

Best proximity points of α - β - ψ -proximal contractive mappings in partially ordered complete metric spaces

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ABSTRACT. In this paper, we define α - β - ψ -proximal contractive mappings in partially ordered metric spaces and prove the existence of best proximity points of these maps in partially ordered complete metric spaces. These results extend/generalize the results of Asgari and Badshah, *J. Nonl. Sci. and Appl.*, 2015. We provide illustrative examples in support of our theorems.

1. INTRODUCTION

In recent research in the field of nonlinear functional analysis, many researchers are interested in dealing with non-selfmaps to determine the distance between two nonempty subsets of a metric space (X, d) . Let A and B be two nonempty subsets of a metric space (X, d) and $T : A \rightarrow B$ be a non-selfmapping. Then $d(x, Tx) \geq d(A, B)$, for all $x \in A$, where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. In general, for a non-selfmapping $T : A \rightarrow B$, the fixed point equation $Tx = x$ may not have a solution. In such cases, one intend to find an approximate solution $x \in A$ such that $d(x, Tx) = d(A, B)$. Best approximation theorems and best proximity point theorems are relevant in this regard. For instance, let us consider the well known classical best approximation theorem by Ky Fan [12].

Theorem 1.1 ([12]). *Let A be a non-empty compact convex subset of a normed linear space X and $T : A \rightarrow X$ be a continuous function. Then there exists $x \in A$ such that*

$$\|x - Tx\| = d(Tx, A) = \inf\{\|Tx - a\| : a \in A\}.$$

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On the other hand, though best approximation theorems only ensure the existence of approximate solutions, in this case, such results need not yield optimal solutions. But, best proximity point theorems provide sufficient conditions that assure the existence of approximate solutions which are optimal. In this regard, the best proximity point evolves as a generalization of the best approximation. The authors Basha [5], Choudhury, Maity and Konar, [9, 10] and Kutbi, Chandok and Sintunavarat [16] tried to reduce the problem of finding approximate solutions to that of finding optimal approximate solutions.

In recent years, the existence of best proximity points is an interesting aspect of optimization theory which attracted the attention of many researchers. For example, Abkar and Gableh [2], Basha [6], Caballaro, Harjani and Sadarangani [7], Eldred [11], Gabeleh [13] and Karapinar [15] and the related references cited in these papers, worked in this area.

A best proximity point becomes a fixed point if the underlying mapping is a selfmapping. Therefore, it is concluded that best proximity point theorems generalize fixed point theorems in a natural way. For more works on the existence of best proximity points, we refer [1, 3, 8, 14, 17] and references therein.

Our purpose here is to establish best proximity point theorems in partially ordered metric spaces.

We recall the following notation and definitions. Let (X, d, \preceq) be a partially ordered metric space and let A and B be nonempty subsets of X .

$$A_0 := \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$

$$B_0 := \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

Definition 1.1 ([6]). A mapping $T : A \rightarrow B$ is said to be proximally increasing if for all $u_1, u_2, x_1, x_2 \in A$, $x_1 \preceq x_2$ and $d(u_1, Tx_1) = d(u_2, Tx_2) = d(A, B)$ imply $u_1 \preceq u_2$.

We denote by Ψ a family of functions $\psi : [0, \infty) \rightarrow [0, \infty)$ such that for each $\psi \in \Psi$ and $t > 0$,

- (i) ψ is nondecreasing,
- (ii) $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$, where ψ^n is the n -th iterate of ψ .

Remark 1.1. If $\psi \in \Psi$, then $\psi^n(t) \rightarrow 0$ as $n \rightarrow \infty$, for all $t \geq 0$ and $\psi(t) < t$, for all $t > 0$.

Definition 1.2 ([4]). Let (X, \preceq) be a partially ordered space with metric d . We say that $f : X \rightarrow X$ is an α - β - ψ -contractive mapping if there exist three functions $\alpha, \beta : X \times X \rightarrow [0, \infty)$, $\psi \in \Psi$ such that

$$\alpha(x, y)d(f(x), f(y)) \leq \beta(x, y)\psi(d(x, y)),$$

for all $x, y \in X$ with $x \succeq y$.

Definition 1.3 ([4]). Let $f : X \rightarrow X$, $\alpha, \beta : X \times X \rightarrow [0, \infty)$ and $C_\alpha > 0$, $C_\beta \geq 0$. We say that f is an α - β -admissible mapping, if for all $x, y \in X$ with $x \succeq y$ hold

- (i) $\alpha(x, y) \geq C_\alpha \implies \alpha(fx, fy) \geq C_\alpha$,
- (ii) $\beta(x, y) \leq C_\beta \implies \beta(fx, fy) \leq C_\beta$,
- (iii) $0 \leq \frac{C_\beta}{C_\alpha} \leq 1$.

In 2015, Asgari and Badehian [4], proved fixed point theorems for α - β - ψ -contractive mappings in partially ordered space with complete metric.

Theorem 1.2 ([4]). Let (X, \preceq) be a partially ordered space with complete metric d . Let $f : X \rightarrow X$ be a nondecreasing, α - β - ψ -contractive mapping satisfying the following conditions:

- (i) f is continuous,
- (ii) f is α - β -admissible,
- (iii) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$,
- (iv) there exist $C_\alpha > 0$, $C_\beta \geq 0$ such that $\alpha(fx_0, x_0) \geq C_\alpha$, $\beta(fx_0, x_0) \leq C_\beta$.

Then, f has a fixed point.

In Section 2, we introduce a notion of α - β -proximal admissible mappings and α - β - ψ -proximal contractive mappings that we consider to prove our main results in Section 3. In Section 4, we draw some corollaries and provide examples in support of our main results.

2. PRELIMINARIES

Definition 2.1. Let (X, d, \preceq) be a partially ordered metric space, A, B be two nonempty subsets of X , $\alpha, \beta : A \times A \rightarrow [0, \infty)$ be functions, $C_\alpha > 0$, $C_\beta \geq 0$ be two constants and $T : A \rightarrow B$ be a non-selfmapping. We say that T is an α - β -proximal admissible, if for all $x, y, u, v \in A$, with $x \preceq y$ hold

- (i) $\alpha(x, y) \geq C_\alpha$ and $d(u, Tx) = d(v, Ty) = d(A, B)$ implies $\alpha(u, v) \geq C_\alpha$,
- (ii) $\beta(x, y) \leq C_\beta$ and $d(u, Tx) = d(v, Ty) = d(A, B)$ implies $\beta(u, v) \leq C_\beta$,
- (iii) $0 \leq \frac{C_\beta}{C_\alpha} \leq 1$.

If (i), (ii) and (iii) hold for $\alpha(x, y) = 1 = \beta(x, y)$, for all $x, y \in A$, then we say that T is proximal admissible.

Remark 2.1. If $A = B = X$ in Definition 2.1, then T is an α - β -admissible mapping.

Example 2.1. Let $X = [0, \infty) \times [0, \infty)$ with the Euclidean metric d . Let $A = \{1\} \times [0, \infty)$, $B = \{2\} \times [0, \infty)$. We define a partial order \preceq on X by $(x, y) \preceq (u, v)$ if and only if $x \leq u$ and $y \leq v$, for all $(x, y), (u, v) \in X$, where \preceq is the usual order on $[0, \infty)$. We define $T : A \rightarrow B$ by

$$T(1, x) = (2, 2^x), \quad \text{if } x \in [0, \infty).$$

We also define functions $\alpha, \beta : A \times A \rightarrow [0, \infty)$ by

$$\alpha((1, x), (1, y)) = \begin{cases} \frac{3}{2}, & \text{if } (1, x) \preceq (1, y), \\ 0, & \text{otherwise.} \end{cases}$$

$$\beta((1, x), (1, y)) = \begin{cases} \frac{1}{2}, & \text{if } (1, x) \preceq (1, y), \\ 0, & \text{otherwise.} \end{cases}$$

We choose $C_\alpha = 1$ and $C_\beta = \frac{2}{3}$. Clearly, $0 \leq \frac{C_\beta}{C_\alpha} \leq 1$.

Let $(1, x), (1, y), (1, u)$ and $(1, v) \in A$, such that

- (i) $\alpha((1, x), (1, y)) = \frac{3}{2} \geq 1 = C_\alpha$ and
 $d((1, u), T(1, x)) = d((1, v), T(1, y)) = d(A, B) = 1$,
- (ii) $\beta((1, x), (1, y)) = \frac{1}{2} \leq \frac{2}{3} = C_\beta$ and
 $d((1, u), T(1, x)) = d((1, v), T(1, y)) = d(A, B) = 1$.

From (i) and (ii), we obtain $u = 2^x$ and $v = 2^y$.

Since $(1, x) \preceq (1, y)$, it follows that $(1, u) \preceq (1, v)$, so $\alpha((1, u), (1, v)) = \frac{3}{2} \geq 1 = C_\alpha$ and $\beta((1, u), (1, v)) = \frac{1}{2} \leq \frac{2}{3} = C_\beta$. Hence T is α - β -proximal admissible.

Definition 2.2. Let (X, d, \preceq) be a partially ordered metric space and A and B be nonempty closed subsets of X . We say that $T : A \rightarrow B$ is an α - β - ψ -proximal contractive mapping if there exist functions $\alpha, \beta : A \times A \rightarrow [0, \infty)$, $\psi \in \Psi$ such that for all $x, y, u, v \in A$ with $x \preceq y$ holds:

$$(1) \quad \left. \begin{array}{l} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array} \right\} \implies \alpha(x, y)d(u, v) \leq \beta(x, y)\psi(d(x, y)).$$

If $\alpha(x, y) = 1 = \beta(x, y)$, for all $x, y \in A$ in (1) then we say that T is a ψ -proximal contractive mappings.

Remark 2.2. Here we observe that if $A = B = X$ in Definition 2.2, then T is an α - β - ψ -contractive mapping.

Example 2.2. Let $A = \{0\} \times [0, \infty)$, $B = \{-1\} \times [0, \infty)$ and $X = A \cup B$, with the Euclidean metric d . We define a partial order \preceq on X by $(x, y) \preceq (u, v)$ if and only if $x \leq u$ and $y \leq v$, for all $(x, y), (u, v) \in X$.

Clearly, holds $d(A, B) = 1$. We define $T : A \rightarrow B$ by

$$T(0, x) = \begin{cases} (-1, \frac{x^2}{1+x}), & \text{if } x \in [0, 1], \\ (-1, 2x - \frac{3}{2}), & \text{if } x \in [1, \infty). \end{cases}$$

We also define functions $\alpha, \beta : A \times A \rightarrow [0, \infty)$ by

$$\alpha((0, x), (0, y)) = \begin{cases} \frac{3}{4} & \text{if } x, y \in [0, 1], \text{ with } (0, x) \preceq (0, y), \\ 0 & \text{otherwise.} \end{cases}$$

$$\beta((0, x), (0, y)) = \begin{cases} \frac{2}{3} & \text{if } x, y \in [0, 1], \text{ with } (0, x) \preceq (0, y), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\psi(t) = \frac{7}{8}t$, for all $t \geq 0$. We will show that T is an α - β - ψ -proximal contractive mapping.

Let $(0, x), (0, y), (0, u)$ and $(0, v) \in A$ with $(0, x) \preceq (0, y)$ such that

$$(2) \quad d((0, u), T(0, x)) = d((0, v), T(0, y)) = d(A, B) = 1.$$

If $x, y \in [0, 1]$, with $(0, x) \preceq (0, y)$, then $\alpha((0, x), (0, y)) = \frac{3}{4}$ and $\beta((0, x), (0, y)) = \frac{2}{3}$.

From (2), we obtain $u = \frac{x^2}{1+x} \in [0, \frac{1}{2}]$ and $v = \frac{y^2}{1+y} \in [0, \frac{1}{2}]$.

Therefore,

$$\begin{aligned} \alpha((0, x), (0, y))d((0, u), (0, v)) &= \frac{3}{4}|u - v| \\ &= \frac{3}{4} \left| \frac{x^2}{1+x} - \frac{y^2}{1+y} \right| = \frac{3}{4} \left(\frac{|x - y|(x + y + xy)}{(1+x)(1+y)} \right) \\ &\leq \frac{7}{12}|x - y| = \beta((0, x), (0, y))\psi(d((0, x), (0, y))). \end{aligned}$$

For the other possible cases, the inequality (1) holds trivially. Hence T is an α - β - ψ -proximal contractive mapping.

Here we observe that the inequality (1) fails to hold for any $\psi \in \Psi$, if $\alpha((0, x), (0, y)) = \beta((0, x), (0, y)) = 1$ such that $x, y \in [1, \infty)$ with $(0, x) \preceq (0, y)$ and $x \neq y$. Let $(0, x), (0, y), (0, u)$ and $(0, v) \in A$, with $(0, x) \preceq (0, y)$ and $x, y \in [1, \infty)$, such that

$$(3) \quad d((0, u), T(0, x)) = d((0, v), T(0, y)) = d(A, B) = 1.$$

From (3), we get $u = 2x - \frac{3}{2}$ and $v = 2y - \frac{3}{2}$.

Therefore,

$$d((0, u), (0, y)) = 2|x - y| \not\leq \psi(|x - y|) = \psi(d((0, x), (0, y))).$$

This shows the importance of α and β in the inequality (1).

In the following, we prove our main results.

3. MAIN RESULTS

Theorem 3.1. *Let (X, d, \preceq) be a partially ordered complete metric space. Let A, B be non-empty closed subsets of X with A_0 is nonempty and closed. Let $T : A \rightarrow B$ be a proximally increasing non-selfmapping such that the following conditions hold:*

- (i) T is continuous,
- (ii) T is an α - β - ψ -proximal contractive mapping,
- (iii) T is an α - β -proximal admissible,
- (iv) $T(A_0) \subseteq B_0$,
- (v) there exist elements $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A, B)$,
- (vi) there exist $C_\alpha > 0, C_\beta \geq 0$ such that $\alpha(x_0, x_1) \geq C_\alpha, \beta(x_0, x_1) \leq C_\beta$.

Then T has a best proximity point in A_0 .

Proof. By condition (v), there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and

$$(4) \quad d(x_1, Tx_0) = d(A, B).$$

Since $T(A_0) \subseteq B_0$, we have $Tx_1 \in B_0$ and hence there exists an element $x_2 \in A$ such that

$$(5) \quad d(x_2, Tx_1) = d(A, B).$$

Since T is proximally increasing on A , from (4) and (5), we have $x_1 \preceq x_2$. On continuing this process, we get a sequence $\{x_n\}$ in A_0 such that

$$(6) \quad \left. \begin{aligned} d(x_n, Tx_{n-1}) &= d(A, B) \\ d(x_{n+1}, Tx_n) &= d(A, B) \end{aligned} \right\}$$

satisfying

$$x_1 \preceq x_2 \preceq x_3 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots, \quad \text{for } n = 1, 2, 3, \dots$$

If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is the best proximity point of T and hence the conclusion of the theorem follows.

Now, we assume with out loss of generality that any two consecutive elements of $\{x_n\}$ are distinct.

From condition (iii), condition (vi) and (6), the following holds:

$$(7) \quad \left. \begin{aligned} \alpha(x_0, x_1) &\geq C_\alpha \text{ and } \beta(x_0, x_1) \leq C_\beta \\ d(x_1, Tx_0) &= d(A, B) \\ d(x_2, Tx_1) &= d(A, B) \end{aligned} \right\} \implies \\ \alpha(x_1, x_2) \geq C_\alpha \quad \text{and} \quad \beta(x_1, x_2) \leq C_\beta.$$

Since T is an α - β - ψ -proximal contractive mapping and by considering (7), we have

$$\begin{aligned} C_\alpha d(x_1, x_2) &\leq \alpha(x_0, x_1) d(x_1, x_2) \\ &\leq \beta(x_0, x_1) \psi(d(x_0, x_1)) \\ &\leq C_\beta \psi(d(x_0, x_1)), \end{aligned}$$

therefore

$$(8) \quad d(x_1, x_2) \leq \frac{C_\beta}{C_\alpha} \psi(d(x_0, x_1)) \leq \psi(d(x_0, x_1)).$$

Again by condition (iii), condition (vi), (6) and (7), we have

$$(9) \quad \left. \begin{aligned} \alpha(x_1, x_2) &\geq C_\alpha, \beta(x_1, x_2) \leq C_\beta \\ d(x_2, Tx_1) &= d(A, B) \\ d(x_3, Tx_2) &= d(A, B) \end{aligned} \right\} \implies \\ \alpha(x_2, x_3) \geq C_\alpha \text{ and } \beta(x_2, x_3) \leq C_\beta.$$

Therefore, by considering (9) and by the fact that T is an α - β - ψ -proximal contractive mapping, we have

$$\begin{aligned} C_\alpha d(x_2, x_3) &\leq \alpha(x_1, x_2) d(x_2, x_3) \\ &\leq \beta(x_1, x_2) \psi(d(x_1, x_2)) \\ &\leq C_\beta \psi(d(x_1, x_2)), \end{aligned}$$

and hence

$$d(x_2, x_3) \leq \frac{C_\beta}{C_\alpha} \psi(d(x_1, x_2)) \leq \psi(d(x_1, x_2)) \leq \psi^2(d(x_0, x_1)).$$

On continuing this process, we obtain

$$\left. \begin{aligned} \alpha(x_{n-1}, x_n) &\geq C_\alpha, \quad \beta(x_{n-1}, x_n) \leq C_\beta \\ d(x_n, Tx_{n-1}) &= d(A, B) \\ d(x_{n+1}, Tx_n) &= d(A, B) \end{aligned} \right\} \implies \\ \alpha(x_n, x_{n+1}) \geq C_\alpha \quad \text{and} \quad \beta(x_n, x_{n+1}) \leq C_\beta,$$

for $n = 1, 2, 3, \dots$ and

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)).$$

Since $\psi \in \Psi$, we have $\psi^n(d(x_0, x_1)) \rightarrow 0$ as $n \rightarrow \infty$.

Now, we show that $\{x_n\}$ is a Cauchy sequence. We fix $\epsilon > 0$ and choose $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^{\infty} \psi^n(d(x_0, x_1)) < \epsilon$. Let $m, n \in \mathbb{N}$ with $m > n > n_0$.

Therefore by applying triangle inequality, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m) \\ &\leq \psi^n(d(x_0, x_1)) + \psi^{n+1}(d(x_0, x_1)) + \dots + \psi^{m-1}(d(x_0, x_1)) \\ &= \sum_{n=n_0}^{m-1} \psi^n(d(x_0, x_1)) \leq \sum_{n=n_0}^{\infty} \psi^n(d(x_0, x_1)) < \epsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence. Since A_0 is a closed subset of a complete metric space and hence it is complete, there exists $x \in A_0$ such that $x_n \rightarrow x$. Since T is continuous, by letting $n \rightarrow \infty$ in (6), we obtain $d(x, Tx) = d(A, B)$. Hence x is a best proximity point of T . \square

If we drop the continuity assumption from Theorem 3.1, we obtain the following result.

Theorem 3.2. *Let (X, d, \preceq) be a partially ordered complete metric space. Let A, B be non-empty closed subsets of X with A_0 is nonempty and closed. Let $T : A \rightarrow B$ be proximally increasing non-selfmapping such that the following conditions hold:*

- (i) T is an α - β - ψ -proximal contractive mapping and T is an α - β -proximal admissible;

- (ii) $T(A_0) \subseteq B_0$;
- (iii) there exist elements $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A, B)$;
- (iv) there exist $C_\alpha > 0$, $C_\beta \geq 0$ such that $\alpha(x_0, x_1) \geq C_\alpha$, $\beta(x_0, x_1) \leq C_\beta$;
- (v) if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq C_\alpha$, $\beta(x_n, x_{n+1}) \leq C_\beta$, for all $n \in \mathbb{N}$, $x_n \rightarrow x$ as $n \rightarrow \infty$, then $\alpha(x_n, x) \geq C_\alpha$ and $\beta(x_n, x) \leq C_\beta$;
- (vi) if $\{x_n\}$ is a nondecreasing sequence in A such that $x_n \rightarrow x$ as $n \rightarrow \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then T has a best proximity point in A_0 .

Proof. From the proof of Theorem 3.1, we have the sequence $\{x_n\}$ is Cauchy and $x_n \rightarrow x \in A_0$. Since $T(A_0) \subseteq B_0$, then $T(x) \in B_0$ and hence there exists $z \in A$ such that

$$(10) \quad d(z, Tx) = d(A, B).$$

In the proof of Theorem 3.1, we obtained that $\{x_n\}$ is a nondecreasing sequence satisfying $\alpha(x_n, x_{n+1}) \geq C_\alpha$ and $\beta(x_n, x_{n+1}) \leq C_\beta$. Therefore, by condition (v), it follows that $\alpha(x_n, x) \geq C_\alpha$ and $\beta(x_n, x) \leq C_\beta$, and condition (vi), we have $x_n \preceq x$ for $n \in \mathbb{N}$.

We now claim that $z = x$. Since $d(x_{n+1}, Tx_n) = d(A, B)$, by combining this equation with (10) and by the fact that T is an α - β - ψ -proximal contractive mapping, we have

$$\begin{aligned} C_\alpha d(x_{n+1}, z) &\leq \alpha(x_n, x) d(x_{n+1}, z) \\ &\leq \beta(x_n, x) \psi(d(x_n, x)) \\ &\leq C_\beta \psi(d(x_n, x)), \end{aligned}$$

and therefore

$$d(x_{n+1}, z) \leq \frac{C_\beta}{C_\alpha} \psi(d(x_n, x)) \leq \psi(d(x_n, x)).$$

Since $\psi \in \Psi$, we get

$$d(x_{n+1}, z) < d(x_n, x).$$

If $n \rightarrow \infty$, we obtain $x = z$. Hence x is a best proximity point of T . \square

Lemma 3.1. *In addition to the hypotheses of Theorem 3.1 (Theorem 3.2), if x is a best proximity point of T with $x \preceq u$, $\alpha(x, u) \geq C_\alpha$ and $\beta(x, u) \leq C_\beta$ for some $u \in A_0$, then there exists a sequence $\{u_n\} \subseteq A_0$ such that $d(u_n, Tu_{n-1}) = d(A, B)$, $x \preceq u_n$, for $n = 1, 2, 3, \dots$ and $u_n \rightarrow x$ as $n \rightarrow \infty$.*

Proof. Let x be a best proximity point of T , i.e.,

$$(11) \quad d(x, Tx) = d(A, B).$$

Let $u \in A_0$ such that $x \preceq u$. We set $u_0 = u$. Since $T(A_0) \subseteq B_0$ and $u = u_0 \in A_0$, we have $Tu_0 \in B_0$. Hence there exists $u_1 \in A$ such that

$$(12) \quad d(u_1, Tu_0) = d(A, B).$$

From (12), by the definition of A_0 and B_0 , we have $u_1 \in A_0$. Since T is proximally increasing on A_0 , from $x \preceq u = u_0$, (11) and (12), we have $x \preceq u_1$.

On continuing this process we can construct a sequence $\{u_n\}$ in A_0 such that

$$(13) \quad d(u_n, Tu_{n-1}) = d(A, B),$$

satisfying

$$(14) \quad x \preceq u_n, \quad \text{for } n = 1, 2, 3, \dots$$

By assumption, from (11) and (12) we have

$$(15) \quad \left. \begin{array}{l} \alpha(x, u_0) \geq C_\alpha \text{ and } \beta(x, u_0) \leq C_\beta \\ d(x, Tx) = d(A, B) \\ d(u_1, Tu_0) = d(A, B) \\ \alpha(x, u_1) \geq C_\alpha \text{ and } \beta(x, u_1) \leq C_\beta. \end{array} \right\} \implies$$

Since T is an α - β - ψ -proximal contractive mapping and by considering (15), we have

$$\begin{aligned} C_\alpha d(x, u_1) &\leq \alpha(x, u_0) d(x, u_1) \\ &\leq \beta(x, u_0) \psi(d(x, u_0)) \\ &\leq C_\beta \psi(d(x, u_0)), \end{aligned}$$

and it follows that

$$d(x, u_1) \leq \frac{C_\beta}{C_\alpha} \psi(d(x, u_0)) \leq \psi(d(x, u_0)) = \psi(d(x, u)).$$

From (11), (13), (14) and (15) we have

$$(16) \quad \left. \begin{array}{l} \alpha(x, u_1) \geq C_\alpha \text{ and } \beta(x, u_1) \leq C_\beta \\ d(x, Tx) = d(A, B) \\ d(u_2, Tu_1) = d(A, B) \\ \alpha(x, u_2) \geq C_\alpha \text{ and } \beta(x, u_2) \leq C_\beta. \end{array} \right\} \implies$$

Since T is an α - β - ψ -proximal contractive mapping and by considering (16), we have

$$\begin{aligned} C_\alpha d(x, u_2) &\leq \alpha(x, u_1) d(x, u_2) \\ &\leq \beta(x, u_1) \psi(d(x, u_1)) \\ &\leq C_\beta \psi(d(x, u_1)). \end{aligned}$$

Now, it follows that

$$\begin{aligned} d(x, u_2) &\leq \frac{C_\beta}{C_\alpha} \psi(d(x, u_1)) \\ &\leq \psi(d(x, u_1)) \\ &\leq \psi^2(d(x, u_0)) \\ &= \psi^2(d(x, u)). \end{aligned}$$

On continuing this process, we obtain

$$d(x, u_n) \leq \psi^n(d(x, u)) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

i.e., $u_n \rightarrow x$ as $n \rightarrow \infty$. □

Theorem 3.3. *Suppose that all the hypotheses of Theorem 3.1 (Theorem 3.2) are satisfied. We Assume the following hypothesis.*

Condition (H): There exists $u \in A_0$ such that for every $x, y \in A_0$ with $x \preceq u, y \preceq u$,

$$(17) \quad \begin{cases} \alpha(x, u) \geq C_\alpha & \text{and } \beta(x, u) \leq C_\beta, \\ \alpha(y, u) \geq C_\alpha & \text{and } \beta(y, u) \leq C_\beta. \end{cases}$$

Then T has a unique best proximity point in A_0 .

Proof. By the proof of Theorem 3.1 (Theorem 3.2), the set of best proximity points of T is nonempty. Let x, y be two best proximity points of T in A_0 . By our assumption, we have there exists $u \in A_0$ such that $x \preceq u, y \preceq u, \alpha(x, u) \geq C_\alpha$ and $\beta(x, u) \leq C_\beta$ and $\alpha(y, u) \geq C_\alpha$ and $\beta(y, u) \leq C_\beta$. Now by applying Lemma 3.1, it follows that there exists a sequence $\{u_n\} \subseteq A_0$ such that $u_n \rightarrow x$ and $u_n \rightarrow y$ as $n \rightarrow \infty$. Hence by the uniqueness of limits we have $x = y$. □

4. COROLLARIES AND EXAMPLES

Corollary 4.1. *Let (X, d, \preceq) be a partially ordered complete metric space. Let A, B be non-empty closed subsets of X with A_0 is nonempty and closed. Let $T : A \rightarrow B$ be proximally increasing non-selfmapping such that for all $x, y, u, v \in A$ with $x \preceq y$ hold:*

- (i) $d(u, Tx) = d(v, Ty) = d(A, B)$ implies that $d(u, v) \leq kd(x, y)$, for some $k \in [0, 1)$,
- (ii) T is continuous and $T(A_0) \subseteq B_0$,
- (iii) T is ψ -proximal contractive mapping and T is proximal admissible,
- (iv) there exist elements $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A, B)$,

Then T has a best proximity point in A_0 .

Proof. Follows by choosing $\psi(t) = kt, t \geq 0$, and $\alpha(x, y) = \beta(x, y) = 1$, for all x, y in A , with $C_\alpha = C_\beta = 1$ in Theorem 3.1. □

If the continuity assumption is removed from Corollary 4.1, we have the following result.

Corollary 4.2. *Let (X, d, \preceq) be a partially ordered complete metric space. Let A, B be non-empty closed subsets of X with A_0 is nonempty and closed. Let $T : A \rightarrow B$ be proximally increasing non-selfmapping such that for all $x, y, u, v \in A$ with $x \preceq y$ hold:*

- (i) $d(u, Tx) = d(v, Ty) = d(A, B)$ implies that $d(u, v) \leq kd(x, y)$, for some $k \in [0, 1)$,
- (ii) $T(A_0) \subseteq B_0$,
- (iii) T is ψ -proximal contractive mapping and T is proximal admissible,
- (iv) if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq C_\alpha$, $\beta(x_n, x_{n+1}) \leq C_\beta$ for all $n \in \mathbb{N}$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, then $\alpha(x_n, x) \geq C_\alpha$ and $\beta(x_n, x) \leq C_\beta$,
- (v) if $\{x_n\}$ is a nondecreasing sequence in A such that $x_n \rightarrow x$ as $n \rightarrow \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then T has a best proximity point in A_0 .

Proof. By choosing $\psi(t) = kt, k \in [0, 1)$ and $\alpha(x, y) = \beta(x, y) = 1$ for all x, y in A with $C_\alpha = C_\beta = 1$ in Theorem 3.2, it follows that all the hypothesis of Theorem 3.2 holds so that the conclusion of this corollary holds. \square

Remark 4.1. If $A = B = X$ in Theorem 3.1, then Theorem 1.2 follows as a corollary to Theorem 3.1.

The following example is in support of Theorem 3.1.

Example 4.1. Let $X = \mathbb{R}^2$, with an Euclidean metric d . We define a partial order \preceq on X by $(x, y) \preceq (u, v)$ if and only if $x \geq u$ and $y \geq v$, for all $(x, y), (u, v) \in X$, where \geq is the usual order on \mathbb{R} .

Let $A = \{-1\} \times [0, 4] = A_0$, $B = \{1\} \times [0, 4] = B_0$. Clearly $d(A, B) = 2$. We define $T : A \rightarrow B$ by

$$T(-1, x) = \begin{cases} (1, \frac{x}{4}), & \text{if } x \in [0, 1], \\ (1, \frac{5}{4}x - 1), & \text{if } x \in [1, 4]. \end{cases}$$

Clearly T is continuous, proximally increasing on A and $T(A_0) \subseteq B_0$.

Now, we define functions $\alpha, \beta : A \times A \rightarrow [0, \infty)$ by

$$\alpha((-1, x), (-1, y)) = \begin{cases} \frac{3}{4}, & \text{if } x, y \in [0, 1] \text{ with } (-1, x) \preceq (-1, y), \\ 0, & \text{otherwise.} \end{cases}$$

$$\beta((-1, x), (-1, y)) = \begin{cases} \frac{1}{2}, & \text{if } x, y \in [0, 1] \text{ with } (-1, x) \preceq (-1, y), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. We now show that T is an α - β - ψ -proximal contractive mapping.

Now, let $(-1, x), (-1, y), (-1, u)$ and $(-1, v) \in A$ with $(-1, x) \preceq (-1, y)$ such that

$$(18) \quad d((-1, u), T(-1, x)) = d((-1, v), T(-1, y)) = d(A, B) = 2.$$

We consider the case for $x, y \in [0, 1]$ with $(-1, x) \leq (-1, y)$.

Then $\alpha((-1, x), (-1, y)) = \frac{3}{4}$ and $\beta((-1, x), (-1, y)) = \frac{1}{2}$. In this case from (18), we obtain $u = \frac{x}{4}$ and $v = \frac{y}{4} \in [0, \frac{1}{4}]$. Therefore

$$\begin{aligned} & \alpha((-1, x), (-1, y))d((-1, u), (-1, v)) \\ &= \frac{3}{4}|u - v| \leq |u - v| = \frac{1}{2} \left(\frac{|x - y|}{2} \right) \\ &= \beta((-1, x), (-1, y))\psi(d((-1, x), (-1, y))). \end{aligned}$$

The inequality (1) holds trivially for the other possible cases. Hence T is an α - β - ψ -proximal contractive mapping.

We now show that T is an α - β -proximal admissible. For this purpose, we choose $C_\alpha = \frac{2}{3}$ and $C_\beta = \frac{1}{2}$. Clearly $0 \leq \frac{C_\beta}{C_\alpha} \leq 1$.

Let $(-1, x), (-1, y), (-1, u)$ and $(-1, v) \in A$ with $(-1, x) \preceq (-1, y)$ with $x, y \in [0, 1]$ such that:

- (i) $\alpha((-1, x), (-1, y)) = \frac{3}{4} \geq \frac{2}{3} = C_\alpha$ and $d((-1, u), T(-1, x)) = d((-1, v), T(-1, y)) = d(A, B) = 2$,
- (ii) $\beta((-1, x), (-1, y)) = \frac{1}{2} \leq \frac{1}{2} = C_\beta$ and $d((-1, u), T(-1, x)) = d((-1, v), T(-1, y)) = d(A, B) = 2$.

From (i) and (ii), we obtain $u = \frac{x}{4}$ and $v = \frac{y}{4}$. Since $x \leq y$, it follows that $u \leq v$. Therefore

$$\begin{aligned} \alpha((-1, u), (-1, v)) &= \frac{3}{4} \geq \frac{2}{3} = C_\alpha, \\ \beta((-1, u), (-1, v)) &= \frac{1}{2} \leq \frac{1}{2} = C_\beta. \end{aligned}$$

Hence T is an α - β -proximal admissible.

We choose $x_0 = (-1, 1)$, $x_1 = (-1, \frac{1}{4})$ in A_0 . Then $x_0 \preceq x_1$ and $d((-1, \frac{1}{4}), T(-1, 1)) = 2 = d(A, B)$. Also,

$$\begin{aligned} \alpha((-1, 1), (-1, \frac{1}{4})) &= \frac{3}{4} \geq \frac{2}{3} = C_\alpha \\ \beta((-1, 1), (-1, \frac{1}{4})) &= \frac{1}{2} \leq \frac{1}{2} = C_\beta. \end{aligned}$$

Hence all the hypotheses of Theorem 3.1 are satisfied and $(-1, 0)$ and $(-1, 4)$ are two best proximity points of T .

Here we observe that Condition (H) of Theorem 3.3 fails to hold: if $u = (-1, x_0) \in A_0$ with $x_0 > 0$, we choose $x = (-1, 0), y = (-1, 4)$ so that $x = (-1, 0) \not\preceq (-1, x_0) = u$; if $u = (-1, 0)$ then $x \preceq u$ and $y = (-1, 4) \preceq (-1, 0) = u$, $\alpha(x, u) = \frac{3}{4} > \frac{2}{3} = C_\alpha$, $\beta(x, u) = \frac{1}{2} = C_\beta$ and $\beta(y, u) = 0 < \frac{1}{2} = C_\beta$. But $\alpha(y, u) = 0 \not\geq \frac{2}{3} = C_\alpha$.

Hence Condition (H) of Theorem 3.3 fails to hold.

The following example is in support of Theorem 3.3 in which T is not continuous.

Example 4.2. Let $X = [0, 2] \times [0, 2]$ with an Euclidean metric d .

We define a partial order on X by $(x, y) \preceq (u, v)$ if and only if $x \geq u$ and $y \geq v$, for all $(x, y), (u, v) \in X$, where \geq is the usual order on \mathbb{R} .

Let $A = \{0\} \times [0, 2] = A_0$, $B = \{1\} \times [0, 2] = B_0$.

We define $T : A \rightarrow B$ by

$$T(0, x) = \begin{cases} (1, \frac{x^2}{1+2x}), & \text{if } x \in [0, 1], \\ (1, \frac{2}{3}x), & \text{if } x \in (1, 2]. \end{cases}$$

Clearly, $d(A, B) = 1$, T is not continuous, T is proximally increasing on A and $T(A_0) \subseteq B_0$.

Now, we define functions $\alpha, \beta : A \times A \rightarrow [0, \infty)$ by

$$\alpha((0, x), (0, y)) = \begin{cases} \frac{3}{4}, & \text{if } x, y \in [0, 1] \text{ with } (0, x) \preceq (0, y), \\ 0, & \text{otherwise.} \end{cases}$$

$$\beta((0, x), (0, y)) = \begin{cases} \frac{2}{5}, & \text{if } x, y \in [0, 1] \text{ with } (0, x) \preceq (0, y), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\psi(t) = \frac{7}{8}t$ for all $t \geq 0$. We now show that T is an α - β - ψ -proximal contractive mapping.

Now, let $(0, x), (0, y), (0, u)$ and $(0, v) \in A$ with $(0, x) \preceq (0, y)$ such that

$$(19) \quad d((0, u), T(0, x)) = d((0, v), T(0, y)) = d(A, B) = 1.$$

Let us consider the case $x, y \in [0, 1]$ with $(0, x) \preceq (0, y)$. Then $\alpha((0, x), (0, y)) = \frac{3}{4}$ and $\beta((0, x), (0, y)) = \frac{2}{5}$.

From (18), we obtain $u = \frac{x^2}{1+2x} \in [0, \frac{1}{3}]$ and $v = \frac{y^2}{1+2y} \in [0, \frac{1}{3}]$.

Therefore

$$\begin{aligned} & \alpha((0, x), (0, y))d((0, u), (0, v)) \\ &= \frac{3}{4}|u - v| = \frac{3}{4} \left| \frac{x^2}{1+2x} - \frac{y^2}{1+2y} \right| \\ &= \frac{3}{4} \left(\frac{|x - y|(x + y + 2xy)|}{(1+2x)(1+2y)} \right) \\ &\leq \frac{7}{20}|x - y| = \beta((0, x), (0, y))\psi(d((0, x), (0, y))). \end{aligned}$$

For the other possible cases the inequality (1) clearly holds. Hence T is an α - β - ψ -proximal contractive mapping.

We now show that T is an α - β -proximal admissible. For this purpose, we choose $C_\alpha = \frac{2}{3}$ and $C_\beta = \frac{1}{2}$. Clearly $0 \leq \frac{C_\beta}{C_\alpha} \leq 1$.

Let $(0, x), (0, y), (0, u)$ and $(0, v) \in A$ with $x \geq y$ such that

- (a) $\alpha((0, x), (0, y)) = \frac{3}{4} \geq \frac{2}{3} = C_\alpha$,
 $d((0, u), T(0, x)) = d((0, v), T(0, y)) = d(A, B) = 1$,
- (b) $\beta((0, x), (0, y)) = \frac{2}{5} \leq \frac{1}{2} = C_\beta$,
 $d((0, u), T(0, x)) = d((0, v), T(0, y)) = d(A, B) = 1$.

From (a) and (b), we obtain $u = \frac{x^2}{1+2x} \in [0, \frac{1}{3}]$ and $v = \frac{y^2}{1+2y} \in [0, \frac{1}{3}]$.

Since $x \geq y$, it follows that $u \geq v$. Therefore $\alpha((0, u), (0, v)) = \frac{3}{4} \geq \frac{2}{3} = C_\alpha$ and $\beta((0, u), (0, v)) = \frac{2}{5} \leq \frac{1}{2} = C_\beta$.

Hence T is an α - β -proximal admissible.

Now, we choose $x_0 = (0, \frac{1}{2})$, $x_1 = (0, \frac{1}{8})$, such that $d((0, \frac{1}{8}), T(0, \frac{1}{2})) = 1$, $\alpha((0, \frac{1}{2}), (0, \frac{1}{8})) = \frac{3}{4} \geq \frac{2}{3} = C_\alpha$, $\beta((0, \frac{1}{2}), (0, \frac{1}{8})) = \frac{2}{5} \leq \frac{1}{2} = C_\beta$ and $x_0 \preceq x_1$.

Finally, if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq C_\alpha$, $\beta(x_n, x_{n+1}) \leq C_\beta$ for all $n \in \mathbb{N}$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, then by definition of α and β , $x_n \in [0, 1]$. Thus $x \in [0, 1]$. i.e., $\alpha(x_n, x) \geq C_\alpha$ and $\beta(x_n, x) \leq C_\beta$. Further, $\{x_n\}$ is a nondecreasing sequence, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Hence all the hypotheses of Theorem 3.3 are satisfied and $(0, 0)$ is the unique best proximity point of T .

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