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TRANSVERSAL SPACES

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Abstract. In this paper we formulate a new structure of spaces which we call it transversal (upper or lower) spaces. We introduced the concept of transversal ordered (upper or lower) spaces as a natural extension of Fréchet's, Kurepa's and Menger's spaces.

1. Fundamental facts

The possibility of defining such notions as limit and continuity in an arbitrary set is an idea which undoubtedly was first put forward by M. Fréchet in 1904, and developed by him in his famous doctoral dissertation 1905.

In 1934 Đ. Kurepa introduced the notion of a pseudo-metric space; and in 1936 also Đ. Kurepa introduced, for a given ordinal α, the notion of (Δ^α) or (D_α) as the class of pseudo-metric spaces. The case $\alpha = 0$ coincides with the class of metric spaces.

In this paper we introduce a new concept by name transversal spaces as a nature extension of Fréchet's, Kurepa's and Menger's spaces.

Let X be a nonempty set. The function $\rho : X \times X \to \mathbf{R}_+^0 := [0, \infty)$ is called an **upper transverse** on X (or upper transversal) if: $\rho[x,y] = \rho[y,x]$, $\rho[x,y] = 0$ iff $x = y$, and if there is a function $g : (\mathbf{R}_+^0)^2 \to \mathbf{R}_+^0$ such that

$$\text{(A)} \qquad \rho[x,y] \leq \max\{\rho[x,z], \rho[z,y], g(\rho[x,z], \rho[z,y])\}$$

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for all $x, y, z \in X$. An **upper transversal space** is a set X together with a given upper transverse on X. The function $g : (\mathbf{R}_+^0)^2 \to \mathbf{R}_+^0$ in (A) is called **upper bisection function.**

From (A) it follows by induction that there is a function $g : (\mathbf{R}_+^0)^n \to$ $\to \mathbf{R}_+^0$ such that

$$\rho[x_0, x_n] \leq \max\left\{\rho[x_0, x_1], \dots, \rho[x_{n-1}, x_n], g\left(\rho[x_0, x_1], \dots, \rho[x_{n-1}, x_n]\right)\right\}$$

for all $x_0, x_1, \dots, x_n \in X$ and for any fixed integer $n \geq 2$.

A fundamental first example of upper transversal space for the upper bisection function $g : (\mathbf{R}_+^0)^2 \to \mathbf{R}_+^0$ defined by $g(a, b) := a + b$ is a *metric space*. Also, if the upper bisection function $g : (\mathbf{R}_+^0)^2 \to \mathbf{R}_+^0$ defined by $g(a, b) := \max\{a, b\}$, then we obtain *ultrametric spaces*.

On the other hand, the function $\rho : X \times X \to \mathbf{R}_+^0$ is called a **lower transverse** on X (or lower transversal) if: $\rho[x, y] = \rho[y, x]$, $\rho[x, y] = 0$ iff $x = y$, and if there is a **lower bisection function** $d : (\mathbf{R}_+^0)^2 \to \mathbf{R}_+^0$ such that

$$\text{(Am)} \qquad \min\left\{\rho[x, z], \rho[z, y], d\left(\rho[x, z], \rho[z, y]\right)\right\} \leq \rho[x, y]$$

for all $x, y, z \in X$. A **lower transversal space** is a set X together with a given lower transverse on X.

In connection with this, an example of lower transversal spaces is also a metric space, because a lower bisection function $d : (\mathbf{R}_+^0)^2 \to \mathbf{R}_+^0$ we can defined by $d(a, b) := \Big||a| - |b|\Big|$.

Thus, any metric space is a transversal space, i.e., an upper and a lower transversal space simultaneous.

In connection with the fixed points we have very interesting architecture of lower transversal spaces.

In this sense, if (X, ρ) a lower transversal space and if T is a self-map of X such that

$$\text{(Id)} \qquad \min\left\{\rho[a, b], d\left(\rho[a, b], \rho[a, b]\right)\right\} = 0 \ \text{ implies } \ a = b\,,$$

then T has at least fixed point in space X.

Any metric space is a lower transversal space, but metric spaces have not property (Id), i.e., not hold that

$$\min\left\{\rho[a, b], |\rho[a, b] - \rho[a, b]|\right\} = 0 \ \text{ implies } \ a = b\,.$$

A special feature in the former notions (of Fréchet and Kurepa) is the "triangular relation" occuring in the elementary geometry and in many other cases.

At the same time, Fréchet considered instead of triangular relation, apparently weaker, **regularity condition**: There exists a self-map f of $\mathbf{R}_+ := (0, +\infty)$ into itself such that $f(x) \to 0$ $(x \to 0)$ and that for any triple (a, b, c) of elements of X one has $\rho(a, b) < x$ and $\rho(b, c) < x$ implies $\rho(a, c) < f(x)$.

Fréchet remarked that metric spaces (X, ρ) and the preceding spaces (X, ρ, f) with the regularity condition have similar properties. In 1910 he asked whether this two classes of spaces should be the same. Chittenden in 1917 confirmed this conjecture. A simple proof was exibited by Frink in 1937.

We remarked that an important example of upper transversal spaces is also and every Frechet's space with the regularity condition. For this an upper bisection function $g : (\mathbf{R}_+^0)^2 \to \mathbf{R}_+^0$ can be defined by $g(\alpha, \beta) := = \max\{x, f(x)\}$.

2. Ordered transversal spaces

Further, as a natural extension of transversal (upper and lower) spaces we have the following notations of spaces.

Let X be a nonempty set and let $P := (P, \preceq)$ be a partially ordered set. The function $\rho : X \times X \to P$ is called an **upper ordered transverse** on X (or upper ordered transversal) if: $\rho[x, y] = \rho[y, x]$, and if there is an upper bisection function $g : P \times P \to P$ such that

(Bs) $$\rho[x, y] \preceq \sup\{\rho[x, z], \rho[z, y], g(\rho[x, z], \rho[z, y])\}$$

for all $x, y, z \in X$. An **upper ordered transversal space** is a set X together with a given upper ordered transverse on X.

Let $k = \aleph_\alpha$ $(\alpha \geq 0)$ be a regular cardinal. Call a topological space X an **upper k-transversal space** or a **$g(D_\alpha)$-space** if there exists $\rho : : X \times X \to \omega_\alpha \cup \{\omega_\alpha\} := W$ such that: $\rho[x, y] = \omega_\alpha$ iff $x = y$, $\rho[x, y] = \rho[y, x]$, and if there is $g : W^2 \to W$ such that (Bs) for all $x, y, z \in X$.

Obviously, Fréchet's ordered spaces are important examples of upper k-transversal spaces.

Open problem 1. Does for every regular cardinal $k \geq \aleph_0$ there exists an upper k-transversal nonlinearly orderable topological space?

In connection with the preceding, the function $\rho : X \times X \to (P, \preceq)$ is called a **lower ordered transverse** on X (or lower ordered transversal) if: $\rho[x,y] = \rho[y,x]$ and if there is a lower bisection function $d : P \times P \to P$ such that

$$\text{(Bi)} \qquad \inf\{\rho[x,z], \rho[z,y], d(\rho[x,z], \rho[z,y])\} \preceq \rho[x,y]$$

for all $x, y, z \in X$. A **lower ordered transversal space** is a set X together with a given lower ordered transverse on X.

On the other hand, let $k = \aleph_\alpha$ ($\alpha \geq 0$) be a regular cardinal. Call a topological space X a **lower k-transversal space** or $d(D_\alpha)$**-space** if there exists the function $\rho : X \times X \to \omega_\alpha \cup \{\omega_\alpha\} := W$ such that: $\rho[x,y] = \omega_\alpha$ iff $x = y$, $\rho[x,y] = \rho[y,x]$ and if there is $d : W^2 \to W$ such that (Bi) for all $x, y, z \in X$.

Open problem 2. Does for every regular cardinal $k \geq \aleph_0$ there exists a lower k-transversal nonlinearly orderable topological space?

We notice, in connection with this problem, that work of Đ. Kurepa in 1963 is very important, where there is result that for every regular cardinal $k \geq \aleph_\alpha$ ($\alpha \geq 0$) there exists a k-metrizable nonlinearly orderable topological space. A proof of this result was exhibit by S. Todorčević in 1981.

Let $k = \aleph_\alpha$ ($\alpha \geq 0$) be a regular cardinal. Call a topological space X a **k-metrizable space** or a D_α**-space** if there exist $\rho : X^2 \to \omega_\alpha \cup \{\omega_\alpha\}$ and $\phi : \omega_\alpha \to \omega_\alpha$ such that: $\rho(x,y) = \omega_\alpha$ iff $x = y$, $\rho(x,y) = \rho(y,x)$, $\rho(x,y) > \phi(\xi)$ and $\rho(y,z) > \phi(\xi)$ implies $\rho(x,z) > \xi$, and that the sets

$$B_\xi(x) = \{y \in X : \rho(x,y) > \xi\}$$

for $x \in X$ and $\xi < \omega_\alpha$ are form a basis of X.

This definition was given by Đ. Kurepa in 1934 using the name pseudo-distancial spaces. The class of all D_0-spaces is just the class of all metrizable spaces. The class of all pseudo-distancial spaces was extensively considered by Đ. Kurepa, Fréchet, Doss, Colmez, Appert, Papić, Mamuzić, Cammaroto, Kočinac, Ky Fan and many others.

This class has also the name "spaces with linearly ordered basis of uniformity". We notice, that, an important example of lower ordered transversal spaces is and every Kurepa's pseudo-distancial space. For this a lower bisection function $d : P \times P \to P$ can be defined by $d(a,b) := = \inf\{\xi, \phi(\xi)\}$.

Obviously, Kurepa's pseudo-distancial spaces are important examples of lower k-transversal spaces.

On the other hand, let $\tau = \omega_\mu$ be a regular cardinal number, X a set and $(G, +, \preceq)$ a linearly ordered abelian group with cofinality $\mathrm{cof}(G) = \omega_\mu$ at the identity element $0 \in G$ (which means that 0 is the infimum of a strictly decreasing τ-sequence $\{x_\alpha : \alpha \in \tau\} \subset G\backslash\{0\}$). An τ-**metric** on X is a function $\rho : X^2 \to (G, \preceq)$ which satisfies all the formal properties of metric.

This definition of space X was given by Đ. Kurepa in 1934 and by Sikorski in 1950 using the name ω_μ-**metrizable topological space** (if its topology can be induced by some ω_μ-metric on X).

Obviously, ω_μ-metrizable topological spaces are fundamental examples of upper transversal ordered spaces with the upper bisection function $g : G^2 \to G$ defined by $g(a,b) := a + b$.

Open problem 3. Find sufficient and necessary conditions such that an upper or a lower k-transversal space is the form of a Kurepa's space (a metric space or an uniform space or a Fréchet's space with the regularity condition)!?

In the theory of metric spaces, as and in the transversal spaces, it is extremely convenient to use a geometrical language inspired by classical geometry.

Thus elements of a transversal space will usually be called **points.** Given an upper transversal space (X, ρ), with the upper bisection function $g : P \times P \to P$ and a point $a \in X$, the **open ball** of center a and radius $r \in P$ is the set

$$g\,(B(a,r)) = \{x \in X :\ \rho[a,x] \prec r\}\ ,$$

till for given a lower transversal space (X, ρ), with the lower bisection function $d : P \times P \to P$ and a point $a \in X$, the **open ball** of center a and radius $r \in P$ is the set

$$d\,(B(a,r)) = \{x \in X :\ r \prec \rho[a,x]\}\ .$$

3. Fuzzy and Menger's spaces are transversal spaces

As an important example of lower ordered transversal spaces we have a Menger's (probabilistic) space.

K. Menger introduced in 1928 and 1942 the notion of probabilistic metric space. O. Kaleva and S. Seikkala proved in 1984 that each Menger space, which is a special probabilistic metric space, can be considered as a fuzzy metric space.

Interesting, every fuzzy metric space is, de facto, also a lower ordered transversal space.

Let X be a nonempty set, E a set of all upper semicontinuous normal convex fuzzy numbers, G a set of all nonnegative fuzzy numbers of E and $m : X \times X \to G$.

A quadruple (X, m, L, R) is called a **fuzzy metric space** with m as a **fuzzy metric** if $L, R : [0,1]^2 \to [0,1]$ are symmetric functions, nondecreasing in both arguments, $L(0,0) = 0$ and $R(1,1) = 1$ such that $m(x,y) = 0$ iff $x = y$, $m(x,y) = m(y,x)$,

$$m(x,y)_{(s+r)} \leq R\left(m(x,z)_{(s)}, m(z,y)_{(r)}\right)$$

for all $x, y, z \in X$, where $s \geq \lambda_1(x,z)$, $r \geq \lambda_1(z,y)$ and $s + r \geq \lambda_1(x,y)$, and such that

$$m(x,y)_{(s+u)} \geq L\left(m(x,z)_{(s)}, m(z,y)_{(u)}\right)$$

for all $x, y, z \in X$, where $s \leq \lambda_1(x,z)$, $u \leq \lambda_1(z,y)$, and $s + u \leq \lambda_1(x,y)$.

In connection with this, let $P := [0,1]$ and we chosen a lower bisection function $d : [0,1]^2 \to [0,1]$ such that $d = L$ (with the peceding properties), then we immediate obtain that every fuzzy metric space is a lower ordered transversal space.

On the other hand, if for the upper bisection function $g : [0,1]^2 \to \to [0,1]$ we chosen that is $g = R$ (with the preceding properties), then we have that every fuzzy metric space is an upper ordered transversal space. This mean, common with the preceding, that every fuzzy metric space, de facto, is an ordered transversal space.

A mapping $F : \mathbf{R} \to \mathbf{R}_+^0 := [0, +\infty)$ is called a distribution function if it is nondecreasing, left continuous with $\inf F = 0$ and $\sup F = 1$. We will denote by $\mathcal{D}$ the set of all distribution functions.

A probabilistic metric space is a pair (E, F), where E is an abstract set of elements and F is a mapping of $E \times E$ into $\mathcal{D}$. We shall denote the distribution function $F(p,q)$ by $F_{p,q}$ and $F_{p,q}(x)$ will represent the value of $F_{p,q}$ at $x \in \mathbf{R}$.

The function $F_{p,q}(p, q \in E)$ are assumed to satisfy the following conditions: $F_{p,q}(x) = 1$ for $x > 0$ iff $p = q$, $F_{p,q}(0) = 0$, $F_{p,q} = F_{q,p}$ and

$$F_{p,q}(x) = 1 \quad \text{and} \quad F_{q,r}(y) = 1 \quad \text{implies} \quad F_{p,r}(x + y) = 1$$

for all $p, q, r \in E$. (This definition suggests that $F_{p,q}(x)$ may be interpreted as probability of the event that the distance between p and q is less than x.).

Let B denote the set of all Δ-norms. A **Menger space** is a triplet (E, F, Δ), where (E, F) is a probabilistic metric space and $\Delta \in B$ satisfies the following inequality:

$$F_{p,r}(x, y) \geq \Delta\left(F_{p,q}(x), F_{q,r}(y)\right)$$

for all $p, q, r \in E$ and for all $x, y \geq 0$. Metric spaces are special cases of Menger spaces with $\Delta(x, x) \geq x$ for all $x \in [0, 1]$.

If the partially ordering set $P := [0,1]$ and if we chosen a lower bisection function $d : [0,1]^2 \to [0,1]$ such that $d = \Delta$ (for $\Delta \in B$), then we immediate obtain that every Menger's space, for $\rho[x, y] = F_{x,y}$, is a lower ordered transversal space.

4. Fixed points on transversal spaces

I admit very interesting of the architecture and behaviour of the lower ordered transversal spaces according to problem of fixed point.

Namely, if (X, ρ) is a lower ordered transversal space and if T is a self-map on X such that

(Mi) $$\inf\left\{\rho[a, b], d\left(\rho[a, b], \rho[a, b]\right)\right\} \preceq \rho(a, a) \text{ implies } a = b\,,$$

then T has at least fixed point on space X.

In this sense, metric spaces are lower ordered transversal spaces but, in general case, they not have the property of fixed point because for them not hold (Mi), i.e.,

$$\min\left\{\rho[a, b], |\rho[a, b] - \rho[a, b]|\right\} = 0$$

not implies $a = b$. This holds and for Menger's spaces.

However, if $g(X, m, L, R)$ is a fuzzy metric space with $L \neq 0$, then every self-map T on X has at least fixed point.

Menger's space are fuzzy metric spaces, but then not is $L \neq 0$, i.e., then $L \equiv 0$.

On the other hand, let (X, ρ) be a lower transversal space with a lower bisection function $d : (\mathbf{R}_+^0)^2 \to \mathbf{R}_+^0$. In a lower transversal space X, a **C-transversal sequence** is an infinite sequence $\{x_n\}_{n\in\mathbf{N}}$ such that $\min\{\rho[x_n, x_m], \rho[x_m, x_{n+1}], d(\rho[x_n, x_m], \rho[x_m, x_{n+1}])\} \to 0$ for $m > n \to \infty$ implies that $\rho[x_n, x_m] \to 0$ for $m > n \to \infty$.

In order that $J = \lim_{n\to\infty} x_n$, a necessary and sufficient condition is that, for $\varepsilon > 0$, there exist an integer n_0 such that the relation $n \geq n_0$ implies $\rho[J, x_n] < \varepsilon$.

A lower transversal space (X, ρ) is called **complete** if any C-transversal sequence $\{x_n\}_{n\in\mathbf{N}}$ in X is convergent (to a point of X, of course).

Theorem 1. *Let T be a self-map on a complete lower transversal space (X, ρ) and let $\rho[T^n x, T^{n+1} x] \to 0$ $(n \to \infty)$ for the sequence of iterates $\{T^n x\}_{n\in\mathbf{N}}$. If $x \mapsto \rho[x, Tx]$ is a lower semicontinuous function, then T has at least fixed point in X.*

As an immediate application of this statement we obtain the following result of Banach's type on a lower transversal space.

Theorem 2. *Let T be a self-map on a complete lower transversal space (X, ρ) and let there exists $\lambda \in [0, 1)$ such that*

$$\text{(Sb)} \qquad \rho[Tx, Ty] \leq \lambda \rho[x, y] \quad \textit{for all } x, y \in X.$$

If $x \mapsto \rho[x, Tx]$ is a lower semicontinuous function, then T has an unique fixed point in X.

Proof. Since the condition (Sb) implies that the following fact holds that $\rho[T^n x, T^{n+1} x] \to 0$ $(n \to \infty)$, thus it follows from Theorem 1 that there is a fixed point $\xi = T\xi$ for some $\xi \in X$. Uniqueness follows immediately from condition (Sb). The proof is complete.

Proof of Theorem 1. Let (X, ρ) be a lower transversal space with the lower bisection function $d : (\mathbf{R}_+^0)^2 \to \mathbf{R}_+^0$, then we obtain directly

$$\min\{\rho[x_n, x_m], \rho[x_m, x_{n+1}], d(\rho[x_n, x_m], \rho[x_m, x_{n+1}])\} \leq \rho[x_n, x_{n+1}]$$

where $x_n := T^n(x)$ for $n \in \mathbf{N}$. Thus we have that $\rho[x_n, x_m] \to 0$ for $m > n \to \infty$.

This implies that the C-transversal sequence $\{T^n x\}_{n \in \mathbf{N}}$ is convergent in X, i.e., there is $\xi \in X$ such that $T^n x \to \xi$ $(n \to \infty)$. Since $x \mapsto \rho[x, Tx]$ is a lower semicontinuous function at ξ, $\rho[\xi, T\xi] = 0$, i.e., ξ is a fixed point of T in X. The proof is complete.

5. References

[1] A. Appert and Ky Fan: *Espaces topologiques intermédiaires. Problème de la distanciation*, Actuelités scient., Paris, 1951.

[2] F. Cammaroto and Lj. Kočinac: *Some results on ω_μ-metrizable and related spaces*, Boll. U. M. I., **7-B** (1993), 607-629.

[3] E. W. Chittenden: *On the equivalence of ecart and voisinage*, Trans. Amer. Math. Soc., **18** (1917), 161-166.

[4] J. Colmez: *Espaces à écart généralisé régulier*, C. R. Acad. Sci. Paris, **224** (1947), p.p. 372.

[5] R. Doss: *Sur la condition de régularité pour l'ecart abstrait. Écart abstrait symétrique et régulier*, C. R. Acad. Sci. Paris, **223** (1946), 14-16, 1087-1088.

[6] M. Fréchet: *La notion d'écart et le calcul fonctionnel*, C. R. Acad. Sci. Paris, **140** (1905), 772-774.

[7] M. Fréchet: *Sur quelques points du calcul fonctionnel*, Thèse, Paris, 1905. Rendiconti Circolo Mat. Palermo, **22** (1906), 1-74.

[8] A. H. Frink: *Distance functions and the metrization problem*, Bull. Amer. Math. Soc., **43** (1937), 133-142.

[9] O. Kaleva and S. Seikkala: *On fuzzy metric spaces*, Fuzzy Sets and Systems, **12** (1984), 215-229.

[10] Đ. R. Kurepa: *Tableaux ramifiés d'ensembles. Espaces pseudo-distanciés*, C. R. Acad. Paris, **198** (1934), 1563-1565.

[11] Đ. R. Kurepa: *Ensembles ordonnés et ramifiés*, Thèse, Paris, 1935, p.p. 1-138; Publ. Inst. Math. Belgrade, **4** (1935), 1-138.

[12] Đ. R. Kurepa: *Sur les classes (E) et (D)*, Publ. Inst. Math. Belgrade, **55** (1936), 124-132.

[13] Đ. R. Kurepa: *Un critère de distanciabilitè*, Mathematica (Cluj), **13** (1937), 59-65.

[14] Đ. R. Kurepa: *Distanca numerica. Distanca non numerica*, Publ. Seminario Matematica, Bari, 1963.

[15] Đ. R. Kurepa: *On the existence of pseudometric non totally orderable spaces*, Glasnik Mat. Fiz. Astr. Zagreb, **18** (1963), 183-194.

[16] Z. Mamuzić: *Introduction to General Topology*, P. Noordhoff, Groningen, 1963.

[17] K. Menger: *Untersuchungen über allgemeine Metrik*, Math. Ann., **100** (1928), 75-163.

[18] K. Menger: *Statistical metric*, Proc. Nat. Acad. Sci. USA, **28** (1942), 535-537.

[19] P. Papić: *Sur une classe d'espaces abstrait*, Glasnik Mat. Fiz. Astr. Zagreb, **9** (1954), 197-216.

[20] R. Sikorski: *Remark on some topological spaces of high power*, Fund. Math., **37** (1950), 125-136.

[21] M. R. Tasković: *Nonlinear Functional Analysis*, Fundamental elements of theory, First Book. Monographs, Zavod za udžbenike i nastavna sredstva, Beograd 1993, 792 p.p. (Serbo-Croation). English summary: *Comments only new main results of this book.* Vol. **1** (1993), 713-752.

[22] S. Todorčević: *On a theorem of Đ. Kurepa*, in *Topology and order structures I*, Math. Centre Tracts **142**, Amsterdam, 1981, 173-176.

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