(fx,hy)}\omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}}\right\}^p\right],$$

for all x, y in $\mathcal{X}$, where $0 < a < 1$, p is an integer such that $p \geq 1$, $F \in \mathcal{F}$ and $\omega \in \Omega$. If h is continuous and the pairs (f, h) and (g, h) are subcompatible, then, f, g and h have a unique common fixed point.

The next result is a generalization of Theorem 3.3.

Theorem 3.4. *Let h, k and $\{f_n\}_{n\in\mathbb{N}^*}$ be self-mappings of a metric space $(\mathcal{X}, d)$ satisfying the inequality*

$$\left(\int_0^{d(f_nx,f_{n+1}y)}\omega(t)\,\mathrm{d}\,t\right)^p$$
$$\leq F\left[a\left(\int_0^{d(hx,ky)}\omega(t)\,\mathrm{d}\,t\right)^p + (1-a)\max\left\{\int_0^{d(f_nx,hx)}\omega(t)\,\mathrm{d}\,t,\right.\right.$$
$$\int_0^{d(f_{n+1}y,ky)}\omega(t)\,\mathrm{d}\,t, \left(\int_0^{d(f_nx,hx)}\omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}}\left(\int_0^{d(f_nx,ky)}\omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}},$$

$$\left(\int_0^{d(hx,f_{n+1}y)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}} \left(\int_0^{d(f_nx,ky)} \omega(t)\,\mathrm{d}\,t\right)^{\frac{1}{2}} \Bigg\}\Bigg)^{p}\Bigg],$$

for all x, y in $\mathcal{X}$, where $0 < a < 1$, p is an integer such that $p \geq 1$, $F \in \mathcal{F}$ and $\omega \in \Omega$. If h and k are continuous and the pairs (f_n, h) and (f_{n+1}, k) are subcompatible, then h, k and $\{f_n\}_{n\in\mathbb{N}^*}$ have a unique common fixed point.

We end our work by establishing another result which improves, extends and generalizes especially the result of [5].

Theorem 3.5. *Let $(\mathcal{X}, d)$ be a metric space, f, g, h and k be mappings from $\mathcal{X}$ into itself and F be an upper semi-continuous function of $[0,\infty)$ into itself such that $F(t) = 0$ if and only if $t = 0$ and satisfying inequality*

$$\begin{aligned}
(3) \qquad & \int_0^{F(d(fx,gy))} \omega(t)\,\mathrm{d}\,t \\
\leq\ & a(d(hx,ky)) \int_0^{F(d(hx,ky))} \omega(t)\,\mathrm{d}\,t \\
& + b(d(hx,ky)) \int_0^{F(d(hx,fx))+F(d(ky,gy))} \omega(t)\,\mathrm{d}\,t \\
& + c(d(hx,ky)) \int_0^{\min\{F(d(hx,gy)),F(d(ky,fx))\}} \omega(t)\,\mathrm{d}\,t,
\end{aligned}$$

for all x, y in $\mathcal{X}$, where $\omega \in \Omega$ and $a, b, c : [0,\infty) \to [0,1)$ are upper semi-continuous and satisfying the condition

$$a(t) + c(t) < 1, \quad t > 0.$$

If the pairs (f,h) and (g,k) are subcompatible and h and k are continuous, then, f, g, h and k have a unique common fixed point.

Proof. Since the pairs (f,h) and (g,k) are subcompatible, then, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = t$ for some $t \in \mathcal{X}$ and $\lim_{n\to\infty} d(fhx_n, hfx_n) = 0$; $\lim_{n\to\infty} gy_n = \lim_{n\to\infty} ky_n = z$ for some $z \in \mathcal{X}$ and $\lim_{n\to\infty} d(gky_n, kgy_n) = 0$.

First, we prove that $z = t$. Suppose that $F(d(t,z)) > 0$, using inequality (3) we get

$$\begin{aligned}
\int_0^{F(d(fx_n,gy_n))} \omega(t)\,\mathrm{d}\,t \ \leq\ & a(d(hx_n,ky_n)) \int_0^{F(d(hx_n,ky_n))} \omega(t)\,\mathrm{d}\,t \\
& + b(d(hx_n,ky_n)) \int_0^{F(d(hx_n,fx_n))+F(d(ky_n,gy_n))} \omega(t)\,\mathrm{d}\,t \\
& + c(d(hx_n,ky_n)) \int_0^{\min\{F(d(hx_n,gy_n)),F(d(ky_n,fx_n))\}} \omega(t)\,\mathrm{d}\,t.
\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$\begin{aligned}\int_0^{F(d(t,z))} \omega(t)\,\mathrm{d}t &\leq [a(d(t,z)) + c(d(t,z))] \int_0^{F(d(t,z))} \omega(t)\,\mathrm{d}t \\ &< \int_0^{F(d(t,z))} \omega(t)\,\mathrm{d}t,\end{aligned}$$

which is a contradiction. Hence $F(d(t,z)) = 0$ which implies that $d(t,z) = 0$, thus $t = z$.

Since h is continuous, then, we have $h^2x_n \to ht$, $hfx_n \to ht$. Also, we have

$$d(fhx_n, ht) \leq d(fhx_n, hfx_n) + d(hfx_n, ht).$$

As f and h are subcompatible, letting n tends to infinity in the above inequality, we obtain $\lim_{n\to\infty} fhx_n = ht$. If $F(d(ht,t)) > 0$, the use of condition (3) gives

$$\begin{aligned}\int_0^{F(d(fhx_n,gy_n))} \omega(t)\,\mathrm{d}t &\leq a(d(h^2x_n, ky_n)) \int_0^{F(d(h^2x_n,ky_n))} \omega(t)\,\mathrm{d}t \\ &+ b(d(h^2x_n, ky_n)) \int_0^{F(d(h^2x_n,fhx_n))+F(d(ky_n,gy_n))} \omega(t)\,\mathrm{d}t \\ &+ c(d(h^2x_n, ky_n)) \int_0^{\min\left\{F(d(h^2x_n,gy_n)),F(d(ky_n,fhx_n))\right\}} \omega(t)\,\mathrm{d}t.\end{aligned}$$

Letting $n \to \infty$ we obtain

$$\begin{aligned}\int_0^{F(d(ht,t))} \omega(t)\,\mathrm{d}t &\leq [a(d(ht,t)) + c(d(ht,t))] \int_0^{F(d(ht,t))} \omega(t)\,\mathrm{d}t \\ &< \int_0^{F(d(ht,t))} \omega(t)\,\mathrm{d}t.\end{aligned}$$

This contradiction implies that $F(d(ht,t)) = 0$ and hence $ht = t$.

Suppose that $F(d(ft,t)) > 0$, using condition (3) we get

$$\begin{aligned}\int_0^{F(d(ft,gy_n))} \omega(t)\,\mathrm{d}t &\leq a(d(ht, ky_n)) \int_0^{F(d(ht,ky_n))} \omega(t)\,\mathrm{d}t \\ &+ b(d(ht, ky_n)) \int_0^{F(d(ht,ft))+F(d(ky_n,gy_n))} \omega(t)\,\mathrm{d}t \\ &+ c(d(ht, ky_n)) \int_0^{\min\{F(d(ht,gy_n)),F(d(ky_n,ft))\}} \omega(t)\,\mathrm{d}t.\end{aligned}$$

We obtain at infinity

$$\int_0^{F(d(ft,t))} \omega(t)\,\mathrm{d}t \leq b(0) \int_0^{F(d(t,ft))} \omega(t)\,\mathrm{d}t < \int_0^{F(d(ft,t))} \omega(t)\,\mathrm{d}t,$$

which is a contradiction, hence $F(d(ft,t)) = 0$ which implies that $ft = t$.

Now, since k is continuous, then, we have $k^2y_n \to kt$, $kgy_n \to kt$ and

$$d(gky_n, kt) \leq d(gky_n, kgy_n) + d(kgy_n, kt).$$

Since the pair (g, k) is subcompatible, we get at infinity $\lim_{n\to\infty} gky_n = kt$. We claim that $kt = t$, if not, then by (3) we have

$$\begin{aligned}
\int_0^{F(d(ft,gky_n))} \omega(t)\,\mathrm{d}\,t &\leq a(d(ht,k^2y_n)) \int_0^{F(d(ht,k^2y_n))} \omega(t)\,\mathrm{d}\,t \\
&+ b(d(ht,k^2y_n)) \int_0^{F(d(ht,ft))+F(d(k^2y_n,gky_n))} \omega(t)\,\mathrm{d}\,t \\
&+ c(d(ht,k^2y_n)) \int_0^{\min\left\{F(d(ht,gky_n)),F(d(k^2y_n,ft))\right\}} \omega(t)\,\mathrm{d}\,t.
\end{aligned}$$

Taking the limit when $n \to \infty$ we have

$$\begin{aligned}
\int_0^{F(d(t,kt))} \omega(t)\,\mathrm{d}\,t &\leq [a(d(t,kt)) + c(d(t,kt))] \int_0^{F(d(t,kt))} \omega(t)\,\mathrm{d}\,t \\
&< \int_0^{F(d(t,kt))} \omega(t)\,\mathrm{d}\,t, \\
\Phi(d(t,kt)) &\leq [a(d(t,kt)) + c(d(t,kt))]\,\Phi(d(t,kt)) \\
&< \Phi(d(t,kt)),
\end{aligned}$$

which is a contradiction, thus $kt = t$.

Suppose that $F(d(t,gt)) > 0$, then the use of inequality (3) yields

$$\begin{aligned}
\int_0^{F(d(t,gt))} \omega(t)\,\mathrm{d}\,t &= \int_0^{F(d(ft,gt))} \omega(t)\,\mathrm{d}\,t \\
&\leq a(d(ht,kt)) \int_0^{F(d(ht,kt))} \omega(t)\,\mathrm{d}\,t \\
&\quad + b(d(ht,kt)) \int_0^{F(d(ht,ft))+F(d(kt,gt))} \omega(t)\,\mathrm{d}\,t \\
&\quad + c(d(ht,kt)) \int_0^{\min\{F(d(ht,gt)),F(d(kt,ft))\}} \omega(t)\,\mathrm{d}\,t \\
&= b(0) \int_0^{F(d(t,gt))} \omega(t)\,\mathrm{d}\,t < \int_0^{F(d(t,gt))} \omega(t)\,\mathrm{d}\,t,
\end{aligned}$$

which is a contradiction, thus $F(d(t,gt)) = 0$ which implies that $d(t,gt) = 0$ i.e. $gt = t$.

Now, assume that there exists another common fixed point z of f, g, h and k such that $z \neq t$. By inequality (3) we obtain

$$\int_0^{F(d(t,z))} \omega(t)\,\mathrm{d}\,t = \int_0^{F(d(ft,gz))} \omega(t)\,\mathrm{d}\,t$$

$$\begin{aligned}
&\leq a(d(ht,kz))\int_0^{F(d(ht,kz))}\omega(t)\,\mathrm{d}\,t \\
&+ b(d(ht,kz))\int_0^{F(d(ht,ft))+F(d(kz,gz))}\omega(t)\,\mathrm{d}\,t \\
&+ c(d(ht,kz))\int_0^{\min\{F(d(ht,gz)),F(d(kz,ft))\}}\omega(t)\,\mathrm{d}\,t \\
&= [a(d(t,z))+c(d(t,z))]\int_0^{F(d(t,z))}\omega(t)\,\mathrm{d}\,t \\
&< \int_0^{F(d(t,z))}\omega(t)\,\mathrm{d}\,t.
\end{aligned}$$

This contradiction implies that $F(d(t,z)) = 0 \Leftrightarrow d(t,z) = 0$, hence $z = t$. □

Remark 3.1. Theorem 3.5 remains valid if we replace inequality (3) by the following one

$$\begin{aligned}
\int_0^{F(d(fx,gy))}\omega(t)\,\mathrm{d}\,t \leq\ & a(d(hx,ky))\int_0^{F(d(hx,ky))}\omega(t)\,\mathrm{d}\,t \\
&+ b(d(hx,ky))\int_0^{\frac{F(d(hx,fx))+F(d(ky,gy))}{2}}\omega(t)\,\mathrm{d}\,t \\
&+ c(d(hx,ky))\int_0^{\frac{F(d(hx,gy))+F(d(ky,fx))}{2}}\omega(t)\,\mathrm{d}\,t.
\end{aligned}$$

Corollary 3.5. *Let f and h be self-mappings of a metric space $(\mathcal{X},d)$. Assume that h is continuous, the pair (f,h) is subcompatible and satisfies the inequality*

$$\begin{aligned}
\int_0^{F(d(fx,fy))}\omega(t)\,\mathrm{d}\,t \leq\ & a(d(hx,hy))\int_0^{F(d(hx,hy))}\omega(t)\,\mathrm{d}\,t \\
&+ b(d(hx,hy))\int_0^{F(d(hx,fx))+F(d(hy,fy))}\omega(t)\,\mathrm{d}\,t \\
&+ c(d(hx,hy))\int_0^{\min\{F(d(hx,fy)),F(d(hy,fx))\}}\omega(t)\,\mathrm{d}\,t,
\end{aligned}$$

for all x, y in $\mathcal{X}$, where F, ω, a, b and c are as in Theorem 3.5. Then, f and h have a unique common fixed point.

Corollary 3.6. *Let f, g, $h : \mathcal{X} \to \mathcal{X}$ be mappings satisfying the following inequality*

$$\int_0^{F(d(fx,gy))}\omega(t)\,\mathrm{d}\,t \leq a(d(hx,hy))\int_0^{F(d(hx,hy))}\omega(t)\,\mathrm{d}\,t$$

$$+\ b(d(hx,hy))\int_0^{F(d(hx,fx))+F(d(hy,gy))}\omega(t)\,\mathrm{d}t$$
$$+\ c(d(hx,hy))\int_0^{\min\{F(d(hx,gy)),F(d(hy,fx))\}}\omega(t)\,\mathrm{d}t,$$

for all x, y in $\mathcal{X}$, where F, ω, a, b and c are as in Theorem 3.5. If h is continuous and the pairs (f,h) and (g,h) are subcompatible, then, f, g and h have a unique common fixed point.

Now, we give a generalization of Theorem 3.5.

Theorem 3.6. *Let $(\mathcal{X},d)$ be a metric space, h, k, $\{f_n\}_{n\in\mathbb{N}^*}$ be mappings from $\mathcal{X}$ into itself and F be an upper semi-continuous function of $[0,\infty)$ into itself such that $F(t)=0$ if and only if $t=0$ and satisfying the inequality*

$$\int_0^{F(d(f_nx,f_{n+1}y))}\omega(t)\,\mathrm{d}t \ \leq\ a(d(hx,ky))\int_0^{F(d(hx,ky))}\omega(t)\,\mathrm{d}t$$
$$+\ b(d(hx,ky))\int_0^{F(d(hx,f_nx))+F(d(ky,f_{n+1}y))}\omega(t)\,\mathrm{d}t$$
$$+\ c(d(hx,ky))\int_0^{\min\{F(d(hx,f_{n+1}y)),F(d(ky,f_nx))\}}\omega(t)\,\mathrm{d}t,$$

for all x, y in $\mathcal{X}$, where $\omega\in\Omega$, a, b, $c:[0,\infty)\to[0,1)$ are upper semi-continuous and satisfying the condition

$$a(t)+c(t)<1,\quad t>0.$$

If the pairs (f_n,h) and (f_{n+1},k) are subcompatible and h and k are continuous, then h, k and $\{f_n\}_{n\in\mathbb{N}^*}$ have a unique common fixed point.

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Hakima Bouhadjera
Laboratory of Applied Mathematics
Badji Mokhtar-Annaba University
P.O. Box 12, 23000 Annaba
Algeria
E-mail address: `b_hakima2000@yahoo.fr`