

# An analytical approach for systems of fractional differential equations by means of the innovative homotopy perturbation method

RAHMAT DARZI AND BAHRAM AGHELI

ABSTRACT. We have applied the new approach of homotopy perturbation method (NAHPM) for partial differential system equations featuring time-fractional derivative. The Caputo-type of fractional derivative is considered in this paper. A combination of NAHPM and multiple fractional power series form has been used the first time to present analytical solution. In order to illustrate the simplicity and ability of the suggested approach, some specific and clear examples have been given. All numerical calculations in this manuscript have been carried out with *Mathematica*.

## 1. INTRODUCTION

In this research work, it has been proposed that the new HPM based on the multiple fractional power series can be engaged to solution of partial differential system equations featuring time-fractional derivative (FPDSEs).

This system equation has frequently appeared in different fields of science and engineering such as; physics, optics, plasma physics, superconductivity and quantum mechanics [1].

There are some more books related to fractional calculus for interested readers [2, 3]. It should be noted that there are no accurate analytical solutions for most fractional differential equations. Consequently, for such equations we have to employ some direct and iterative methods. Researchers have used various methods to solve systems equations in recent years. Some familiar methods are: Adomian's decomposition method [4, 5, 6], homotopy perturbation method [7, 8], homotopy analysis method [9, 10, 11] and so on [12, 13, 14, 15, 16, 17, 18].

This paper is organized: in Section 2, fundamental idea of the new method is presented. We explained convergence of this method in section 3. In

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Section 4, the application of innovative *HPM* to FPDSEs is illustrated, and some numerical examples are presented. And conclusions are drawn in Section 5.

## 2. FUNDAMENTAL IDEA OF THE NEW METHOD

In this part of the paper, we present and define Riemann-Liouville fractional integral and Caputo's fractional derivative that are presented [2]. Then the new approach of homotopy perturbation method (NAHPM) is introduced and explained in detail.

**Definition 2.1.** A real function  $f(x)$ ,  $x > 0$ , is considered to be in the space  $C_\nu$ , ( $\nu \in \mathbb{R}$ ), if there exists a real number  $n (> \nu)$ , so that  $f(x) = x^n f_1(x)$ , where  $f_1(x) \in C[0, \infty)$ , and it is said to be in the space  $C_\nu^k$  if and only if  $f^{(k)} \in C_\nu$ ,  $k \in \mathbb{N}$ .

**Definition 2.2.** The Riemann-Liouville fractional integral operator of order of  $\alpha > 0$ , of a function  $f \in C_\nu$ ,  $\nu \geq -1$ , is given by

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-r)^{\alpha-1} f(r) dr,$$

$$I^\alpha f(x) = I_0^\alpha f(x), \quad I^0 f(x) = f(x).$$

**Definition 2.3.** The Caputo's fractional derivative of  $f$  is defined as

$$D^\alpha f(x) = I^{k-\alpha} D^k f(x) = \frac{1}{\Gamma(k-\alpha)} \int_0^x (x-r)^{k-\alpha-1} f^{(k)}(r) dr, \quad x > 0.$$

where,  $f \in C_{-1}^k$ ,  $k-1 < \alpha \leq k$  and  $k \in \mathbb{N}$ .

**Remark 2.1.** For  $k-1 < \alpha \leq k$ ,  $k \in \mathbb{N}$ ,  $f \in C_\nu^k$ ,  $\nu \geq -1$  and  $x > 0$ , the following properties satisfy

$$(i) \quad D_a^\alpha I_a^\alpha f(x) = f(x),$$

$$(ii) \quad I_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{j=0}^{k-1} f^{(j)}(a^+) \frac{(x-a)^j}{j!}.$$

To describe the fundamental ideas of the NAHPM method for partial differential system equations featuring time-fractional derivative:

$$(1) \quad D_\tau^\mu u_i + N_i(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau, u_1, u_2, \dots, u_k) = h_i(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau),$$

in which  $n-1 < \mu \leq n$  and  $i = 1, \dots, k$ , with the following initial conditions for  $i = 1, 2, \dots, k$  and  $j = 1, 2, \dots, n-1$ :

$$(2) \quad u_i^{(j)}(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau_0) = g_{ij}(\zeta_1, \zeta_2, \dots, \zeta_{k-1}).$$

where  $N_1, \dots, N_k$  are nonlinear operators, which usually depend on the functions  $u_i$  and derivatives,  $D^\mu$  denotes that Caputo fractional and  $h_1, \dots, h_k$ , are inhomogeneous term.

For the solution of (1), by using NAHPM, we make the under homotopy for  $i = 1, 2, \dots, k$ :

$$\begin{aligned} & * (1 - q)(D_\tau^\mu U_i - u_{i0}) \\ & + q (D_\tau^\mu U_i + N(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau, u_1, u_2, \dots, u_k) - h_i) = 0, \end{aligned}$$

or

$$(3) \quad D_\tau^\mu U_i = u_{i0} - p(u_{i0} + N(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau, u_1, u_2, \dots, u_k) - h_i).$$

Using the inverse operator,  $L^{-1} = I_\tau^\mu(\cdot)$  to both sides of (3), then we gain

$$(4) \quad \begin{aligned} U_i(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau_0) &= U_i(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau_0) \\ &+ I_\tau^\mu u_{i0} - p I_\tau^\mu (u_{i0}(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau_0) \\ &+ N_i(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau, u_1, u_2, \dots, u_k) - h_i), \end{aligned}$$

where

$$(5) \quad U_i(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau_0) = \sum_{j=0}^{n-1} g_{ij} \frac{\tau^j}{j!}, \quad i = 1, \dots, k.$$

Now assume we introduce the solution of (4) in the next form

$$(6) \quad U_i = U_{i0} + q U_{i1} + q^2 U_{i2} + \dots$$

where  $U_{ij}$ ,  $i = 1, \dots, k$ ,  $j = 0, 1, 2, 3, \dots$  are functions which should be calculated.

**Definition 2.4.** A series expansion of the next form

$$\sum_{n=0}^{\infty} c_n (\tau - \tau_0)^{n\mu} = c_0 + c_1 (\tau - \tau_0)^\mu + c_2 (\tau - \tau_0)^{2\mu} + \dots$$

for  $0 \leq n-1 < \mu \leq n$ ,  $t \leq \tau_0$ , is called fractional power series around  $\tau = \tau_0$ .

**Definition 2.5.** A series expansion of the below form

$$\sum_{n=0}^{\infty} h_n(\zeta) (\tau - \tau_0)^{n\mu}, \quad 0 \leq n-1 < \mu \leq n, \quad t \leq \tau_0$$

is called multiple fractional power series around  $\tau = \tau_0$ .

Assume that the initial approximation of the solution of relation (1) is in the following form:

$$(7) \quad u_{i0}(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau) = \sum_{j=0}^{\infty} a_{ij}(\zeta_1, \zeta_2, \dots, \zeta_{k-1}) p_j(\tau), \quad i = 1, \dots, k$$

where  $a_{ij}(\zeta)$ ,  $i = 1, \dots, k$ ,  $j = 0, 1, 2, 3, \dots$ , are unfamiliar coefficients, and  $p_j(\tau)$ ,  $j = 0, 1, 2, 3, \dots$ , are particular functions.

It is deserving to consider that if  $h(\zeta, \tau)$ , and  $u_0(\zeta, \tau)$  are analytic around  $\tau = 0$ , then their Taylor series can be written as

$$(8) \quad u_0(\zeta, \tau) = \sum_{k=0}^{\infty} a_k(\zeta) \tau^{k\mu}.$$

With considering (4) and substituting (6) and (7) into that and equating the coefficients of the same power  $q$ , for  $i = 1, \dots, k$ , with

$$\begin{aligned}
 q^0 &: U_{i0}(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau) \\
 &= U_i(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau_0) + \sum_{j=0}^{\infty} a_{ij}(\zeta) I_{\tau}^{\mu}(p_j(\tau)), \\
 q^1 &: U_{i1}(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau) = - \sum_{j=0}^{\infty} a_{ij}(\zeta) I_{\tau}^{\mu}(p_j(\tau)) \\
 &\quad - I_{\tau}^{\mu}(N_i(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau, U_{10}, U_{20}, \dots, U_{k0}) - h_i), \\
 q^2 &: U_{i2}(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau) \\
 &= -I_{\tau}^{\mu}\left(N_i(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau, U_{10}, U_{20}, \dots, U_{k0}, U_{11}, U_{21}, \dots, U_{k1})\right), \\
 &\quad \vdots \\
 q^j &: U_{ij}(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau) = I_{\tau}^{\mu}\left(N_i(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau, \right. \\
 &\quad \left. U_{10}, U_{20}, \dots, U_{k0}, U_{1j-1}, \dots, U_{2j-1}, \dots, U_{kj-1})\right).
 \end{aligned}
 \tag{9}$$

By solving these equations in such a manner that

$$U_{i1}(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau) = 0, i = 1, \dots, k,$$

then relations (9) yield to

$$U_{i2}(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau) = U_{i3}(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau) = \dots = 0.$$

Therefore, the numerical analytical solution may be gained as follows:

$$\begin{aligned}
 U_i(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau) &= U_{i0}(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau) \\
 &= U_i(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau_0) + \sum_{j=0}^{\infty} a_{ij}(\zeta) I_{\tau}^{\mu}(p_j(\tau)), \quad i = 1, \dots, k.
 \end{aligned}
 \tag{10}$$

It should be noted that if  $h_i(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau)$ ,  $u_i(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau)$ , are analytic around  $\tau = \tau_0$ , then Taylor series can be written as

$$\begin{aligned}
 u_{i0}(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau) &= \sum_{j=0}^{\infty} a_{ij}(\zeta_1, \zeta_2, \dots, \zeta_{k-1})(\tau - \tau_0)^{k\mu}, \\
 h_i(\zeta_1, \zeta_2, \dots, \zeta_{k-1}, \tau) &= \sum_{j=0}^{\infty} a_{ij}^*(\zeta_1, \zeta_2, \dots, \zeta_{k-1})(\tau - \tau_0)^{k\mu},
 \end{aligned}$$

can be used in relations (9), where  $a_{ij}(\zeta_1, \zeta_2, \dots, \zeta_{k-1})$  are unknown coefficients which must be computed, and  $a_{ij}^*(\zeta_1, \zeta_2, \dots, \zeta_{k-1})$ ,  $i = 1, \dots, k$ , and  $j = 0, 1, 2, \dots$ , are known ones.

## 3. CONVERGENCE ANALYSIS

A large number of problems can be treated by NAHPM through applying the methodology that has been elaborated in the previous sections.

**Theorem 3.1.** *Presume that  $S$  and  $T$  are Banach spaces and  $\mathcal{A} : S \rightarrow T$  is a contractive nonlinear mapping which is*

$$\forall v, v^* \in S, \quad \|\mathcal{A}(v) - \mathcal{A}(v^*)\| \leq \lambda \|v - v^*\|, \quad 0 < \lambda < 1.$$

*Then due to Banach's fixed point theorem  $\mathcal{A}$  has a unique fixed point  $u$ , which is  $\mathcal{A}(u) = u$ .*

*Assume that the sequence provided by new HPM is stated that*

$$w_k = \mathcal{A}(w_{k-1}), \quad w_{k-1} = \sum_{i=0}^{k-1} v_i, \quad k = 1, 2, \dots,$$

*assume that  $w_0 = v_0 \in B_r(v)$ , where  $B_r(v) = \{v^* \in S; \ \|v^* - v\| < r\}$ , then we have*

- (i)  $w_k \in B_r(v)$ ,
- (ii)  $\lim_{k \rightarrow \infty} w_k = v$ .

*Proof.* (i) By inductive way featuring  $k = 1$

$$\|w_1 - v\| = \|\mathcal{A}(w_0) - \mathcal{A}(v)\| \leq \lambda \|v_0 - v\|.$$

Allow that  $\|w_{k-1} - v\| \leq \lambda^{k-1} \|v_0 - v\|$ , while induction hypothesis, hence

$$\|w_k - v\| = \|\mathcal{A}(w_{k-1}) - \mathcal{A}(v)\| \leq \lambda \|w_{k-1} - v\| \leq \|w_{k-1} - v\| \leq \lambda^k \|v_0 - v\|.$$

So

$$\|w_k - v\| \leq \lambda^k \|v_0 - v\| \leq \lambda^k r < r,$$

in this manner  $w_k \in B_r(v)$ .

(ii) Due to  $\|w_k - v\| \leq \lambda^k \|v_0 - v\|$  and  $\lim_{k \rightarrow \infty} \lambda^k = 0$ ,  $\lim_{k \rightarrow \infty} \|w_k - v\| = 0$ , that is,  $\lim_{k \rightarrow \infty} w_k = v$ .  $\square$

## 4. TEST EXAMPLES

Now, we apply NAHPM based on the multiple fractional power series to solve FPDSEs. All of the plots and computations for this equations have been done with *Mathematica*<sup>®</sup>.

**Example 4.1.** We purpose the following FPDSEs:

$$(11) \quad \begin{cases} D_t^\mu u - v u_{xx} - u v_{xx} = \frac{3e^x t^{2\mu}}{\Gamma(2\mu)} + \frac{2t^\mu \sin(x)}{\Gamma(\mu)}, & 1 < \mu \leq 2, \\ D_t^\mu v - v v_{xx} + u u_{xx} = -\frac{9e^{2x} t^{4\mu}}{\Gamma(2\mu)^2} - \frac{t^{4\mu} \sin^2(x)}{\Gamma(2\mu)^2} + \frac{t^{2\mu} \sin(x)}{\Gamma(2\mu)} + \frac{6e^x t^\mu}{\Gamma(\mu)} \end{cases}$$

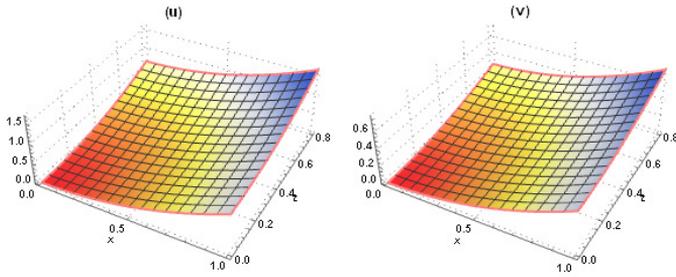


FIGURE 1. Approximate result of test example 4.1.

The estimate answers featuring  $\mu = 2$  acquired for several amounts of  $x$  and  $t$  applying NAHPM, is shown in Table 1. featuring the primary conditions:

$$(12) \quad \begin{aligned} u(x, 0) &= \frac{x^2}{\Gamma(\mu + 1)}, & u_t(x, 0) &= 0, \\ v(x, 0) &= \frac{x^2}{2\Gamma(\mu + 1)}, & v_t(x, 0) &= 0. \end{aligned}$$

TABLE 1. Approximate result of test example 4.1.

$t$	$x$	$u_{NAHPM}=u_{Exact}$	$v_{NAHPM}=v_{Exact}$
0.1	0.50	0.130000	0.065000
	0.75	0.286250	0.143125
	1.00	0.505000	0.252500
0.3	0.50	0.170000	0.085000
	0.75	0.326250	0.163125
	1.00	0.545000	0.272500
0.5	0.50	0.250000	0.125000
	0.75	0.406250	0.203125
	1.00	0.625000	0.312500

Assume

$$\begin{aligned} u_0(x, t) &= \sum_{k=0}^m a_k(x)t^{k\mu}, & U(x, 0) &= g(x), \\ v_0(x, t) &= \sum_{k=0}^m b_k(x)t^{k\mu}, & V(x, 0) &= h(x). \end{aligned}$$

Solving equations (11)-(12) for  $U_1(x, t)$ ,  $V_1(x, t)$  guidance to the following conclusion:

$$(13) \quad U_1(x, t) = t^\mu \left( \frac{1}{\mu\Gamma(\mu)} - \frac{a_0(x)}{\mu\Gamma(\mu)} \right) + t^{2\mu} \left( -\frac{1}{\mu^2\Gamma(\mu)\Gamma(2\mu)} + \frac{a_0(x)}{\mu\Gamma(\mu)\Gamma(2\mu+1)} - \frac{\sqrt{\pi}2^{-2\mu}a_1(x)}{\Gamma(\mu+\frac{1}{2})} + \frac{x^2a_0''(x)}{4\mu^2\Gamma(\mu)\Gamma(2\mu)} + \frac{x^2b_0''(x)}{\mu\Gamma(\mu)\Gamma(2\mu+1)} + \frac{b_0(x)}{\mu^2\Gamma(\mu)\Gamma(2\mu)} \right) + \dots$$

$$(14) \quad V_1(x, t) = t^\mu \left( \frac{1}{2\mu\Gamma(\mu)} - \frac{b_0(x)}{\mu\Gamma(\mu)} \right) + t^{2\mu} \left( \frac{3}{4\mu^2\Gamma(\mu)\Gamma(2\mu)} - \frac{a_0(x)}{\mu^2\Gamma(\mu)\Gamma(2\mu)} + \frac{b_0(x)}{\mu\Gamma(\mu)\Gamma(2\mu+1)} - \frac{x^2a_0''(x)}{\mu\Gamma(\mu)\Gamma(2\mu+1)} + \frac{x^2b_0''(x)}{4\mu^2\Gamma(\mu)\Gamma(2\mu)} + \frac{\sqrt{\pi}2^{-2\mu}b_1(x)}{\Gamma(\mu+\frac{1}{2})} \right) + \dots$$

When  $U_1(x, t)$  and  $V_1(x, t)$  are vanished, the coefficients  $a_k(x)$ ,  $k = 1, 2, 3, \dots$ , will be gained is state as:

$$a_0(x) = 1, \quad a_1(x) = 0, \quad a_2(x) = 0, \dots$$

and

$$b_0(x) = \frac{1}{2}, \quad b_1(x) = 0, \quad b_2(x) = 0, \dots$$

This outcomes that

$$(15) \quad u(x, t) = U_0(x, t) = \frac{t^\mu + x^2}{\Gamma(\mu + 1)},$$

$$(16) \quad v(x, t) = V_0(x, t) = \frac{t^\mu + x^2}{2\Gamma(\mu + 1)}.$$

In Figure 4.1, we may view the estimate answers featuring  $\mu = 2$  which is concluded for several amounts of  $t$  and  $x$  utilizing NAHPM.

**Example 4.2.** Next, we consider the system of coupled fractional equations [19]:

$$(17) \quad \begin{cases} D_t^\mu u - u_{xx} + u u_x + (uv)_x = \frac{(2x^3 + 1)t^{2\mu}}{\Gamma(\mu + 1)} - \frac{2t^\mu}{\Gamma(\mu + 1)} + x^2, & 0 < \mu \leq 1 \\ D_t^\mu v - v_{xx} + v v_x - (uv)_x = \frac{(\frac{1}{x^3} + 1)t^{2\mu}}{\Gamma(\mu + 1)} - \frac{2t^\mu}{x^3\Gamma(\mu + 1)} + \frac{1}{x}, \end{cases}$$

along with:

$$(18) \quad u(x, 0) = 0, \quad v(x, 0) = 0.$$

With considering  $U_1(x, t)$  and  $V_1(x, t)$ , the result shows

(19)

$$\begin{aligned}
 U_1(x, t) = & t^\mu \left( \frac{x^2}{\mu\Gamma(\mu)} - \frac{a_0(x)}{\mu\Gamma(\mu)} \right) + \\
 & t^{2\mu} \left( \frac{\sqrt{\pi}2^{-2\mu}a_0''(x)}{\mu\Gamma(\mu)\Gamma(\mu + \frac{1}{2})} - \frac{\sqrt{\pi}2^{-2\mu}a_1(x)}{\Gamma(\mu + \frac{1}{2})} - \frac{\sqrt{\pi}2^{1-2\mu}}{\mu\Gamma(\mu)\Gamma(\mu + \frac{1}{2})} \right) + \\
 & t^{3\mu} \left( \frac{2\Gamma(2\mu)}{\Gamma(\mu)\Gamma(3\mu + 1)} + \frac{4x^3\Gamma(2\mu)}{\Gamma(\mu)\Gamma(3\mu + 1)} - \frac{a_2(x)\Gamma(2\mu + 1)}{\Gamma(3\mu + 1)} - \right. \\
 & \quad \frac{a_0(x)\Gamma(2\mu + 1)a_0'(x)}{\mu^2\Gamma(\mu)^2\Gamma(3\mu + 1)} - \frac{\Gamma(2\mu + 1)b_0(x)a_0'(x)}{\mu^2\Gamma(\mu)^2\Gamma(3\mu + 1)} - \\
 & \quad \left. \frac{a_0(x)\Gamma(2\mu + 1)b_0'(x)}{\mu^2\Gamma(\mu)^2\Gamma(3\mu + 1)} + \frac{\Gamma(\mu + 1)a_1''(x)}{\Gamma(3\mu + 1)} \right) + \dots
 \end{aligned}$$

(20)

$$\begin{aligned}
 V_1(x, t) = & t^\mu \left( \frac{1}{\mu x\Gamma(\mu)} - \frac{b_0(x)}{\mu\Gamma(\mu)} \right) + \\
 & t^{2\mu} \left( \frac{\sqrt{\pi}2^{-2\mu}b_0''(x)}{\mu\Gamma(\mu)\Gamma(\mu + \frac{1}{2})} - \frac{\sqrt{\pi}2^{-2\mu}b_1(x)}{\Gamma(\mu + \frac{1}{2})} - \frac{\sqrt{\pi}2^{1-2\mu}}{\mu x^3\Gamma(\mu)\Gamma(\mu + \frac{1}{2})} \right) + \\
 & t^{3\mu} \left( \frac{2\Gamma(2\mu)}{\Gamma(\mu)\Gamma(3\mu + 1)} + \frac{2\Gamma(2\mu)}{x^3\Gamma(\mu)\Gamma(3\mu + 1)} - \frac{\Gamma(2\mu + 1)b_2(x)}{\Gamma(3\mu + 1)} - \right. \\
 & \quad \frac{\Gamma(2\mu + 1)b_0(x)a_0'(x)}{\mu^2\Gamma(\mu)^2\Gamma(3\mu + 1)} - \frac{a_0(x)\Gamma(2\mu + 1)b_0'(x)}{\mu^2\Gamma(\mu)^2\Gamma(3\mu + 1)} - \\
 & \quad \left. \frac{\Gamma(2\mu + 1)b_0(x)b_0'(x)}{\mu^2\Gamma(\mu)^2\Gamma(3\mu + 1)} + \frac{\Gamma(\mu + 1)b_1''(x)}{\Gamma(3\mu + 1)} \right) + \dots
 \end{aligned}$$

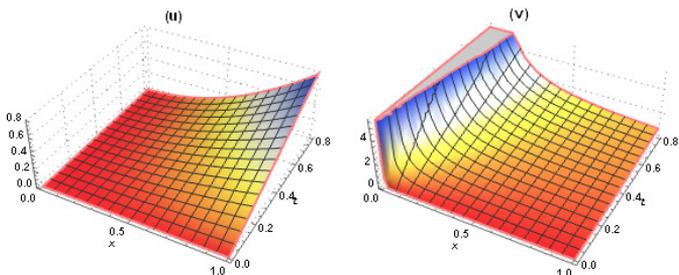


FIGURE 2. The estimate solution for  $\mu = 1$ .

Considering the hypothesis  $U_1(x, t) = 0, V_1(x, t) = 0$ , coefficients  $a_k(x), b_k(x), k = 1, 2, 3, \dots$ , will be determined as follows:

$$a_0(x) = x^2, \quad a_1(x) = 0, \quad a_2(x) = 0, \quad \dots,$$

and

$$b_0(x) = \frac{1}{x}, \quad b_1(x) = 0, \quad b_2(x) = 0, \quad \dots$$

Therefore we provide the solution of (17) which that comes next:

$$(21) \quad u(x, t) = \frac{x^2 t^\mu}{\mu \Gamma(\mu)}, \quad v(x, t) = \frac{t^\mu}{x \Gamma(\mu + 1)}.$$

In Figure 2, we can view the approximate answers featuring  $\mu = 1$ .

In Table 2, we may view the approximate answers featuring  $\mu = 1$ , which is concluded for several amounts of  $t$  and  $x$  applying NAHPM.

TABLE 2. Approximate result of test example 4.2.

$t$	$x$	$u_{NAHPM}=u_{Exact}$	$v_{NAHPM}=v_{Exact}$
0.1	0.50	0.02500	0.200000
	0.75	0.05625	0.133333
	1.00	0.10000	0.100000
0.3	0.50	0.07500	0.600000
	0.75	0.16875	0.400000
	1.00	0.30000	0.300000
0.5	0.50	0.12500	1.000000
	0.75	0.28125	0.666667
	1.00	0.50000	0.500000

**Example 4.3.** We choose the non-linear system of inhomogeneous *FPDEs*:

$$(22) \quad \begin{cases} D_t^\mu u - w_x v_x - \frac{1}{2} w_x u_{xx} = -\frac{2\mu t^\mu}{\Gamma(\mu+1)} - \frac{4x^2}{\Gamma(2\mu)^2} - \frac{2x}{\Gamma(2\mu)^2}, & 0 < \mu \leq 1, \\ D_t^\mu v - w_x u_{xx} = \frac{2\mu t^\mu}{\Gamma(\mu+1)} - \frac{4x}{\Gamma(2\mu)^2}, \\ D_t^\mu w - u_{xx} - v_x w_x = -\frac{2}{\Gamma(2\mu)} - \frac{2\mu t^\mu}{\Gamma(\mu+1)} - \frac{4x^2}{\Gamma(2\mu)^2}, \end{cases}$$

with the primary conditions:

$$(23) \quad u(x, 0) = \frac{1 + x^2}{\Gamma(2\mu)}, \quad v(x, 0) = \frac{x^2 + 2}{\Gamma(2\mu)}, \quad w(x, 0) = \frac{3x^2}{\Gamma(2\mu)}.$$

Assume

$$u_0(x, t) = \sum_{k=0}^m a_k(x) t^{k\mu}, \quad v_0(x, t) = \sum_{k=0}^m b_k(x) t^{k\mu}, \quad w_0(x, t) = \sum_{k=0}^m c_k(x) t^{k\mu},$$

and  $U(x, 0) = g(x)$ ,  $V(x, 0) = h(x)$ ,  $W(x, 0) = k(x)$ . With considering  $U_1(x, t)$ ,  $V_1(x, t)$  and  $W_1(x, t)$ , we have

$$U_1(x, t) = \frac{a_0(x)t^\mu}{\mu\Gamma(\mu)} + t^{2\mu} \left( -\frac{\sqrt{\pi}2^{1-2\mu}}{\Gamma(\mu)\Gamma(\mu + \frac{1}{2})} - \frac{\sqrt{\pi}2^{-2\mu}a_1(x)}{\Gamma(\mu + \frac{1}{2})} + \frac{\sqrt{\pi}2^{1-2\mu}xb'_0(x)}{\mu\Gamma(\mu)\Gamma(2\mu)\Gamma(\mu + \frac{1}{2})} + \frac{\sqrt{\pi}2^{-2\mu}c'_0(x)}{\mu\Gamma(\mu)\Gamma(2\mu)\Gamma(\mu + \frac{1}{2})} + \frac{\sqrt{\pi}2^{-2\mu}xa''_0(x)}{\mu\Gamma(\mu)\Gamma(2\mu)\Gamma(\mu + \frac{1}{2})} + \frac{\sqrt{\pi}2^{1-2\mu}xc'_0(x)}{\mu\Gamma(\mu)\Gamma(2\mu)\Gamma(\mu + \frac{1}{2})} \right) + \dots,$$

$$V_1(x, t) = -\frac{b_0(x)t^\mu}{\mu\Gamma(\mu)} + t^{2\mu} \left( \frac{\sqrt{\pi}2^{1-2\mu}}{\Gamma(\mu)\Gamma(\mu + \frac{1}{2})} - \frac{\sqrt{\pi}2^{-2\mu}b_1(x)}{\Gamma(\mu + \frac{1}{2})} + \frac{\sqrt{\pi}2^{1-2\mu}xa''_0(x)}{\mu\Gamma(\mu)\Gamma(2\mu)\Gamma(\mu + \frac{1}{2})} + \frac{\sqrt{\pi}2^{1-2\mu}c'_0(x)}{\mu\Gamma(\mu)\Gamma(2\mu)\Gamma(\mu + \frac{1}{2})} \right) + \dots,$$

$$W_1(x, t) = -\frac{c_0(x)t^\mu}{\mu\Gamma(\mu)} + \frac{\Gamma(2\mu + 1)^2\Gamma(6\mu + 1)t^{7\mu}b'_2(x)c'_2(x)}{\Gamma(3\mu + 1)^2\Gamma(7\mu + 1)} + t^{2\mu} \left( -\frac{\sqrt{\pi}2^{1-2\mu}}{\Gamma(\mu)\Gamma(\mu + \frac{1}{2})} - \frac{\sqrt{\pi}2^{-2\mu}c_1(x)}{\Gamma(\mu + \frac{1}{2})} + \frac{\sqrt{\pi}2^{1-2\mu}xb'_0(x)}{\mu\Gamma(\mu)\Gamma(2\mu)\Gamma(\mu + \frac{1}{2})} + \frac{\sqrt{\pi}2^{-2\mu}a''_0(x)}{\mu\Gamma(\mu)\Gamma(\mu + \frac{1}{2})} + \frac{\sqrt{\pi}2^{1-2\mu}xc'_0(x)}{\mu\Gamma(\mu)\Gamma(2\mu)\Gamma(\mu + \frac{1}{2})} \right) + \dots.$$

Accordingly, by vanishing of  $U_1(x, t)$ ,  $V_1(x, t)$  and  $W_1(x, t)$  the coefficients  $a_k(x)$ ,  $b_k(x)$  and  $c_k(x)$ ,  $k = 1, 2, \dots$ , will be gained:

$$a_0(x) = 0, \quad a_1(x) = -\frac{2}{\Gamma(\mu)}, \quad a_2(x) = 0, \quad \dots,$$

and

$$b_0(x) = 0, \quad b_1(x) = \frac{2}{\Gamma(\mu)}, \quad b_2(x) = 0, \quad \dots,$$

and

$$c_0(x) = 0, \quad c_1(x) = -\frac{2}{\Gamma(\mu)}, \quad c_2(x) = 0, \quad \dots$$

Therefore we obtain approximate solution of Eq.(22)

$$u(x, t) = U_0(x, t) = \frac{1}{\Gamma(2\mu)} - \frac{\sqrt{\pi}2^{1-2\mu}t^{2\mu}}{\Gamma(\mu)\Gamma(\mu + \frac{1}{2})} + \frac{x^2}{\Gamma(2\mu)},$$

$$v(x, t) = V_0(x, t) = \frac{2}{\Gamma(2\mu)} + \frac{\sqrt{\pi}2^{1-2\mu}t^{2\mu}}{\Gamma(\mu)\Gamma(\mu + \frac{1}{2})} + \frac{x^2}{\Gamma(2\mu)},$$

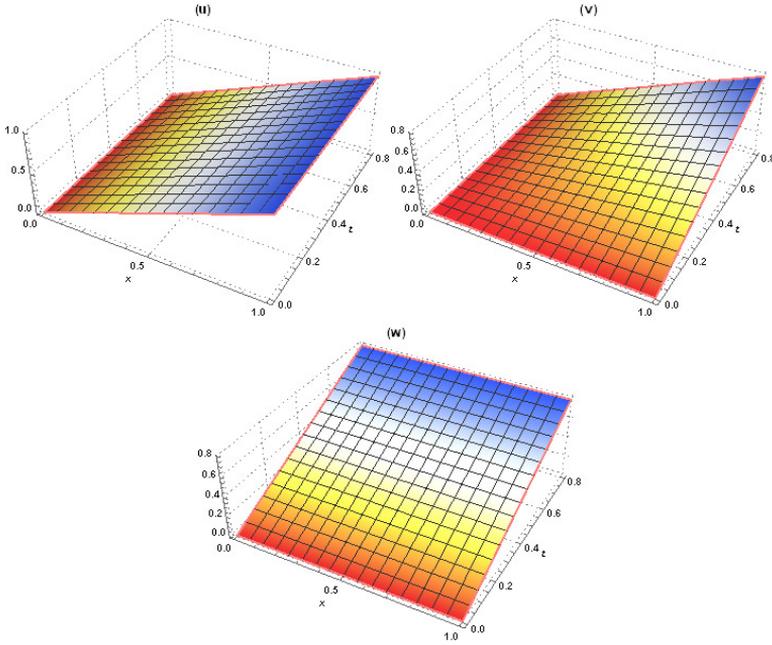


FIGURE 3. The estimate solution for  $\mu = 1$  .

and

$$w(x, t) = W_0(x, t) = \frac{3}{\Gamma(2\mu)} - \frac{\sqrt{\pi}2^{1-2\mu}t^{2\mu}}{\Gamma(\mu)\Gamma(\mu + \frac{1}{2})} + \frac{x^2}{\Gamma(2\mu)}.$$

We can see the analytical answers toward  $\mu = 1$ , in figure 3.

The analytical answers featuring  $\mu = 1$  acquired for several amounts of  $x$ , and  $t$  applying NAHPM, can be seen in Table 3.

TABLE 3. Approximate result of test example 4.3.

$t$	$x$	$u_{NAHPM}=u_{Exact}$	$v_{NAHPM}=v_{Exact}$	$w_{NAHPM}=w_{Exact}$
0.1	0.50	1.2400	2.2600	3.2400
	0.75	1.5525	2.5725	3.5525
	1.00	1.9900	3.0100	3.9900
0.3	0.50	1.1600	2.3400	3.1600
	0.75	1.4725	2.6525	3.4725
	1.00	1.9100	3.0900	3.9100
0.5	0.50	1.0000	2.5000	3.0000
	0.75	1.4725	2.6525	3.4725
	1.00	1.7500	3.2500	3.7500

## 5. CONCLUSION

In this paper, we have successfully applied NAHPM to obtain series solution of partial differential system equations featuring time-fractional derivative. The result indicated that a few iteration of NAHPM will result in some solution.

Finally, it should be added that the suggested approach has the potentials to be applied in solving other similar nonlinear problems in partial differential equations of fractional order.

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**RAHMAT DARZI**

DEPARTMENT OF MATHEMATICS

NEKA BRANCH

ISLAMIC AZAD UNIVERSITY

NEKA

IRAN

*E-mail address:* r.darzi@iauneka.ac.ir**BAHRAM AGHELI**

DEPARTMENT OF MATHEMATICS

QAEMSHAHR BRANCH

ISLAMIC AZAD UNIVERSITY

QAEMSHAHR

IRAN

*E-mail address:* b.agheli@qaemiau.ac.ir