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ON $\{1, n\}$–NEUTRAL, INVERSING AND SKEW OPERATIONS OF n–GROUP

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Abstract. The skew operation in n–group ($n \geq 3$) has been introduced in [1]. Using this operation, n–group has been described in [2] as a variety of type $< n, 1 >$. Aim of this note is to connect the skew operation with neutral and inversing operations which have been introduced in [6] and [7].

1. Preliminaries

1.1. Definition: *Let $n \geq 2$ and let (Q, A) be an n–groupoid. We say that (Q, A) is a Dörnte n–group [briefly: n–group] iff is an n–semigroup and an n–quasigroup as well [: see, e. g. [3–5]].*[1)]

1.2. Definition [6]: *Let $n \geq 2$ and let (Q, A) be an n–groupoid. Further on, let* **e** *be an mappings of the set Q^{n-2} into the set Q. Let also $\{i, j\} \subseteq \{1, \ldots, n\}$ and $i < j$. Then:* **e** *is an $\{i, j\}$–neutral operation of the n–groupoid (Q, A) iff the following formula holds*

$$(\forall a_i \in Q)_1^{n-2}(\forall x \in Q)\left(A\left(a_1^{i-1}, \mathbf{e}(a_1^{n-2}), a_i^{j-2}, x, a_{n-2}^{j-2}\right) = x\right)$$
$$\wedge \left(A\left(a_1^{i-1}, x, a_i^{j-2}, \mathbf{e}(a_1^{n-2}), a_{n-2}^{j-2}\right) = x\right) \ ^{2)}$$

1.3. Proposition [6]: *Let $n \geq 2, \{1, \ldots, n\}$ and $i < j$. Then in every n–groupoid there is* **at most one** *$\{i, j\}$– neutral operation.*

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1) A notion of an n–group was introduced by W. Dörnte in [1] as a generalization of the notion of a group.

2) For $n = 2, \mathbf{e}(a_1^{n-2})$ $[= \mathbf{e}(\emptyset)$ $] = e \in Q$ is a neutral element of the groupoid (Q, A).

1.4. Proposition [6]: *In every $n-$group, $n \geq 2$, there is a $\{1,n\}$-neutral operation.*

1.5. Proposition [2]: *For $n \geq 3$, an $n-$semigroup (Q,A) is an $n-$group iff there is a unary operation $^-$ in Q such that the following laws are satisfied:*

$$A\left(x,\overset{n-2}{a},\overline{a}\right)=x,\quad A\left(\overline{a},\overset{n-2}{a},x\right)=x,$$
$$A\left(x,\overset{n-3}{a},\overline{a},a\right)=x,\quad A\left(a,\overline{a},\overset{n-3}{a},x\right)=x\ ^{3)}.$$

By 1.1, 1.2, 1.4 and 1.5, we conclude that also the following proposition holds:

1.6. Proposition : *Let $n \geq 3$. Also let (Q,A) be an $n-$group,* $\mathbf{e}$ *its $\{1,n\}-$neutral operation and $^-$ its unary operation "skew element." Then for every $a \in Q$ the following equality holds*

$$\overline{a}=\mathbf{e}\left(\overset{n-2}{a}\right).$$

1.7. Proposition [8]: *Let $n \geq 2$ and let (Q,A) be an $n-$groupoid. Then the following statements are equivalent: (i) (Q,A) is an $n-$group; (ii) there are mappings $^{-1}$ and* $\mathbf{e}$ *respectively of the sets Q^{n-1} and Q^{n-2} into the set Q such that the following laws hold in the algebra $(Q,\{A,^{-1},\mathbf{e})$ [of the type $< n, n-1, n-2 >$]*

(a) $A\left(x_1^{n-2},A\left(x_{n-1}^{2n-2}\right),x_{2n-1}\right)=A\left(x_1^{n-1},A\left(x_n^{2n-1}\right)\right)$,

(b) $A\left(\mathbf{e}\left(a_1^{n-2}\right),a_1^{n-2},x\right)=x$ and

(c) $A\left(\left(a_1^{n-2},a\right)^{-1},a_1^{n-2},a\right)=\mathbf{e}\left(a_1^{n-2}\right)$; and

(iii) there are mappings $^{-1}$ and $\mathbf{e}$ *respectively of the sets Q^{n-1} and Q^{n-2} into the set Q such that the following laws hold in the algebra $(Q,\{A,^{-1},\mathbf{e})$ [of the type $< n, n-1, n-2 >$]*

($\hat{a}$) $A\left(A\left(x_1^n\right),x_{n+1}^{2n-1}\right)=A\left(x_1,A\left(x_2^{n+1}\right),x_{n+2}^{2n-1}\right)$,

3) $\overline{a} \in Q$ is said to be a **skew element** of the element $a \in Q$ [1] [:see, e.g. [3–5]].

$(\hat{b})$ $A\left(x, a_1^{n-2}, \mathbf{e}\left(a_1^{n-2}\right)\right) = x$ and

$(\hat{c})$ $A\left(a, a_1^{n-2}, \left(a_1^{n-2}, a\right)^{-1}\right) = \mathbf{e}\left(a_1^{n-2}\right)$. [4)]

1.8. Proposition: *Let (Q,B) be an $n-$groupoid and $n \geq 2$. Then: 1) $\overset{1}{B} \overset{def}{=} B$, and 2) for every $k \in N$ and for every $x_1^{(k+1)(n-1)+1} \in Q$*

$$\overset{k+1}{B}\left(x_1^{(k+1)(n-1)+1}\right) \overset{def}{=} B\left(\overset{k}{B}\left(x_1^{k(n-1)+1}\right), x_{k(n-1)+2}^{(k+1)(n-1)+1}\right).$$

1.9. Proposition: *Let (Q,B) be an $n-$semigroup, $n \geq 2$ and $(i,j) \in N^2$. Then, for every $x_1^{(i+j)(n-1)+1} \in Q$ and for every $t \in \{1,\ldots,i(n-1)+ +1\}$ the following equality holds*

$$\overset{i+j}{B}\left(x_1^{(i+j)(n-1)+1}\right) = \overset{i}{B}\left(x_1^{t-1}, \overset{j}{B}\left(x_t^{t+j(n-1)}\right), x_{t+j(n-1)+1}^{(i+j)(n-1)+1}\right).$$

2. Results

2.1. Theorem: *Let $n \geq 3$. Also let (Q,A) be an $n-$group, $\mathbf{e}$ its $\{1,n\}-$ neutral operation and $^{-}$ its unary operation "skew element". Then for every sequence a_1^{n-2} over Q the following equality holds*

$$\mathbf{e}\left(a_1^{n-2}\right) = \overset{n-3}{A}\left(\bar{a}_{n-2}, \overset{n-3}{a}_{n-2}, \ldots, \bar{a}_1, \overset{n-3}{a_1}\right).$$

Proof. By 1.6, 1.8, 1.9 and 1.4, we conclude that for every sequence a_1^{n-2} over Q and for all $x \in Q$ the following series of equalities holds

$$A\left(\overset{n-3}{A}\left(\bar{a}_{n-2}, \overset{n-3}{a}_{n-2}, \ldots, \bar{a}_1, \overset{n-3}{a_1}\right), a_1^{n-2}, x\right) =$$

$$A\left(\overset{n-3}{A}\left(\mathbf{e}\left(\overset{n-3}{a}_{n-2}\right), \overset{n-3}{a}_{n-2}, \ldots, \mathbf{e}\left(\overset{n-2}{a_1}\right), \overset{n-3}{a_1}\right), a_1^{n-2}, x\right) =$$

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$$A\left(\mathbf{e}\left(\overset{n-2}{a}_{n-2}\right), \overset{n-3}{a}_{n-2}, a_{n-2}, x\right) = x,$$

[4)] $(a_1^{n-2}, a)^{-1} = \mathsf{E}(a_1^{n-2}, a, a_1^{n-2})$, where E is $A\{1, 2n-1\}-$neutral operation of a $(2n-1)-$ group $(Q, \overset{2}{A})$; $\overset{2}{A}(x_1^{2n-1}) \overset{def}{=} A(A(x_1^n), x_{n+1}^{2n-1})$ [7]. For $n = 2, a^{-1} = \mathsf{E}(a); a^{-1}$ is the inverse element of the element a with respect to the neutral element $\mathbf{e}(\emptyset)$ of the group (Q, A).

whence, by 1.2 and 1.4, we conclude that for every sequence a_1^{n-2} over Q and for all $x \in Q$ the following equality holds

$$A\left(\overset{n-3}{A}\left(\overline{a}_{n-2}, \overset{n-3}{a}_{n-2}, \ldots, \overline{a}_1, \overset{n-3}{a_1}\right), a_1^{n-2}, x\right) = A\left(\mathbf{e}(a_1^{n-2}, a_1^{n-2}, x\right).$$

Whence, by 1.1, we conclude that for every sequence a_1^{n-2} over Q the following equality holds

$$\mathbf{e}\left(a_1^{n-2}\right) = \overset{n-3}{A}\left(\overline{a}_{n-2}, \overset{n-3}{a}_{n-2}, \ldots, \overline{a}_1, \overset{n-3}{a_1}\right). \qquad \square$$

2.2. Theorem: *Let $n \geq 3$. Also let (Q, A) be an $n-$group, $\mathbf{e}$ its $\{1, n\}-$neutral operation, $^{-1}$ its inversing operation*[5] *and $^{-}$ its unary operation " skew element". Then for every sequence a_1^{n-2} over Q, for every sequence c_1^{n-3} over Q and for all $a \in Q$ the following equality holds*

$$\left(a_1^{n-2}, a\right)^{-1} = \overset{3n-8}{A}\left(\overline{a}_{n-2}, \overset{n-3}{a}_{n-2}, \ldots, \overline{a_1}, \overset{n-3}{a_1}, c_1^{n-3}, \overline{c}_{n-3}, \overset{n-3}{c}_{n-3}, \ldots\right.$$
$$\left.\ldots, \overline{c}_1, \overset{n-3}{c_1}, \overline{a}, \overset{n-3}{a}, \overline{a}_{n-2}, \overset{n-3}{a}_{n-2}, \ldots, \overline{a}_1, \overset{n-3}{a_1}\right).$$

Proof. By 1.1, 1.4 and 1.2, we conclude that for every sequence a_1^{n-2} over Q, for every sequence c_1^{n-3} over Q and for all $a \in Q$ the following series of equivalences holds

$$A\left(\left(a_1^{n-2}, a\right)^{-1}, a_1^{n-2}, a\right) = \mathbf{e}\left(a_1^{n-2}\right) \Leftrightarrow$$
$$A\left(A\left(\left(a_1^{n-2}, a\right)^{-1}, a_1^{n-2}, a\right), c_1^{n-3}, \mathbf{e}\left(a, c_1^{n-3}\right), \mathbf{e}\left(a_1^{n-2}\right)\right) =$$
$$A\left(\mathbf{e}\left(a_1^{n-2}\right), c_1^{n-3}, \mathbf{e}\left(a, c_1^{n-3}\right), \mathbf{e}\left(a_1^{n-2}\right)\right) \Leftrightarrow$$
$$A\left(\left(a_1^{n-2}, a\right)^{-1}, a_1^{n-3}, A\left(a_{n-2}, a, c_1^{n-3}, \mathbf{e}\left(a, c_1^{n-3}\right)\right), \mathbf{e}\left(a_1^{n-2}\right)\right) =$$
$$A\left(\mathbf{e}\left(a_1^{n-2}\right), c_1^{n-3}, \mathbf{e}\left(a, c_1^{n-3}\right), \mathbf{e}\left(a_1^{n-2}\right)\right) \Leftrightarrow$$
$$A\left(\left(a_1^{n-2}, a\right)^{-1}, a_1^{n-3}, a_{n-2}, \mathbf{e}\left(a_1^{n-2}\right)\right) =$$
$$A\left(\mathbf{e}\left(a_1^{n-2}\right), c_1^{n-3}, \mathbf{e}\left(a, c_1^{n-3}\right), \mathbf{e}\left(a_1^{n-2}\right)\right) \Leftrightarrow$$
$$\left(a_1^{n-2}, a\right)^{-1} = A\left(\mathbf{e}\left(a_1^{n-2}\right), c_1^{n-3}, \mathbf{e}\left(a, c_1^{n-3}\right), \mathbf{e}\left(a_1^{n-2}\right)\right),$$

[5] [7]; see 1.7 and footnote 4.

whence, by 1.7, we conclude that for every sequence a_1^{n-2} over Q, for every sequence c_1^{n-3} over Q and for all $a \in Q$ the following equality holds:

$$\left(a_1^{n-2}, a\right)^{-1} = A\left(\mathbf{e}\left(a_1^{n-2}\right), c_1^{n-3}, \mathbf{e}\left(a, c_1^{n-3}\right), \mathbf{e}\left(a_1^{n-2}\right)\right).$$

Hence, by 2.1 and 1.9, we conclude that the proposition holds.

Remark: Moreover, for $c_1 = \ldots = c_{n-3} = a$, for every sequence a_1^{n-2} over Q and for all $a \in Q$ the following equality holds

$$\left(a_1^{n-2}, a\right)^{-1} = \overset{2n-5}{A}\left(\overline{a}_{n-2}, \overset{n-3}{a}_{n-2}, \ldots \right.$$

$$\left. \ldots, \overline{a_1}, \overset{n-3}{a_1}, \overset{n-3}{a}, \overline{a}, \overline{a}_{n-2}, \overset{n-3}{a}_{n-2}, \ldots, \overline{a}_1, \overset{n-3}{a_1}\right). \square$$

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