

Hermite-Hadamard type inequalities for (m, M) - Ψ -convex functions when $\Psi = -\ln$

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ABSTRACT. In this paper we establish some Hermite-Hadamard type inequalities for (m, M) - Ψ -convex functions when $\Psi = -\ln$. Applications for power functions and weighted arithmetic mean and geometric mean are also provided.

1. INTRODUCTION

The following integral inequality

$$(1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a)+f(b)}{2},$$

which holds for any convex function $f : [a, b] \rightarrow \mathbb{R}$, is well known in the literature as the Hermite-Hadamard inequality.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, for which we would like to refer the reader to [1]-[4], [15]-[30], the monograph [13] and the references therein.

Assume that the function $\Psi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ (I is an interval) is convex on I and $m \in \mathbb{R}$. We shall say that the function $\Phi : I \rightarrow \mathbb{R}$ is m - Ψ -lower convex if $\Phi - m\Psi$ is a convex function on I . We may introduce (see [6]) the class of functions

$$(2) \quad \mathcal{L}(I, m, \Psi) := \{\Phi : I \rightarrow \mathbb{R} \mid \Phi - m\Psi \text{ is convex on } I\}.$$

Similarly, for $M \in \mathbb{R}$ and Ψ as above, we can introduce the class of M - Ψ -upper convex functions (see [6])

$$(3) \quad \mathcal{U}(I, M, \Psi) := \{\Phi : I \rightarrow \mathbb{R} \mid M\Psi - \Phi \text{ is convex on } I\}.$$

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The intersection of these two classes will be called the class of (m, M) - Ψ -convex functions and will be denoted by [6]

$$(4) \quad \mathcal{B}(I, m, M, \Psi) := \mathcal{L}(I, m, \Psi) \cap \mathcal{U}(I, M, \Psi).$$

Remark 1.1. If $\Psi \in \mathcal{B}(I, m, M, \Psi)$, then $\Phi - m\Psi$ and $M\Psi - \Phi$ are convex and then $(\Phi - m\Psi) + (M\Psi - \Phi)$ is also convex which shows that $(M - m)\Psi$ is convex, implying that $M \geq m$ (as Ψ is assumed not to be the trivial convex function $\Psi(t) = 0, t \in I$).

The above concepts may be introduced in the general case of a convex subset in a real linear space, but we do not consider this extension here.

In [12], S. S. Dragomir and N. M. Ionescu introduced the concept of g -convex dominated functions, for a function $f : I \rightarrow \mathbb{R}$. We recall this, by saying, for a given convex function $g : I \rightarrow \mathbb{R}$, the function $f : I \rightarrow \mathbb{R}$ is g -convex dominated iff $g + f$ and $g - f$ are convex functions on I . In [12], the authors pointed out a number of inequalities for convex dominated functions related to Jensen's, Fuch's, Pečarić's, Barlow-Proschan and Vasić-Mijalković results, etc.

We observe that the concept of g -convex dominated functions can be obtained as a particular case from (m, M) - Ψ -convex functions by choosing $m = -1, M = 1$ and $\Psi = g$.

The following lemma holds [6].

Lemma 1.1. Let $\Psi, \Phi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable functions on $\overset{\circ}{I}$, the interior of I and Ψ is a convex function on $\overset{\circ}{I}$.

(i) For $m \in \mathbb{R}$, the function $\Phi \in \mathcal{L}(\overset{\circ}{I}, m, \Psi)$ if and only if

$$(5) \quad m [\Psi(t) - \Psi(s) - \Psi'(s)(t - s)] \leq \Phi(t) - \Phi(s) - \Phi'(s)(t - s),$$

for all $t, s \in \overset{\circ}{I}$.

(ii) For $M \in \mathbb{R}$, the function $\Phi \in \mathcal{U}(\overset{\circ}{I}, M, \Psi)$ if and only if

$$(6) \quad \Phi(t) - \Phi(s) - \Phi'(s)(t - s) \leq M [\Psi(t) - \Psi(s) - \Psi'(s)(t - s)],$$

for all $t, s \in \overset{\circ}{I}$.

(iii) For $M, m \in \mathbb{R}$ with $M \geq m$, the function $\Phi \in \mathcal{B}(\overset{\circ}{I}, m, M, \Psi)$ if and only if both (5) and (6) hold.

Another elementary fact for twice differentiable functions also holds [6].

Lemma 1.2. Let $\Psi, \Phi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable on $\overset{\circ}{I}$ and Ψ is convex on $\overset{\circ}{I}$.

(i) For $m \in \mathbb{R}$, the function $\Phi \in \mathcal{L}(\overset{\circ}{I}, m, \Psi)$ if and only if

$$(7) \quad m\Psi''(t) \leq \Phi''(t) \quad \text{for all } t \in \overset{\circ}{I}.$$

(ii) For $M \in \mathbb{R}$, the function $\Phi \in \mathcal{U}(\overset{\circ}{I}, M, \Psi)$ if and only if

$$(8) \quad \Phi''(t) \leq M\Psi''(t) \text{ for all } t \in \overset{\circ}{I}.$$

(iii) For $M, m \in \mathbb{R}$ with $M \geq m$, the function $\Phi \in \mathcal{B}(\overset{\circ}{I}, m, M, \Psi)$ if and only if both (7) and (8) hold.

For various inequalities concerning these classes of function, see the survey paper [11].

In what follows, we are considering the class of functions $\mathcal{B}(I, m, M, -\ln)$ for $M, m \in \mathbb{R}$ with $M \geq m$ that is obtained for $\Psi : I \subseteq (0, \infty) \rightarrow \mathbb{R}$, $\Psi(t) = -\ln t$.

If $\Phi : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ is a differentiable function on $\overset{\circ}{I}$ then by Lemma 1.1 we have $\Phi \in \mathcal{B}(I, m, M, -\ln)$ if and only if

$$(9) \quad m \left[\ln s - \ln t - \frac{1}{s}(s-t) \right] \leq \Phi(t) - \Phi(s) - \Phi'(s)(t-s) \\ \leq M \left[\ln s - \ln t - \frac{1}{s}(s-t) \right],$$

for any $s, t \in \overset{\circ}{I}$.

If $\Phi : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ is a twice differentiable function on $\overset{\circ}{I}$ then by Lemma 1.2 we have $\Phi \in \mathcal{B}(I, m, M, -\ln)$ if and only if

$$(10) \quad m \leq t^2\Phi''(t) \leq M,$$

which is a convenient condition to verify in applications.

In this paper we establish some Hermite-Hadamard type inequalities for (m, M) - Ψ -convex functions when $\Psi = -\ln$. Applications for power functions and weighted arithmetic mean and geometric mean are also provided.

2. HERMITE-HADAMARD TYPE INEQUALITIES

In 2002, Barnett, Cerone and Dragomir [5] obtained the following refinement of the Hermite-Hadamard inequality for the convex function $f : [a, b] \rightarrow \mathbb{R}$:

$$(11) \quad f\left(\frac{a+b}{2}\right) \leq \nu f\left(a + \nu\frac{b-a}{2}\right) + (1-\nu)f\left(\frac{a+b}{2} + \nu\frac{b-a}{2}\right) \\ \leq \frac{1}{b-a} \int_a^b f(u) du \\ \leq \frac{1}{2} [f((1-\nu)a + \nu b) + \nu f(a) + (1-\nu)f(b)] \leq \frac{f(a) + f(b)}{2},$$

for all $\nu \in [0, 1]$.

The inequality was also rediscovered in 2010 by A. E. Farissi in [14].

We give a simple proof by following [5].

Applying the Hermite-Hadamard inequality for the convex function f on each subinterval $[a, (1 - \nu)a + \nu b]$, $[(1 - \nu)a + \nu b, b]$, $\nu \in (0, 1)$, we have,

$$\begin{aligned} & f\left(\frac{a + (1 - \nu)a + \nu b}{2}\right) [(1 - \nu)a + \nu b - a] \\ & \leq \int_a^{(1 - \nu)a + \nu b} f(u) du \\ & \leq \frac{f((1 - \nu)a + \nu b) + f(a)}{2} [(1 - \nu)a + \nu b - a] \end{aligned}$$

and

$$\begin{aligned} & f\left(\frac{(1 - \nu)a + \nu b + b}{2}\right) [b - (1 - \nu)a - \nu b] \\ & \leq \int_{(1 - \nu)a + \nu b}^b f(u) du \\ & \leq \frac{f(b) + f((1 - \nu)a + \nu b)}{2} [b - (1 - \nu)a - \nu b], \end{aligned}$$

which are clearly equivalent to

$$(12) \quad \begin{aligned} \nu f\left(a + \nu \frac{b - a}{2}\right) & \leq \frac{1}{b - a} \int_a^{(1 - \nu)a + \nu b} f(u) du \\ & \leq \frac{\nu f((1 - \nu)a + \nu b) + \nu f(a)}{2} \end{aligned}$$

and

$$(13) \quad \begin{aligned} (1 - \nu) f\left(\frac{a + b}{2} + \nu \frac{b - a}{2}\right) & \leq \frac{1}{b - a} \int_{(1 - \nu)a + \nu b}^b f(u) du \\ & \leq \frac{(1 - \nu) f(b) + (1 - \nu) f((1 - \nu)a + \nu b)}{2} \end{aligned}$$

respectively.

Summing (12) and (13), we obtain the second and third inequality in (11).

By the convexity property of f , we obtain

$$\begin{aligned} & \nu f\left(a + \nu \frac{b - a}{2}\right) + (1 - \nu) f\left(\frac{a + b}{2} + \nu \frac{b - a}{2}\right) \\ & \geq f\left[\nu \left(a + \nu \frac{b - a}{2}\right) + (1 - \nu) \left(\frac{a + b}{2} + \nu \frac{b - a}{2}\right)\right] = f\left(\frac{a + b}{2}\right) \end{aligned}$$

and the first inequality in (11) is proved.

The latter inequality in (11) is obvious by the convexity property of f .

Let us recall the following means:

(i) The arithmetic mean

$$A = A(a, b) := \frac{a+b}{2}, \quad a, b \geq 0;$$

(ii) The geometric mean:

$$G = G(a, b) := \sqrt{ab}, \quad a, b \geq 0;$$

(iii) The harmonic mean:

$$H = H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}, \quad a, b \geq 0;$$

(iv) The logarithmic mean:

$$L = L(a, b) := \begin{cases} a, & \text{if } a = b; \\ \frac{b-a}{\ln b - \ln a}, & \text{if } a \neq b; \end{cases} \quad a, b > 0;$$

(v) The identric mean:

$$I := I(a, b) = \begin{cases} a, & \text{if } a = b; \\ \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}}, & \text{if } a \neq b; \end{cases} \quad a, b > 0;$$

(vi) The p -logarithmic mean:

$$L_p = L_p(a, b) := \begin{cases} \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}}, & \text{if } a \neq b; \\ a, & \text{if } a = b; \end{cases}$$

where $p \in \mathbb{R} \setminus \{-1, 0\}$ and $a, b > 0$.

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$.

In particular, we have the inequalities $H \leq G \leq L \leq I \leq A$. We also notice that

$$\frac{1}{b-a} \int_a^b t^p dt = L_p^p(a, b), \quad p \in \mathbb{R} \setminus \{-1, 0\}, \quad \frac{1}{b-a} \int_a^b \frac{dt}{t} = L^{-1}(a, b)$$

and

$$\frac{1}{b-a} \int_a^b \ln t dt = \ln I(a, b).$$

We define the weighted arithmetic and geometric means

$$A_\nu(a, b) := (1-\nu)a + \nu b \quad \text{and} \quad G_\nu(a, b) := a^{1-\nu} b^\nu$$

where $\nu \in [0, 1]$ and $a, b > 0$. If $\nu = \frac{1}{2}$, then we recapture $A(a, b)$ and $G(a, b)$.

Theorem 2.1. Let $M, m \in \mathbb{R}$ with $M > m$ and $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$. Then for any $a, b > 0$ and $\nu \in [0, 1]$ we have

$$(14) \quad \begin{aligned} & \ln \left[\frac{(a + \nu \frac{b-a}{2})^\nu \left(\frac{a+b}{2} + \nu \frac{b-a}{2}\right)^{1-\nu}}{I(a, b)} \right]^m \\ & \leq \frac{1}{b-a} \int_a^b \Phi(u) du - \left[\nu \Phi \left(a + \nu \frac{b-a}{2} \right) + (1-\nu) \Phi \left(\frac{a+b}{2} + \nu \frac{b-a}{2} \right) \right] \\ & \leq \ln \left[\frac{(a + \nu \frac{b-a}{2})^\nu \left(\frac{a+b}{2} + \nu \frac{b-a}{2}\right)^{1-\nu}}{I(a, b)} \right]^M \end{aligned}$$

and

$$(15) \quad \begin{aligned} & \ln \left[\frac{I(a, b)}{\sqrt{A_\nu(a, b) G_{1-\nu}(a, b)}} \right]^m \\ & \leq \frac{1}{2} [\Phi((1-\nu)a + \nu b) + \nu \Phi(a) + (1-\nu)\Phi(b)] - \frac{1}{b-a} \int_a^b \Phi(u) du \\ & \leq \ln \left[\frac{I(a, b)}{\sqrt{A_\nu(a, b) G_{1-\nu}(a, b)}} \right]^M. \end{aligned}$$

Proof. Since $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$, then $\Phi_m := \Phi + m \ln$ is convex and by the second inequality in (11) we have

$$(16) \quad \begin{aligned} & \nu \Phi \left(a + \nu \frac{b-a}{2} \right) + (1-\nu) \Phi \left(\frac{a+b}{2} + \nu \frac{b-a}{2} \right) \\ & + m \ln \left[\left(a + \nu \frac{b-a}{2} \right)^\nu \left(\frac{a+b}{2} + \nu \frac{b-a}{2} \right)^{1-\nu} \right] \\ & \leq \frac{1}{b-a} \int_a^b f(u) du + m \frac{1}{b-a} \int_a^b \ln u du \\ & = \frac{1}{b-a} \int_a^b f(u) du + m \ln I(a, b), \end{aligned}$$

while from the third inequality in (11) we have

$$(17) \quad \begin{aligned} & \frac{1}{b-a} \int_a^b f(u) du + m \frac{1}{b-a} \int_a^b \ln u du \\ & \leq \frac{1}{2} [\Phi((1-\nu)a + \nu b) + \nu \Phi(a) + (1-\nu)\Phi(b)] \\ & + \frac{1}{2} m \ln [A_\nu(a, b) G_{1-\nu}(a, b)], \end{aligned}$$

for any $a, b > 0$ and $\nu \in [0, 1]$.

Since $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$, then also $f_M := -\Phi - M \ln$ is convex and by the second inequality in (11) we have

$$(18) \quad \begin{aligned} & -\nu\Phi\left(a + \nu\frac{b-a}{2}\right) - (1-\nu)\Phi\left(\frac{a+b}{2} + \nu\frac{b-a}{2}\right) \\ & - M \ln \left[\left(a + \nu\frac{b-a}{2}\right)^\nu \left(\frac{a+b}{2} + \nu\frac{b-a}{2}\right)^{1-\nu} \right] \\ & \leq -\frac{1}{b-a} \int_a^b \Phi(u) du - M \ln I(a, b), \end{aligned}$$

while from the third inequality in (11) we have

$$(19) \quad \begin{aligned} & -\frac{1}{b-a} \int_a^b \Phi(u) du - M \ln I(a, b) \\ & \leq -\frac{1}{2} [\Phi((1-\nu)a + \nu b) + \nu\Phi(a) + (1-\nu)\Phi(b)] \\ & \quad - \frac{1}{2} M \ln [A_\nu(a, b) G_{1-\nu}(a, b)], \end{aligned}$$

for any $a, b > 0$ and $\nu \in [0, 1]$.

Making use of (16)-(19) we deduce the desired results (14) and (15). \square

Remark 2.1. If we write the second inequality in (11) for the convex function $-\ln$ we have

$$\ln I(a, b) \leq \ln \left[\left(a + \nu\frac{b-a}{2}\right)^\nu \left(\frac{a+b}{2} + \nu\frac{b-a}{2}\right)^{1-\nu} \right],$$

which implies that

$$I(a, b) \leq \left(a + \nu\frac{b-a}{2}\right)^\nu \left(\frac{a+b}{2} + \nu\frac{b-a}{2}\right)^{1-\nu}$$

showing that

$$\ln \left[\frac{\left(a + \nu\frac{b-a}{2}\right)^\nu \left(\frac{a+b}{2} + \nu\frac{b-a}{2}\right)^{1-\nu}}{I(a, b)} \right] \geq 0.$$

If we use the third inequality in (11) for the convex function $-\ln$ we have

$$\ln \sqrt{A_\nu(a, b) G_{1-\nu}(a, b)} \leq \ln I(a, b),$$

which implies that

$$\sqrt{A_\nu(a, b) G_{1-\nu}(a, b)} \leq I(a, b),$$

showing that

$$\ln \left[\frac{I(a, b)}{\sqrt{A_\nu(a, b) G_{1-\nu}(a, b)}} \right] \geq 0.$$

Corollary 2.1. *With the assumptions of Theorem 2.1 we have*

$$(20) \quad \ln \left[\frac{A(a, b)}{I(a, b)} \right]^m \leq \frac{1}{b-a} \int_a^b \Phi(u) du - \Phi \left(\frac{a+b}{2} \right) \leq \ln \left[\frac{A(a, b)}{I(a, b)} \right]^M$$

and

$$(21) \quad \ln \left[\frac{I(a, b)}{G(a, b)} \right]^m \leq \frac{\Phi(a) + \Phi(b)}{2} - \frac{1}{b-a} \int_a^b \Phi(u) du \leq \ln \left[\frac{I(a, b)}{G(a, b)} \right]^M.$$

The inequality (20) was obtained in 2002 by Dragomir in [7], see also [11, p. 197].

Corollary 2.2. *With the assumptions of Theorem 2.1 we have*

$$(22) \quad \begin{aligned} & \ln \left[\frac{\sqrt{\left(\frac{3a+b}{4}\right) \left(\frac{a+3b}{4}\right)}}{I(a, b)} \right]^m \\ & \leq \frac{1}{b-a} \int_a^b \Phi(u) du - \frac{1}{2} \left[\Phi \left(\frac{3a+b}{4} \right) + \Phi \left(\frac{a+3b}{4} \right) \right] \\ & \leq \ln \left[\frac{\sqrt{\left(\frac{3a+b}{4}\right) \left(\frac{a+3b}{4}\right)}}{I(a, b)} \right]^M \end{aligned}$$

and

$$(23) \quad \begin{aligned} & \ln \left[\frac{I(a, b)}{\sqrt{A(a, b)G(a, b)}} \right]^m \\ & \leq \frac{1}{2} \left[\Phi \left(\frac{a+b}{2} \right) + \frac{\Phi(a) + \Phi(b)}{2} \right] - \frac{1}{b-a} \int_a^b \Phi(u) du \\ & \leq \ln \left[\frac{I(a, b)}{\sqrt{A(a, b)G(a, b)}} \right]^M. \end{aligned}$$

For related results see [8] and [11, p. 197].

Theorem 2.2. *Let $M, m \in \mathbb{R}$ with $M > m$ and $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$. Then for any $a, b > 0$ and $\nu \in [0, 1]$ we have*

$$(24) \quad \begin{aligned} & m \left[\frac{(b-a)^2}{8ab} - \ln \left(\frac{I(a, b)}{G(a, b)} \right) \right] \\ & \leq \frac{1}{8} [\Phi_-(b) - \Phi_+(a)] (b-a) - \left[\frac{\Phi(a) + \Phi(b)}{2} - \frac{1}{b-a} \int_a^b \Phi(u) du \right] \\ & \leq M \left[\frac{(b-a)^2}{8ab} - \ln \left(\frac{I(a, b)}{G(a, b)} \right) \right] \end{aligned}$$

and

$$\begin{aligned}
 (25) \quad & m \left[\frac{(b-a)^2}{8ab} - \ln \left(\frac{A(a,b)}{I(a,b)} \right) \right] \\
 & \leq \frac{1}{8} [\Phi_-(b) - \Phi_+(a)](b-a) - \left[\frac{1}{b-a} \int_a^b \Phi(x) dx - \Phi \left(\frac{a+b}{2} \right) \right] \\
 & \leq M \left[\frac{(b-a)^2}{8ab} - \ln \left(\frac{A(a,b)}{I(a,b)} \right) \right].
 \end{aligned}$$

Proof. The following reverses of the Hermite-Hadamard inequality hold [9] and [10]: Let $h : [a, b] \rightarrow \mathbb{R}$ be a convex function on $[a, b]$. Then

$$\begin{aligned}
 (26) \quad & (0 \leq) \frac{h(a) + h(b)}{2} - \frac{1}{b-a} \int_a^b h(x) dx \\
 & \leq \frac{1}{8} [h_-(b) - h_+(a)](b-a)
 \end{aligned}$$

and

$$\begin{aligned}
 (27) \quad & (0 \leq) \frac{1}{b-a} \int_a^b h(x) dx - h \left(\frac{a+b}{2} \right) \\
 & \leq \frac{1}{8} [h_-(b) - h_+(a)](b-a).
 \end{aligned}$$

The constant $\frac{1}{8}$ is best possible in (26) and (27).

Since $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$, then $\Phi_m := \Phi + m \ln$ is convex and by (26) we have

$$\begin{aligned}
 & \frac{\Phi(a) + \Phi(b)}{2} - \frac{1}{b-a} \int_a^b \Phi(x) dx + m \ln \left(\frac{G(a,b)}{I(a,b)} \right) \\
 & \leq \frac{1}{8} [\Phi_-(b) - \Phi_+(a)](b-a) - \frac{m}{8ab} (b-a)^2,
 \end{aligned}$$

which proves the first inequality in (24).

Since $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$, then also $f_M := -\Phi - M \ln$ is convex and by (26) we have

$$\begin{aligned}
 (0 \leq) & \frac{-\Phi(a) - \Phi(b)}{2} + \frac{1}{b-a} \int_a^b \Phi(x) dx - M \ln \left(\frac{G(a,b)}{I(a,b)} \right) \\
 & \leq \frac{1}{8} [-\Phi_-(b) + \Phi_+(a)](b-a) + \frac{M}{8ab} (b-a)^2,
 \end{aligned}$$

which proves the second inequality in (24).

Further on, since $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$, then $\Phi_m := \Phi + m \ln$ is convex and by (27) we have

$$(0 \leq) \frac{1}{b-a} \int_a^b \Phi(x) dx - \Phi \left(\frac{a+b}{2} \right) - m \ln \left(\frac{A(a,b)}{I(a,b)} \right)$$

$$\leq \frac{1}{8} [\Phi_-(b) - \Phi_+(a)] (b-a) - \frac{m}{8ab} (b-a)^2,$$

which is equivalent to the first inequality in (25).

Since $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$, then also $f_M := -\Phi - M \ln$ is convex and by (27) we have

$$\begin{aligned} (0 \leq) & -\frac{1}{b-a} \int_a^b \Phi(x) dx + \Phi\left(\frac{a+b}{2}\right) + M \ln\left(\frac{A(a,b)}{I(a,b)}\right) \\ & \leq \frac{1}{8} [-\Phi_-(b) + \Phi_+(a)] (b-a) + \frac{M}{8ab} (b-a)^2, \end{aligned}$$

which is equivalent to the second inequality in (25). \square

Remark 2.2. If we write the inequality (26) for the convex function $-\ln$ we have

$$(0 \leq) \ln I(a,b) - \ln G(a,b) \leq \frac{1}{8} \left(\frac{1}{a} - \frac{1}{b}\right) (b-a),$$

which shows that

$$\frac{(b-a)^2}{8ab} - \ln\left(\frac{I(a,b)}{G(a,b)}\right) \geq 0.$$

Also, if we write the inequality (27) for the convex function $-\ln$ we have

$$(0 \leq) \ln A(a,b) - \ln I(a,b) \leq \frac{1}{8} \left(\frac{1}{a} - \frac{1}{b}\right) (b-a),$$

which shows that

$$\frac{(b-a)^2}{8ab} - \ln\left(\frac{A(a,b)}{I(a,b)}\right) \geq 0.$$

3. APPLICATIONS FOR SPECIAL MEANS

For m, M with $M > m > 0$ we define

$$(28) \quad M_p := \begin{cases} M^p & \text{if } p > 1 \\ m^p & \text{if } p < 0 \end{cases} \quad \text{and} \quad m_p := \begin{cases} m^p & \text{if } p > 1 \\ M^p & \text{if } p < 0 \end{cases}.$$

Consider the function $\Phi(t) = t^p$, $p \in (-\infty, 0) \cup (1, \infty)$. This is a convex function and $\Phi''(t) = p(p-1)t^{p-2}$, $t > 0$. Consider $\kappa(t) := t^2\Phi''(t) = p(p-1)t^p$. We observe that

$$\max_{t \in [m, M]} \kappa(t) = p(p-1)M_p \quad \text{and} \quad \min_{t \in [m, M]} \kappa(t) = p(p-1)m_p.$$

By making use of the inequalities (14) and (15) for the function $\Phi(t) = t^p$, $p \in (-\infty, 0) \cup (1, \infty)$, then for any $a, b \in [m, M]$ and $\nu \in [0, 1]$ we have

$$(29) \quad \ln \left[\frac{(a + \nu \frac{b-a}{2})^\nu (\frac{a+b}{2} + \nu \frac{b-a}{2})^{1-\nu}}{I(a,b)} \right]^{p(p-1)m_p}$$

$$\begin{aligned} &\leq L_p^p(a, b) - \left[\nu \left(a + \nu \frac{b-a}{2} \right)^p + (1-\nu) \left(\frac{a+b}{2} + \nu \frac{b-a}{2} \right)^p \right] \\ &\leq \ln \left[\frac{\left(a + \nu \frac{b-a}{2} \right)^\nu \left(\frac{a+b}{2} + \nu \frac{b-a}{2} \right)^{1-\nu}}{I(a, b)} \right]^{p(p-1)M_p} \end{aligned}$$

and

$$\begin{aligned} (30) \quad &\ln \left[\frac{I(a, b)}{\sqrt{A_\nu(a, b) G_{1-\nu}(a, b)}} \right]^{p(p-1)m_p} \\ &\leq \frac{1}{2} \left[((1-\nu)a + \nu b)^p + \nu a^p + (1-\nu)b^p \right] - L_p^p(a, b) \\ &\leq \ln \left[\frac{I(a, b)}{\sqrt{A_\nu(a, b) G_{1-\nu}(a, b)}} \right]^{p(p-1)M_p}, \end{aligned}$$

where m_p and M_p are defined by (28).

If we take $p = 2$ in (30), then we get

$$\begin{aligned} (31) \quad &\ln \left[\frac{I(a, b)}{\sqrt{A_\nu(a, b) G_{1-\nu}(a, b)}} \right]^{2m^2} \\ &\leq \frac{1}{2} \left[((1-\nu)a + \nu b)^2 + \nu a^2 + (1-\nu)b^2 \right] - L_2^2(a, b) \\ &\leq \ln \left[\frac{I(a, b)}{\sqrt{A_\nu(a, b) G_{1-\nu}(a, b)}} \right]^{2M^2}. \end{aligned}$$

Since

$$\begin{aligned} &\frac{1}{2} \left[((1-\nu)a + \nu b)^2 + \nu a^2 + (1-\nu)b^2 \right] - L_2^2(a, b) \\ &= \frac{1}{2} \left[(1-\nu)^2 a^2 + 2\nu(1-\nu)ab + \nu^2 b^2 + \nu a^2 + (1-\nu)b^2 \right] \\ &\quad - \frac{1}{b-a} \int_a^b t^2 dt \\ &= \frac{1}{2} \left[(1-\nu)^2 a^2 + 2\nu(1-\nu)ab + \nu^2 b^2 + \nu a^2 + (1-\nu)b^2 \right] \\ &\quad - \frac{1}{3} (a^2 + ab + b^2) = \frac{1}{6} (b-a)^2 (3\nu^2 - 3\nu + 1), \end{aligned}$$

then by (31) we get

$$\ln \left[\frac{I(a, b)}{\sqrt{A_\nu(a, b) G_{1-\nu}(a, b)}} \right]^{2m^2} \leq \frac{1}{6} (b-a)^2 (3\nu^2 - 3\nu + 1)$$

$$\leq \ln \left[\frac{I(a, b)}{\sqrt{A_\nu(a, b) G_{1-\nu}(a, b)}} \right]^{2M^2},$$

which is equivalent to

$$(32) \quad \exp \left[\left(\frac{1}{12} - \frac{1}{4} \nu (1 - \nu) \right) \frac{(b-a)^2}{M^2} \right] \leq \frac{I(a, b)}{\sqrt{A_\nu(a, b) G_{1-\nu}(a, b)}} \\ \leq \exp \left[\left(\frac{1}{12} - \frac{1}{4} \nu (1 - \nu) \right) \frac{(b-a)^2}{m^2} \right],$$

for any $a, b \in [m, M]$ and $\nu \in [0, 1]$.

If we take in (32) $\nu = 0$, then we get

$$(33) \quad \exp \left[\frac{1}{12} \frac{(b-a)^2}{M^2} \right] \leq \frac{I(a, b)}{G(a, b)} \leq \exp \left[\frac{1}{12} \frac{(b-a)^2}{m^2} \right],$$

for any $a, b \in [m, M]$.

If we take in (32) $\nu = \frac{1}{2}$, then we get

$$(34) \quad \exp \left[\frac{1}{48} \frac{(b-a)^2}{M^2} \right] \leq \frac{I(a, b)}{\sqrt{A(a, b) G(a, b)}} \leq \exp \left[\frac{1}{48} \frac{(b-a)^2}{m^2} \right],$$

for any $a, b \in [m, M]$.

If $a, b > 0$ then by taking $M = \max \{a, b\}$ and $m = \min \{a, b\}$ in (33) and (34) and since

$$\frac{(b-a)^2}{\max^2 \{a, b\}} = \left(\frac{\min \{a, b\}}{\max \{a, b\}} - 1 \right)^2 \quad \text{and} \quad \frac{(b-a)^2}{\min^2 \{a, b\}} = \left(\frac{\max \{a, b\}}{\min \{a, b\}} - 1 \right)^2,$$

for any $a, b > 0$, then we have

$$(35) \quad \exp \left[\frac{1}{12} \left(\frac{\min \{a, b\}}{\max \{a, b\}} - 1 \right)^2 \right] \leq \frac{I(a, b)}{G(a, b)} \\ \leq \exp \left[\frac{1}{12} \left(\frac{\max \{a, b\}}{\min \{a, b\}} - 1 \right)^2 \right]$$

and

$$(36) \quad \exp \left[\frac{1}{48} \left(\frac{\min \{a, b\}}{\max \{a, b\}} - 1 \right)^2 \right] \leq \frac{I(a, b)}{\sqrt{A(a, b) G(a, b)}} \\ \leq \exp \left[\frac{1}{48} \left(\frac{\max \{a, b\}}{\min \{a, b\}} - 1 \right)^2 \right],$$

for any $a, b > 0$.

By making use of the inequalities (24) and (25) for the function $\Phi(t) = t^p$, $p \in (-\infty, 0) \cup (1, \infty)$, then for any $a, b \in [m, M]$ and $\nu \in [0, 1]$ we have

$$\begin{aligned}
 (37) \quad & p(p-1)m_p \left[\frac{(b-a)^2}{8ab} - \ln \left(\frac{I(a,b)}{G(a,b)} \right) \right] \\
 & \leq \frac{1}{8}p(b^{p-1} - a^{p-1})(b-a) - \left[\frac{a^p + b^p}{2} - L_p^p(a,b) \right] \\
 & \leq p(p-1)M_p \left[\frac{(b-a)^2}{8ab} - \ln \left(\frac{I(a,b)}{G(a,b)} \right) \right]
 \end{aligned}$$

and

$$\begin{aligned}
 (38) \quad & p(p-1)m_p \left[\frac{(b-a)^2}{8ab} - \ln \left(\frac{A(a,b)}{I(a,b)} \right) \right] \\
 & \leq \frac{1}{8}p(b^{p-1} - a^{p-1})(b-a) - [L_p^p(a,b) - A^p(a,b)] \\
 & \leq p(p-1)M_p \left[\frac{(b-a)^2}{8ab} - \ln \left(\frac{A(a,b)}{I(a,b)} \right) \right],
 \end{aligned}$$

where m_p and M_p are defined by (28).

The case $p = 2$ provides some simpler inequalities, however the details are left to the interested reader.

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